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Invariant regions and global existence of solutions for reaction-diffusion systems with a general full matrix

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Abstract
The aim of this study is to prove the global existence in time of solutions for reaction-diffusion systems. We make use of the appropriate techniques which are based on invariant regions and Lyapunov functional methods. We consider a full matrix of diffusion coefficients and we show the global existence of the solutions.
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1 Introduction

We are mainly interested in the global existence in time of solutions to a reaction-diffusion system of the form

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v = \Pi - f(u,v) - \sigma u \quad \text{in }]0,+\infty[\, \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - d\Delta v = f(u,v) - \sigma v \quad \text{in }]0,+\infty[\, \times \Omega \tag{1.2}$$

with the following boundary conditions:

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{in }]0,+\infty[\, \times \partial\Omega \tag{1.3}$$

and the initial data

$$u(0,x) = u_0, \qquad v(0,x) = v_0 \quad \text{in } \Omega, \tag{1.4}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^1, $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$, Δ denotes the Laplacian operator with respect to the x variable, a, b, c, d, σ are positive constants satisfying the condition $(b+c)^2 < 4ad$, which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion is positive definite, $\Pi \geq 0$. The eigenvalues λ_1 and λ_2 ($\lambda_1 < \lambda_2$) of the matrix are positive. We assume that

$$\lambda_1 < a < d < \lambda_2 < a + c,$$

and the initial data are assumed to be in the following region:

$$\Sigma = \left\{(u_0, v_0) \in \mathbb{R}^2 \text{ such that } \frac{a-\lambda_2}{c} v_0 \le u_0 \le \frac{a-\lambda_1}{c} v_0\right\}. \tag{1.5}$$

For more details, one may consult [1].

The function f is a nonnegative continuously differentiable function on Σ such that

$$f\left(\frac{a-\lambda_2}{c}\eta, \eta\right) = 0 \quad \text{and} \quad f\left(\frac{a-\lambda_1}{c}\eta, \eta\right) \ge \frac{\Pi}{(1 + \frac{a-\lambda_1}{c})} \quad \text{for all } \eta \ge 0. \tag{1.6}$$

In addition we suppose that

$$(\xi, \eta) \in \Sigma \quad \Longrightarrow \quad 0 \le f(\xi, \eta) \le \varphi(\xi)(1+\eta)^\beta, \tag{1.7}$$

where $\beta \ge 1$ and φ is a nonnegative function of class $C(\mathbb{R})$ such that

$$\lim_{\xi \to -\infty} \frac{\varphi(\xi)}{\xi} = 0. \tag{1.8}$$

Melkemi *et al.* [2] established the existence of global solutions (eventually uniformly bounded in time) using a novel approach that involved the use of a Lyapunov function for system (1.1)-(1.4) when $c = b = 0$. Along the same lines, Rebai [3] has proved the global existence of solutions for system (1.1)-(1.4), in the case $b = 0$, $c > 0$ (triangular matrix). The present investigation is a continuation of results obtained in [3]. Here, we follow the same reasoning as in [2], in the study of system (1.1)-(1.4), when $b > 0$, $c > 0$, that is, for a model that involves a general full matrix.

The components $u(t,x)$ and $v(t,x)$ represent either chemical concentrations or biological population densities and system (1.1)-(1.2) is a mathematical model describing various chemical and biological phenomena (see, *e.g.*, Cussler [4]).

Remark 1 If $a < d$, then we have $\lambda_1 < a < d < \lambda_2$. We note that the condition of parabolicity implies that $\det(A) = ad - bc > 0$, where A is the matrix of diffusion.

2 Local existence and invariant regions

Throughout the text we shall denote by $\| \|_p$ the norm in $L^p(\Omega)$, and by $\| \|_\infty$ the norm in $L^\infty(\Omega)$ or $C(\overline{\Omega})$.

For any initial data in $C(\overline{\Omega})$ or $L^p(\Omega)$, $p \in]1, +\infty[$, local existence and uniqueness of solutions to the initial value problem (1.1)-(1.4) follow from the basic existence theory for abstract semilinear differential equations (see Henry [5] and Pazy [6]). The solutions are classical on $]0; T^*[$, where T^* denotes the eventual blowing-up time in $L^\infty(\Omega)$.

Furthermore, if $T^* < +\infty$, then

$$\lim_{t \uparrow T^*} \left(\|u(t)\|_\infty + \|v(t)\|_\infty\right) = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \le C, \quad \forall t \in \left]0, T^*\right[,$$

then $T^* = +\infty$.

