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New refinements of generalized Aczél inequality

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Abstract

In this article, we present several new refinements of the generalized Aczél inequality. As an application, an integral type of the generalized Aczél-Vasić-Pečarić inequality is refined.

MSC: Primary 26D15; secondary 26D10

Keywords: Aczél's inequality; Aczél-Vasić-Pečarić inequality; refinement; generalization

1 Introduction

In 1956, Aczél [1] established the following inequality, which is called the Aczél inequality.

Theorem A *Let* $a_i > 0$, $b_i > 0$ $(i = 1, 2, \ldots, n)$, $a_1^2 - \sum_{i=2}^n a_i^2 > 0$, $b_1^2 - \sum_{i=2}^n b_i^2 > 0$. *Then*

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \le \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2. \tag{1}$$

As is well known, the Aczél inequality plays an important role in the theory of functional equations in non-Euclidean geometry, and many authors (see [2–6] and references therein) have given considerable attention to this inequality and its refinements.

In 1959, Popoviciu [3] generalized the Aczél inequality (1) in the form asserted by Theorem B below.

Theorem B *Let* $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *let* $a_i > 0$, $b_i > 0$ $(i = 1, 2, \ldots, n)$, $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$. *Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}\left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \le a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{2}$$

Later, in 1982, Vasić and Pečarić [7] presented the reversed version of inequality (2), which is stated in the following theorem. The inequality is called the Aczél-Vasić-Pečarić inequality.

Theorem C *Let* $p < 1$ $(p \neq 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, *and let* $a_i > 0$, $b_i > 0$ $(i = 1, 2, \dots, n)$, $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$. *Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \geq a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{3}$$

In another paper, Vasić and Pečarić [8] presented an interesting generalization of inequality (2). The inequality is called the generalized Aczél-Vasić-Pečarić inequality.

Theorem D *Let* $a_{rj} > 0$, $\lambda_j > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, *and let* $\sum_{j=1}^m \frac{1}{\lambda_j} \geq 1$. *Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \tag{4}$$

In 2012, Tian [5] gave the reversed version of inequality (4) in the following form.

Theorem E *Let* $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2, 3, \dots, m)$, $\sum_{j=1}^m \frac{1}{\lambda_j} \leq 1$, *and let* $a_{rj} > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \dots, n$, $j = 1, 2, \dots, m$. *Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \tag{5}$$

Moreover, in [5] Tian established an integral type of generalized Aczél-Vasić-Pečarić inequality.

Theorem F *Let* $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2, 3, \dots, m)$, $\sum_{j=1}^m \lambda_j = 1$, *let* $A_j > 0$ $(j = 1, 2, \dots, m)$, *and let* $f_j(x)$ $(j = 1, 2, \dots, m)$ *be positive Riemann integrable functions on* $[a, b]$ *such that* $A_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\,\mathrm{d}x > 0$. *Then*

$$\prod_{j=1}^m \left(A_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\,\mathrm{d}x\right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^m A_j - \int_a^b \prod_{j=1}^m f_j(x)\,\mathrm{d}x. \tag{6}$$

The main object of this paper is to give several new refinements of inequality (4) and (5). As an application, a new refinement of inequality (6) is given.

2 New refinements of generalized Aczél inequality

In order to prove the main results in this section, we need the following lemmas.

Lemma 2.1 [5] *Let* $a_{rj} > 0$ $(r = 1, 2, \dots, n$, $j = 1, 2, \dots, m)$, *let* λ_1 *be a real number*, $\lambda_j \leq 0$ $(j = 2, 3, \dots, m)$, *and let* $\beta = \max\{\sum_{j=1}^m \lambda_j, 1\}$. *Then*

$$\sum_{r=1}^n \prod_{j=1}^m a_{rj}^{\lambda_j} \geq n^{1-\beta} \prod_{j=1}^m \left(\sum_{r=1}^n a_{rj}\right)^{\lambda_j}. \tag{7}$$