Multiplying (1.2) first through by $-\frac{a-\lambda_2}{c}$ and adding (1.1) and then by $\frac{a-\lambda_1}{c}$ and subtracting (1.1), we get

$$\frac{\partial w}{\partial t} - \lambda_2 \Delta w = \Pi - \left(1 + \frac{a-\lambda_2}{c}\right) F(w,z) - \sigma w \quad \text{in }]0, T^*[\times \Omega, \tag{2.1}$$

$$\frac{\partial z}{\partial t} - \lambda_1 \Delta z = -\Pi + \left(1 + \frac{a-\lambda_1}{c}\right) F(w,z) - \sigma z \quad \text{in }]0, T^*[\times \Omega, \tag{2.2}$$

with the boundary conditions

$$\frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 \quad \text{in }]0, T^*[\times \partial\Omega, \tag{2.3}$$

and the initial data

$$w(0,x) = w_0(x), \qquad z(0,x) = z_0(x) \quad \text{in } \Omega, \tag{2.4}$$

where

$$\begin{aligned} w(t,x) &= u(t,x) - \frac{a-\lambda_2}{c} v(t,x), \\ z(t,x) &= -u(t,x) + \frac{a-\lambda_1}{c} v(t,x) \end{aligned} \tag{2.5}$$

for any (t,x) in $]0, T^*[\times \Omega$ and

$$F(w,z) = f(u,v) \quad \text{for all } (u,v) \text{ in } \Sigma. \tag{2.6}$$

To prove that Σ is an invariant region for system (1.1)-(1.4) it suffices to prove that the region

$$\Sigma_1 = \left\{(w_0, z_0) \in \mathbb{R}^2 \text{ such that } w_0 \geq 0, z_0 \geq 0\right\}$$

is invariant for system (2.1)-(2.4).

Now, to prove that the region Σ_1 is invariant for system (2.1)-(2.4), it suffices to show that $(\Pi - (1 + \frac{a-\lambda_2}{c})F(0,z)) \geq 0$ for $z \geq 0$, and $(-\Pi + (1 + \frac{a-\lambda_1}{c})F(w,0)) \geq 0$, for $w \geq 0$, see [7].

From (1.6), its clear that the region Σ_1 is invariant for system (2.1)-(2.4) and from (2.5) we have

$$\begin{aligned} v(t,x) &= \frac{c}{\lambda_2 - \lambda_1}\big(w(t,x) + z(t,x)\big), \\ u(t,x) &= \frac{a-\lambda_1}{\lambda_2 - \lambda_1} w(t,x) + \frac{a-\lambda_2}{\lambda_2 - \lambda_1} z(t,x). \end{aligned} \tag{2.7}$$

Remark 2 We note that if $(\xi, \eta) \in \Sigma$, then $\xi \in \mathbb{R}$ and $\eta \geq 0$.

3 Existence of global solutions

A simple application of the comparison theorem [7, Theorem 10.1] to system (2.1)-(2.4) implies that for any initial conditions $w_0 \geq 0$ and $z_0 \geq 0$, we have

$$0 \leq w(t,x) \leq \max\left(\|w_0\|_\infty, \frac{\Pi}{\sigma}\right) = K. \tag{3.1}$$

To prove the global existence of the solutions of problem (1.1)-(1.4), one needs to prove it for problem (2.1)-(2.4). As regards this subject, it is well known that it suffices to derive a uniform estimate of $\|-\Pi + (1 + \frac{a-\lambda_1}{c})F(w,z) - \sigma z\|_p$ for some $p > \frac{n}{2}$, *i.e.*

$$\left\| -\Pi + \left(1 + \frac{a-\lambda_1}{c}\right) F(w,z) - \sigma z \right\|_p \le C,$$

where C is a nonnegative constant independent of t.

From the assumptions (1.7) and (1.8), we are led to establish the uniform boundedness of the $\|z\|_p$ on $]0, T^*[$ in order to get that of $\|z\|_\infty$ on $]0, T^*[$.

For $p \ge 2$, we put

$$\alpha = \frac{(\lambda_2 - \lambda_1)^2}{4\lambda_1\lambda_2}, \qquad \alpha(p) = \frac{p\alpha + 1}{p-1}, \qquad M_p = K + \frac{\Pi}{\sigma\alpha(p)}. \tag{3.2}$$

We firstly introduce the following lemmas, which are useful in our main results.