Lemma 2.2 [9] *Let* $a_{rj} > 0$ $(r = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$, *let* $\lambda_j \geq 0$ $(j = 1, 2, \ldots, m)$, *and let* $\gamma = \min\{\sum_{j=1}^{m} \lambda_j, 1\}$. *Then*

$$\sum_{r=1}^{n} \prod_{j=1}^{m} a_{rj}^{\lambda_j} \leq n^{1-\gamma} \prod_{j=1}^{m} \left(\sum_{r=1}^{n} a_{rj} \right)^{\lambda_j}. \tag{8}$$

Lemma 2.3 [10] *If* $x > -1$, $\alpha > 1$ *or* $\alpha < 0$, *then*

$$(1 + x)^{\alpha} \geq 1 + \alpha x. \tag{9}$$

The inequality is reversed for $0 < \alpha < 1$.

Lemma 2.4 [10] *Let* $A_1, A_2, \ldots, A_m$ *be real numbers, let* m *be a natural number, and let* $m \geq 2$. *Then*

$$\sum_{1 \leq i < j \leq m} (A_i - A_j)^2 = m \left(\sum_{i=1}^{m} A_i^2 \right) - \left(\sum_{i=1}^{m} A_i \right)^2. \tag{10}$$

Lemma 2.5 *Let* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m < 0$, *let* $X_j > 1$ $(j = 1, 2, \ldots, m)$, *and let* $m \geq 2$. *Then*

$$\prod_{j=1}^{m} \left(1 - X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{m} X_j \geq \left\{ 1 - \frac{2}{m(m-1)} \left[m \left(\sum_{j=1}^{m} X_j^{2\lambda_j} \right) - \left(\sum_{j=1}^{m} X_j^{\lambda_j} \right)^2 \right] \right\}^{\frac{m}{2\lambda_1}}. \tag{11}$$

Proof From the assumptions in Lemma 2.5, we find

$$\frac{1}{(m-1)\lambda_i} < 0, \qquad \frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i} \leq 0 \quad (1 \leq i < j \leq m),$$

and

$$\begin{aligned} &\sum_{1 \leq i < j \leq m} \left[\frac{1}{(m-1)\lambda_i} + \frac{1}{(m-1)\lambda_i} + \frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i} \right] \\ &\quad = \sum_{1 \leq i < j \leq m} \left[\frac{1}{(m-1)\lambda_i} + \frac{1}{(m-1)\lambda_j} \right] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_m}. \end{aligned} \tag{12}$$

Thus, by using inequality (7) we have

$$\begin{aligned} &\prod_{1 \leq i < j \leq m} \left[1 - \left(X_i^{\lambda_i} - X_j^{\lambda_j} \right)^2 \right]^{\frac{1}{(m-1)\lambda_i}} \\ &\quad = \prod_{1 \leq i < j \leq m} \left\{ \left[X_i^{\lambda_i} + \left(1 - X_j^{\lambda_j}\right) \right]^{\frac{1}{(m-1)\lambda_i}} \left[X_j^{\lambda_j} + \left(1 - X_i^{\lambda_i}\right) \right]^{\frac{1}{(m-1)\lambda_i}} \right. \\ &\qquad \left. \times \left[X_j^{\lambda_j} + \left(1 - X_j^{\lambda_j}\right) \right]^{\frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i}} \right\} \\ &\quad \leq \prod_{1 \leq i < j \leq m} \left[\left(X_i^{\lambda_i}\right)^{\frac{1}{(m-1)\lambda_i}} \left(X_j^{\lambda_j}\right)^{\frac{1}{(m-1)\lambda_i}} \left(X_j^{\lambda_j}\right)^{\frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i}} \right] \end{aligned}$$

$$+\prod_{1\le i<j\le m}\left[\left(1-X_j^{\lambda_j}\right)^{\frac{1}{(m-1)\lambda_i}}\left(1-X_i^{\lambda_i}\right)^{\frac{1}{(m-1)\lambda_i}}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{(m-1)\lambda_j}-\frac{1}{(m-1)\lambda_i}}\right]$$

$$=\prod_{1\le i<j\le m}X_i^{\frac{1}{m-1}}X_j^{\frac{1}{m-1}}+\prod_{1\le i<j\le m}\left[\left(1-X_i^{\lambda_i}\right)^{\frac{1}{(m-1)\lambda_i}}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{(m-1)\lambda_j}}\right]$$

$$=\prod_{j=1}^{m}X_j+\prod_{j=1}^{m}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \tag{13}$$

Noting the fact that there are $\frac{m(m-1)}{2}$ product terms in the expression $\prod_{1\le i<j\le m}[1-(X_i^{\lambda_i}-X_j^{\lambda_j})^2]$, and using the arithmetic-geometric mean's inequality, we obtain

$$\prod_{1\le i<j\le m}\left[1-\left(X_i^{\lambda_i}-X_j^{\lambda_j}\right)^2\right]\le\left\{\frac{2}{m(m-1)}\sum_{1\le i<j\le m}\left[1-\left(X_i^{\lambda_i}-X_j^{\lambda_j}\right)^2\right]\right\}^{\frac{m(m-1)}{2}}$$

$$=\left[1-\frac{2}{m(m-1)}\sum_{1\le i<j\le m}\left(X_i^{\lambda_i}-X_j^{\lambda_j}\right)^2\right]^{\frac{m(m-1)}{2}}. \tag{14}$$

Therefore, we have

$$\prod_{1\le i<j\le m}\left[1-\left(X_i^{\lambda_i}-X_j^{\lambda_j}\right)^2\right]^{\frac{1}{(m-1)\lambda_i}}$$

$$\ge\left\{\prod_{1\le i<j\le m}\left[1-\left(X_i^{\lambda_i}-X_j^{\lambda_j}\right)^2\right]\right\}^{\frac{1}{(m-1)\lambda_1}}$$

$$\ge\left[1-\frac{2}{m(m-1)}\sum_{1\le i<j\le m}\left(X_i^{\lambda_i}-X_j^{\lambda_j}\right)^2\right]^{\frac{m}{2\lambda_1}}. \tag{15}$$

On the other hand, from Lemma 2.4 we have

$$\left[1-\frac{2}{m(m-1)}\sum_{1\le i<j\le m}\left(X_i^{\lambda_i}-X_j^{\lambda_j}\right)^2\right]^{\frac{m}{2\lambda_1}}$$

$$=\left\{1-\frac{2}{m(m-1)}\left[m\left(\sum_{j=1}^{m}X_j^{2\lambda_j}\right)-\left(\sum_{j=1}^{m}X_j^{\lambda_j}\right)^2\right]\right\}^{\frac{m}{2\lambda_1}}. \tag{16}$$

Consequently, from (13), (15), and (16), we obtain the desired inequality (11). □

Lemma 2.6 *Let* $\lambda_m>0$, $\lambda_1\le\lambda_2\le\cdots\le\lambda_{m-1}<0$, *let* $0<X_m<1$, $X_j>1$ $(j=1,2,\ldots,m-1)$, *and let* $\alpha=\max\{\sum_{j=1}^{m}\frac{1}{\lambda_j},1\}$. *If* $m>2$, *then*

$$\prod_{j=1}^{m}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\prod_{j=1}^{m}X_j$$

$$\ge n^{1-\alpha}\left\{1-\frac{2}{(m-1)(m-2)}\left[(m-1)\left(\sum_{j=1}^{m-1}X_j^{2\lambda_j}\right)-\left(\sum_{j=1}^{m-1}X_j^{\lambda_j}\right)^2\right]\right\}^{\frac{m-1}{2\lambda_1}}. \tag{17}$$

If $m = 2$, *then*

$$\prod_{j=1}^{2}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\prod_{j=1}^{2}X_j \ge n^{1-\alpha}\left\{1-\left[2\left(\sum_{j=1}^{2}X_j^{2\lambda_j}\right)-\left(\sum_{j=1}^{2}X_j^{\lambda_j}\right)^2\right]\right\}^{\frac{1}{\lambda_1}}. \tag{18}$$