Lemma 1 *Let* (w,z) *be a solution of* (2.1)-(2.4). *Then*

$$\frac{d}{dt}\int_\Omega w\,dx + \left(1 + \frac{a-\lambda_2}{c}\right)\int_\Omega F(w,z)\,dx + \sigma\int_\Omega w\,dx = \Pi|\Omega|. \tag{3.3}$$

Proof We integrate both sides of (2.1), satisfied by w, which is positive and then we obtain

$$\frac{d}{dt}\int_\Omega w\,dx = \Pi|\Omega| - \left(1 + \frac{a-\lambda_2}{c}\right)\int_\Omega F(w,z)\,dx - \sigma\int_\Omega w\,dx. \qquad \square$$

Lemma 2 *Assume that* $p \ge 2$ *and let*

$$G_q(t) = \int_\Omega \left[qw + \exp\left(-\frac{p-1}{p\alpha+1} \ln\bigl(\alpha(p)(M_p - w)\bigr)\right) z^p \right] dt, \tag{3.4}$$

where (w,z) *is the solution of* (2.1)-(2.4) *on* $]0, T^*[$. *Then under the assumptions* (1.7)-(1.8) *there exist two positive constants* $q > 0$ *and* $s > 0$ *such that*

$$\frac{d}{dt} G_q(t) \le -(p-1)\sigma G_q + s. \tag{3.5}$$

Proof The proof is similar to that in Melkemi *et al.* [2].

Let

$$h(w) = -\frac{p-1}{p\alpha+1} \ln\bigl(\alpha(p)(M_p - w)\bigr), \tag{3.6}$$

then

$$G_q(t) = q\int_\Omega w\,dx + N(t), \tag{3.7}$$

where

$$N(t) = \int_\Omega e^{h(w)} z^p\,dx. \tag{3.8}$$

Differentiating $N(t)$ with respect to t and using the Green formula one obtains

$$\frac{d}{dt}N = H + S, \tag{3.9}$$

where

$$\begin{aligned} H = {}& -\lambda_2 \int_\Omega \big((h'(w))^2 + h''(w) \big) e^{h(w)} z^p (\nabla w)^2 \, dx \\ & - p(\lambda_2 + \lambda_1) \int_\Omega h'(w) e^{h(w)} z^{p-1} \nabla w \nabla z \, dx \\ & - \lambda_1 \int_\Omega p(p-1) e^{h(w)} z^{p-2} (\nabla z)^2 \, dx \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} S = {}& \Pi \int_\Omega h'(w) e^{h(w)} z^p \, dx \\ & + \int_\Omega \left[p z^{p-1} \left(1 + \frac{a - \lambda_1}{c} \right) F(w,z) - \left(1 + \frac{a - \lambda_2}{c} \right) h'(w) z^p F(w,z) \right] e^{h(w)} \, dx \\ & - \sigma \int_\Omega h'(w) w e^{h(w)} z^p \, dx - p\sigma \int_\Omega e^{h(w)} z^p \, dx - p\Pi \int_\Omega e^{h(w)} z^{p-1} \, dx. \end{aligned} \tag{3.11}$$

We observe that H is given by

$$H = -\int_\Omega Q e^{h(w)} \, dx,$$

where

$$\begin{aligned} Q = {}& \lambda_2 \big((h'(w))^2 + h''(w) \big) z^p (\nabla w)^2 + p(\lambda_2 + \lambda_1) h'(w) z^{p-1} \nabla w \nabla z \\ & + \lambda_1 p(p-1) z^{p-2} (\nabla z)^2 \end{aligned} \tag{3.12}$$

is a quadratic form with respect to ∇w and ∇z, which is nonnegative if

$$\big(p(\lambda_2 + \lambda_1) h'(w) z^{p-1} \big)^2 - 4\lambda_1 \lambda_2 p(p-1) \big((h'(w))^2 + h''(w) \big) z^{2p-2} \le 0, \tag{3.13}$$

and we have chosen $h(w)$ such that

$$h'(w) = \frac{1}{\alpha(p)(M_p - w)}, \qquad h''(w) = \frac{\alpha(p)}{[\alpha(p)(M_p - w)]^2}. \tag{3.14}$$