Proof Case I. When $m > 2$. Let us consider the following product:

$$\begin{aligned}&\prod_{1\le i<j\le m-1}\left\{\left[X_i^{\lambda_i}+\left(1-X_j^{\lambda_j}\right)\right]^{\frac{1}{(m-2)\lambda_i}}\left[X_j^{\lambda_j}+\left(1-X_i^{\lambda_i}\right)\right]^{\frac{1}{(m-2)\lambda_i}}\right.\\&\quad\left.\times\left[X_j^{\lambda_j}+\left(1-X_j^{\lambda_j}\right)\right]^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right\}.\end{aligned} \tag{19}$$

From the hypotheses of Lemma 2.6, it is easy to see that

$$\frac{1}{(m-2)\lambda_i}<0,\qquad \frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}\le 0\quad (1\le i<j\le m-1),$$

and

$$\begin{aligned}&\sum_{1\le i<j\le m-1}\left[\frac{1}{(m-2)\lambda_i}+\frac{1}{(m-2)\lambda_i}+\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}\right]\\&\quad=\sum_{1\le i<j\le m-1}\left[\frac{1}{(m-2)\lambda_i}+\frac{1}{(m-2)\lambda_j}\right]=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\cdots+\frac{1}{\lambda_{m-1}}.\end{aligned} \tag{20}$$

Then, applying inequality (7), we have

$$\begin{aligned}&\prod_{1\le i<j\le m-1}\left[1-\left(X_i^{\lambda_i}-X_j^{\lambda_j}\right)^2\right]^{\frac{1}{(m-2)\lambda_i}}\\&\quad=\left[X_m^{\lambda_m}+\left(1-X_m^{\lambda_m}\right)\right]^{\frac{1}{\lambda_m}}\prod_{1\le i<j\le m-1}\left\{\left[X_i^{\lambda_i}+\left(1-X_j^{\lambda_j}\right)\right]^{\frac{1}{(m-2)\lambda_i}}\right.\\&\qquad\left.\times\left[X_j^{\lambda_j}+\left(1-X_i^{\lambda_i}\right)\right]^{\frac{1}{(m-2)\lambda_i}}\left[X_j^{\lambda_j}+\left(1-X_j^{\lambda_j}\right)\right]^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right\}\\&\quad\le n^{\alpha-1}\left\{X_m^{\frac{\lambda_m}{\lambda_m}}\prod_{1\le i<j\le m-1}\left[\left(X_i^{\lambda_i}\right)^{\frac{1}{(m-2)\lambda_i}}\left(X_j^{\lambda_j}\right)^{\frac{1}{(m-2)\lambda_i}}\left(X_j^{\lambda_j}\right)^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right]\right.\\&\qquad+\left(1-X_m^{\lambda_m}\right)^{\frac{1}{\lambda_m}}\prod_{1\le i<j\le m-1}\left[\left(1-X_j^{\lambda_j}\right)^{\frac{1}{(m-2)\lambda_i}}\left(1-X_i^{\lambda_i}\right)^{\frac{1}{(m-2)\lambda_i}}\right.\\&\qquad\left.\left.\times\left(1-X_j^{\lambda_j}\right)^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right]\right\}\\&\quad=n^{\alpha-1}\left\{X_m\prod_{1\le i<j\le m-1}X_i^{\frac{1}{m-2}}X_j^{\frac{1}{m-2}}\right.\\&\qquad\left.+\left(1-X_m^{\lambda_m}\right)^{\frac{1}{\lambda_m}}\prod_{1\le i<j\le m-1}\left[\left(1-X_i^{\lambda_i}\right)^{\frac{1}{(m-2)\lambda_i}}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{(m-2)\lambda_j}}\right]\right\}\\&\quad=n^{\alpha-1}\left[\prod_{j=1}^{m}X_j+\prod_{j=1}^{m}\left(1-X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right].\end{aligned} \tag{21}$$

There are $\frac{(m-1)(m-2)}{2}$ product terms in the expression $\prod_{1\le i<j\le m-1}[1-(X_i^{\lambda_i}-X_j^{\lambda_j})^2]$, and then we derive from the arithmetic-geometric mean's inequality that

$$\begin{aligned}
&\prod_{1\le i<j\le m-1}\bigl[1-\bigl(X_i^{\lambda_i}-X_j^{\lambda_j}\bigr)^2\bigr] \\
&\quad\le\left\{\frac{2}{(m-1)(m-2)}\sum_{1\le i<j\le m-1}\bigl[1-\bigl(X_i^{\lambda_i}-X_j^{\lambda_j}\bigr)^2\bigr]\right\}^{\frac{(m-1)(m-2)}{2}} \\
&\quad=\left[1-\frac{2}{(m-1)(m-2)}\sum_{1\le i<j\le m-1}\bigl(X_i^{\lambda_i}-X_j^{\lambda_j}\bigr)^2\right]^{\frac{(m-1)(m-2)}{2}}. \qquad (22)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\prod_{1\le i<j\le m-1}\bigl[1-\bigl(X_i^{\lambda_i}-X_j^{\lambda_j}\bigr)^2\bigr]^{\frac{1}{(m-2)\lambda_i}} \\
&\quad\ge\left\{\prod_{1\le i<j\le m-1}\bigl[1-\bigl(X_i^{\lambda_i}-X_j^{\lambda_j}\bigr)^2\bigr]\right\}^{\frac{1}{(m-2)\lambda_1}} \\
&\quad\ge\left[1-\frac{2}{(m-1)(m-2)}\sum_{1\le i<j\le m-1}\bigl(X_i^{\lambda_i}-X_j^{\lambda_j}\bigr)^2\right]^{\frac{m-1}{2\lambda_1}}. \qquad (23)
\end{aligned}$$

On the other hand, from Lemma 2.4 we find

$$\begin{aligned}
&\left[1-\frac{2}{(m-1)(m-2)}\sum_{1\le i<j\le m-1}\bigl(X_i^{\lambda_i}-X_j^{\lambda_j}\bigr)^2\right]^{\frac{m-1}{2\lambda_1}} \\
&\quad=\left\{1-\frac{2}{(m-1)(m-2)}\left[(m-1)\left(\sum_{j=1}^{m-1}X_j^{2\lambda_j}\right)-\left(\sum_{j=1}^{m-1}X_j^{\lambda_j}\right)^2\right]\right\}^{\frac{m-1}{2\lambda_1}}. \qquad (24)
\end{aligned}$$

Combining inequalities (21), (23), and (24) yields the desired inequality (17).

Case II. When $m=2$. By the same method as in Lemma 2.5, it is easy to obtain the desired inequality (18). So we omit the proof. The proof of Lemma 2.6 is completed. □

Lemma 2.7 *Let* $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_m>0$, *let* $0<X_j<1$ $(j=1,2,\ldots,m)$, *and let* $m\ge 2$, $\rho=\min\{\sum_{j=1}^m\frac{1}{\lambda_j},1\}$. *Then*

$$\begin{aligned}
&\prod_{j=1}^m\bigl(1-X_j^{\lambda_j}\bigr)^{\frac{1}{\lambda_j}}+\prod_{j=1}^m X_j \\
&\quad\le n^{1-\rho}\left\{1-\frac{2}{m(m-1)}\left[m\left(\sum_{j=1}^m X_j^{2\lambda_j}\right)-\left(\sum_{j=1}^m X_j^{\lambda_j}\right)^2\right]\right\}^{\frac{m}{2\lambda_1}}. \qquad (25)
\end{aligned}$$

Proof By the same method as in Lemma 2.5, applying Lemma 2.2, it is easy to obtain the desired inequality (25). So we omit the proof. □

Lemma 2.8 *Let $\lambda_1, \lambda_2, \dots, \lambda_m < 0$, let $X_j > 1$ ($j = 1, 2, \dots, m$), and let $m \geq 2$. Then*

$$\begin{aligned}
&\prod_{j=1}^{m}\left(1 - X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{m} X_j \\
&\quad \geq \left[1 - \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \left(X_i^{\lambda_i} - X_j^{\lambda_j}\right)^2\right]^{\frac{m}{2\min\{\lambda_1, \lambda_2, \dots, \lambda_m\}}}.
\end{aligned} \tag{26}$$

Proof After simply rearranging, we write by $\lambda_{j_1} \leq \lambda_{j_2} \leq \cdots \leq \lambda_{j_m}$ the component of $\lambda_1, \lambda_2, \dots, \lambda_m$ in increasing order, where $j_1, j_2, \dots, j_m$ is a permutation of $1, 2, \dots, m$.