It is easy to see that the left-hand side of (3.13) can be written as

$$\begin{aligned} & 4\lambda_1 \lambda_2 p z^{2p-2} \left\{ p \left[\alpha \frac{1}{(\alpha(p)(M_p - w))^2} - \frac{\alpha(p)}{(\alpha(p)(M_p - w))^2} \right] \right. \\ & \left. \quad + \frac{1 + \alpha(p)}{(\alpha(p)(M_p - w))^2} \right\} = 0, \end{aligned} \tag{3.15}$$

since

$$p\alpha - p\alpha(p) + 1 + \alpha(p) = 0,$$

the inequality (3.13) holds, $Q \geq 0$, and consequently

$$H = -\int_\Omega Qe^{h(w)}\,dx \leq 0, \tag{3.16}$$

and the second term S can be estimated as

$$\begin{aligned} S &\leq \int_\Omega \big(\Pi h'(w) - \sigma p\big)e^{h(w)}z^p\,dx \\ &\quad + \int_\Omega \left[pz^{p-1}\left(1 + \frac{a-\lambda_1}{c}\right)F(w,z) - h'(w)z^p\left(1 + \frac{a-\lambda_2}{c}\right)F(w,z)\right]e^{h(w)}\,dx \\ &\leq -(p-1)\sigma \int_\Omega e^{h(w)}z^p\,dx \\ &\quad + \int_\Omega \left[\left(1 + \frac{a-\lambda_1}{c}\right)pz^{p-1}F(w,z) - \left(1 + \frac{a-\lambda_2}{c}\right)h'(w)z^pF(w,z)\right]e^{h(w)}\,dx, \end{aligned} \tag{3.17}$$

since

$$h'(w) = \frac{1}{\alpha(p)(M_p - w)} \leq \frac{1}{\alpha(p)(M_p - K)} = \frac{\sigma}{\Pi}. \tag{3.18}$$

On the other hand

$$\begin{aligned} -h'(w) &= \frac{-1}{\alpha(p)(M_p - w)} \leq \frac{-1}{\alpha(p)M_p}, \\ h(w) &\leq \frac{-1}{\alpha(p)} \ln \frac{\Pi}{\sigma}. \end{aligned} \tag{3.19}$$

Taking into account the fact that $z \geq 0$, and using (3.19), we observe that

$$\begin{aligned} &p\left(1 + \frac{a-\lambda_1}{c}\right)z^{p-1}F(w,z) - \left(1 + \frac{a-\lambda_2}{c}\right)h'(w)z^pF(w,z) \\ &\quad \leq \left(p\left(1 + \frac{a-\lambda_1}{c}\right)z^{p-1} - \frac{1}{\alpha(p)M_p}\left(1 + \frac{a-\lambda_2}{c}\right)z^p\right)F(w,z). \end{aligned}$$

Then for $\eta_0 = p(1 + \frac{a-\lambda_1}{c})(\frac{1}{(1+\frac{a-\lambda_2}{c})} + 1)\alpha(p)M_p > 0$, and for $0 \leq \xi \leq K$, $\eta \geq \eta_0$, we have

$$\begin{aligned} &\left(p\left(1 + \frac{a-\lambda_1}{c}\right)\eta^{p-1} - \frac{1}{\alpha(p)M_p}\left(1 + \frac{a-\lambda_2}{c}\right)\eta^p\right)F(\xi,\eta) \\ &\quad = \left[\frac{p(1 + \frac{a-\lambda_1}{c})}{\eta} - \frac{(1 + \frac{a-\lambda_2}{c})}{\alpha(p)M_p}\right]\eta^pF(\xi,\eta) \leq 0. \end{aligned}$$

On the other hand, we deduce that the function

$$(\xi,\eta) \to p\left(1 + \frac{a-\lambda_1}{c}\right)\eta^{p-1} - \frac{1}{\alpha(p)M_p}\left(1 + \frac{a-\lambda_2}{c}\right)\eta^p$$

is bounded on the compact interval $[0, \eta_0]$; then there exists $c_1 > 0$ such that

$$p\left(1 + \frac{a - \lambda_1}{c}\right) z^{p-1} F(w,z) - \left(1 + \frac{a - \lambda_2}{c}\right) h'(w) z^p F(w,z) \le c_1 F(w,z). \tag{3.20}$$