Then from Lemma 2.5 and Lemma 2.4 we get

$$\begin{aligned}
&\prod_{j=1}^{m}\left(1 - X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{m} X_j \\
&\quad = \left(1 - X_{j_1}^{\lambda_{j_1}}\right)^{\frac{1}{\lambda_{j_1}}}\left(1 - X_{j_2}^{\lambda_{j_2}}\right)^{\frac{1}{\lambda_{j_2}}} \cdots \left(1 - X_{j_m}^{\lambda_{j_m}}\right)^{\frac{1}{\lambda_{j_m}}} + X_{j_1} X_{j_2} \cdots X_{j_m} \\
&\quad \geq \left\{1 - \frac{2}{m(m-1)}\left[m\left(\sum_{k=1}^{m} X_{j_k}^{2\lambda_{j_k}}\right) - \left(\sum_{k=1}^{m} X_{j_k}^{\lambda_{j_k}}\right)^2\right]\right\}^{\frac{m}{2\lambda_{j_1}}} \\
&\quad = \left\{1 - \frac{2}{m(m-1)}\left[m\left(\sum_{k=1}^{m} X_{j_k}^{2\lambda_{j_k}}\right) - \left(\sum_{k=1}^{m} X_{j_k}^{\lambda_{j_k}}\right)^2\right]\right\}^{\frac{m}{2\min\{\lambda_1, \lambda_2, \dots, \lambda_m\}}} \\
&\quad = \left[1 - \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \left(X_i^{\lambda_i} - X_j^{\lambda_j}\right)^2\right]^{\frac{m}{2\min\{\lambda_1, \lambda_2, \dots, \lambda_m\}}}.
\end{aligned} \tag{27}$$

The proof of Lemma 2.8 is completed. □

By the same method as in Lemma 2.8, we obtain the following two lemmas.

Lemma 2.9 *Let $\lambda_m > 0$, $\lambda_1, \lambda_2, \dots, \lambda_{m-1} < 0$, let $0 < X_m < 1$, $X_j > 1$ ($j = 1, 2, \dots, m-1$), and let $\alpha = \max\{\sum_{j=1}^{m} \frac{1}{\lambda_j}, 1\}$. If $m > 2$, then*

$$\begin{aligned}
&\prod_{j=1}^{m}\left(1 - X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{m} X_j \\
&\quad \geq n^{1-\alpha}\left[1 - \frac{2}{(m-1)(m-2)} \sum_{1 \leq i < j \leq m-1} \left(X_i^{\lambda_i} - X_j^{\lambda_j}\right)^2\right]^{\frac{m-1}{2\min\{\lambda_1, \lambda_2, \dots, \lambda_m\}}}.
\end{aligned} \tag{28}$$

If $m = 2$, then

$$\prod_{j=1}^{2}\left(1 - X_j^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{2} X_j \geq n^{1-\alpha}\left[1 - \sum_{1 \leq i < j \leq 2} \left(X_i^{\lambda_i} - X_j^{\lambda_j}\right)^2\right]^{\frac{1}{\lambda_1}}. \tag{29}$$

Lemma 2.10 *Let* $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$, *let* $0 < X_j < 1$ $(j = 1, 2, \ldots, m)$, *and let* $m \geq 2$, $\rho = \min\{\sum_{j=1}^{m} \frac{1}{\lambda_j}, 1\}$. *Then*

$$\prod_{j=1}^{m} \bigl(1 - X_j^{\lambda_j}\bigr)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{m} X_j \leq n^{1-\rho} \left[1 - \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \bigl(X_i^{\lambda_i} - X_j^{\lambda_j}\bigr)^2\right]^{\frac{m}{2\max\{\lambda_1, \lambda_2, \ldots, \lambda_m\}}}. \tag{30}$$

Now, we give the refinement and generalization of inequality (5).