From (3.17) and (3.20), we deduce immediately the following inequality:

$$S \le -(p-1)\sigma N + c_1 \int_\Omega F(w,z) e^{h(w)}\, dx \le -(p-1)\sigma N + c_1 e^{\frac{-1}{\alpha(p)} \ln \frac{\Pi}{\sigma}} \int_\Omega F(w,z)\, dx,$$

and putting

$$q = \frac{c_1}{(1 + \frac{a - \lambda_2}{c})} e^{\frac{-1}{\alpha(p)} \ln \frac{\Pi}{\sigma}}, \tag{3.21}$$

by (3.3), we have

$$S \le -(p-1)\sigma N + q\Pi|\Omega| - q\frac{d}{dt}\int_\Omega w(t,x)\, dx, \tag{3.22}$$

and from (3.7), it follows that

$$S \le -(p-1)\sigma G_q + q\big((p-1)\sigma K + \Pi\big)|\Omega| - q\frac{d}{dt}\int_\Omega w(t,x)\, dx, \tag{3.23}$$

and from (3.7) and (3.9), we conclude that

$$\frac{d}{dt} G_q \le -(p-1)\sigma G_q + s, \tag{3.24}$$

where

$$s = q\big((p-1)\sigma K + \Pi\big)|\Omega|. \tag{3.25}$$

□

Now we can establish the global existence and uniform boundedness of the solutions of (2.1)-(2.4).

Theorem 1 *Under the assumptions* (1.7) *and* (1.8), *the solutions of* (2.1)-(2.4) *are global and uniformly bounded on* $[0, +\infty[\times \Omega$.

Proof Multiplying the inequality (3.24) by $e^{(p-1)\sigma t}$ and then integrating, we deduce that there exists a positive constant $C > 0$ independent of t, such that

$$G_q(t) \le C. \tag{3.26}$$

From (3.6), we observe that

$$e^{h(w)} \ge e^{\frac{-1}{\alpha(p)} \ln \alpha(p) M_p}, \tag{3.27}$$

and it follows, for all $p \geq 2$, that

$$\int_\Omega z^p\,dx \leq e^{\frac{1}{\alpha(p)}\ln(K\alpha(p)+\frac{\Pi}{\sigma})} G_q(t) \leq C_1(p), \tag{3.28}$$

where

$$C_1(p) = Ce^{\frac{1}{\alpha(p)}\ln(K\alpha(p)+\frac{\Pi}{\sigma})}, \tag{3.29}$$

and as we select $p > \frac{n}{2}$ we can proceed to bound $\|-\Pi + (1 + \frac{a-\lambda_1}{c})F(w,z) - \sigma z\|_p$.

Let

$$A = \max_{\xi_0 \leq \xi \leq K_1} \varphi(\xi), \tag{3.30}$$

where

$$K_1 = \frac{a-\lambda_1}{\lambda_2-\lambda_1}K,$$

and ξ_0 is such that

$$\xi \leq \xi_0 \quad \Longrightarrow \quad \varphi(\xi) < |\xi|, \tag{3.31}$$

using (1.7), we deduce

$$F(w,z) = f(u,v) \leq \varphi(u)(1+v)^\beta,$$

which implies

$$\begin{aligned} \int_\Omega F^p(w,z)\,dx &\leq \int_\Omega \big(\varphi(u)\big)^p(1+v)^{\beta p}\,dx \\ &= \int_{u\leq\xi_0} \big(\varphi(u)\big)^p(1+v)^{\beta p}\,dx + \int_{\xi_0\leq u} \big(\varphi(u)\big)^p(1+v)^{\beta p}\,dx \\ &\leq \int_{u\leq\xi_0} |u|^p(1+v)^{\beta p}\,dx + A^p \int_{\xi_0\leq u} (1+v)^{\beta p}\,dx. \end{aligned}$$

From (2.7), we have

$$\begin{aligned} |u|^p &= \left|\frac{a-\lambda_1}{\lambda_2-\lambda_1}w(t,x) + \frac{a-\lambda_2}{\lambda_2-\lambda_1}z(t,x)\right|^p \leq \left(\frac{a-\lambda_1}{\lambda_2-\lambda_1}w(t,x) + \frac{\lambda_2-a}{\lambda_2-\lambda_1}z(t,x)\right)^p \\ &\leq \left(\frac{\lambda_2-a}{\lambda_2-\lambda_1}\right)^p\big(w(t,x)+z(t,x)\big)^p, \end{aligned}$$

then

$$\begin{aligned} \int_\Omega F^p(w,z)\,dx \leq{}& \int_{u\leq\xi_0} \left(\frac{\lambda_2-a}{\lambda_2-\lambda_1}\right)^p (w+z)^p \left(1+\frac{c}{\lambda_2-\lambda_1}(w+z)\right)^{\beta p} dx \\ &+ A^p \int_{\xi_0\leq u} \left(1+\frac{c}{\lambda_2-\lambda_1}(w+z)\right)^{\beta p} dx \end{aligned}$$

$$\begin{aligned}
&\leq \max\left(A^p, \left(\frac{\lambda_2 - a}{\lambda_2 - \lambda_1}\right)^p\right)\left(\int_{u\leq\xi_0}(w+z)^p\left(1+\frac{c}{\lambda_2-\lambda_1}(w+z)\right)^{\beta p}dx\right.\\
&\quad\left.+\int_{\xi_0\leq u}\left(1+\frac{c}{\lambda_2-\lambda_1}(w+z)\right)^{\beta p}dx\right)\\
&\leq \max\left(A^p, \left(\frac{\lambda_2 - a}{\lambda_2 - \lambda_1}\right)^p\right)\left(\int_{\Omega}(w+z)^p\left(1+\frac{c}{\lambda_2-\lambda_1}(w+z)\right)^{\beta p}dx\right.\\
&\quad\left.+\int_{\Omega}\left(1+\frac{c}{\lambda_2-\lambda_1}(w+z)\right)^{\beta p}dx\right),
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega}(w+z)^p\left(1+\frac{c}{\lambda_2-\lambda_1}(w+z)\right)^{\beta p}dx\\
&\quad\leq 2^{\beta p-1}\left(\int_{\Omega}(w+z)^p+\left(\frac{c}{\lambda_2-\lambda_1}\right)^{\beta p}(w+z)^{(\beta+1)p}\,dx\right)\\
&\quad\leq 2^{(\beta+1)p-2}\big(K^p|\Omega|+C_1(p)\big)+2^{(2\beta+1)p-2}\left(\frac{c}{\lambda_2-\lambda_1}\right)^{\beta p}\big(K^{(\beta+1)p}|\Omega|+C_1\big((\beta+1)p\big)\big)\\
&\quad= C_2(\beta,p,K,\Omega),
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega}\left(1+\frac{c}{\lambda_2-\lambda_1}(w+z)\right)^{\beta p}dx\\
&\quad\leq 2^{\beta p-1}\left(|\Omega|+\left(\frac{c}{\lambda_2-\lambda_1}\right)^{\beta p}\times 2^{\beta p-1}\big(K^{\beta p}|\Omega|+C_1(\beta p)\big)\right)\\
&\quad= C_3(\beta,p,K,\Omega).
\end{aligned}$$

Consequently

$$\int_{\Omega}F^p(w,z)\,dx\leq C_4(A,\beta,p,K,\Omega).$$

Finally

$$\begin{aligned}
\left\|-\Pi+\frac{a-\lambda_1}{c}F(w,z)-\sigma z\right\|_p &= \left\|\frac{a-\lambda_1}{c}F(w,z)-(\sigma z+\Pi)\right\|_p\\
&\leq \frac{a-\lambda_1}{c}\big\|F(w,z)\big\|_p+\sigma\|z\|_p+\Pi|\Omega|\\
&\leq \frac{a-\lambda_1}{c}\sqrt[p]{C_4(A,\beta,p,K)}+\sigma\sqrt[p]{C_1(p)}+\Pi|\Omega|\\
&= C_5(A,\beta,p,K,\Omega,\sigma).
\end{aligned}\tag{3.32}$$

Using the regularity results for the solutions of the parabolic equations in [5], we conclude that the solutions of problem (2.1)-(2.4) are uniformly bounded on $[0,+\infty[\times\Omega$. □

By (2.7), it is easy to see that the solutions of problem (1.1)-(1.4) are also uniformly bounded on $[0,+\infty[\times\Omega$.

Remark 3 Because $0\leq w(t,x)\leq K$ and $z(t,x)\geq 0$, we deduce that

$$-\infty\leq u(t,x)\leq\frac{a-\lambda_1}{\lambda_2-\lambda_1}K=K_1.$$

Remark 4 We note that $\frac{a-\lambda_2}{\lambda_2-\lambda_1} < 0$ and $\frac{\lambda_2-a}{\lambda_2-\lambda_1} \geq \frac{a-\lambda_1}{\lambda_2-\lambda_1}$, because $\lambda_2 + \lambda_1 = a + d$ and $d > a$.

We conclude by noting that the study of the global existence of strongly coupled systems has been a major development, and several articles are devoted to this subject. In our opinion, many other systems with non-constant diffusion matrix which are in the actual scope of the results previously given, should be taken in consideration and studied with more interest.

Competing interests
The author declares that they have no competing interests.

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