Theorem 2.11 *Let* $a_{rj} > 0$, $\lambda_j < 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *and let* $m \geq 2$. *Then*

$$\begin{aligned} \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} &\geq \left\{1 - \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \left[\sum_{r=2}^{n} \left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2\right\}^{\frac{m}{2\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}}} \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \\ &\geq \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj}. \end{aligned} \tag{31}$$

Proof From the assumptions in Theorem 2.11, it is easy to verify that

$$\frac{(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j})^{\frac{1}{\lambda_j}}}{(a_{1j}^{\lambda_j})^{\frac{1}{\lambda_j}}} > 1 \quad (j = 1, 2, \ldots, m). \tag{32}$$

It thus follows from Lemma 2.8 with the substitution $X_j = (\frac{a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}})^{\frac{1}{\lambda_j}}$ in (26) that

$$\begin{aligned} &\prod_{j=1}^{m} \left(\frac{\sum_{r=2}^{n} a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}} + \prod_{j=1}^{m} \left(\frac{a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)^{\frac{1}{\lambda_j}} \\ &\quad\geq \left\{1 - \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \left[\left(1 - \frac{\sum_{r=2}^{n} a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}}\right) - \left(1 - \frac{\sum_{r=2}^{n} a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2\right\}^{\frac{m}{2\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}}} \\ &\quad= \left\{1 - \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \left[\sum_{r=2}^{n} \left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2\right\}^{\frac{m}{2\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}}}, \end{aligned} \tag{33}$$

which implies

$$\begin{aligned} \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} &\geq \left\{1 - \frac{2}{m(m-1)} \sum_{1 \leq i < j \leq m} \left[\sum_{r=2}^{n} \left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2\right\}^{\frac{m}{2\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}}} \\ &\quad\times \prod_{j=1}^{m} a_{1j} - \prod_{j=1}^{m} \left(\sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \end{aligned} \tag{34}$$

On the other hand, it follows from Lemma 2.1 that

$$\prod_{j=1}^{m}\left(\sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leq \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}. \tag{35}$$

Combining inequalities (34) and (35) yields inequality (31).

The proof of Theorem 2.11 is completed. □

Theorem 2.12 *Let* $\lambda_m > 0$, $\lambda_j < 0$ $(j = 1,2,\ldots,m-1)$, *let* $a_{rj} > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, *and let* $\alpha = \max\{\sum_{j=1}^{m}\frac{1}{\lambda_j}, 1\}$. *If* $m > 2$, *then*

$$\begin{aligned}
&\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \\
&\quad \geq n^{1-\alpha}\left\{1 - \frac{2}{(m-1)(m-2)}\sum_{1\leq i<j\leq m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2\right\}^{\frac{m-1}{2\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}} \\
&\qquad \times \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj} \\
&\quad \geq n^{1-\alpha}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}.
\end{aligned} \tag{36}$$

If $m = 2$, *then*

$$\begin{aligned}
\prod_{j=1}^{2}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} &\geq n^{1-\alpha}\left\{1 - \sum_{1\leq i<j\leq 2}\left[\sum_{r=2}^{n}\left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2\right\}^{\frac{1}{\lambda_1}}\prod_{j=1}^{2} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{2} a_{rj} \\
&\geq n^{1-\alpha}\prod_{j=1}^{2} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{2} a_{rj}.
\end{aligned} \tag{37}$$

Proof From the hypotheses of Theorem 2.12, we find that

$$0 < \frac{(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j})^{\frac{1}{\lambda_j}}}{(a_{1j}^{\lambda_j})^{\frac{1}{\lambda_j}}} < 1 \quad (j = 1,2,\ldots,m-1),$$

and

$$\frac{(a_{1m}^{\lambda_m} - \sum_{r=2}^{n} a_{rm}^{\lambda_m})^{\frac{1}{\lambda_m}}}{(a_{1m}^{\lambda_m})^{\frac{1}{\lambda_m}}} > 1.$$

Consequently, by the same method as in Theorem 2.11, and using Lemma 2.9 with a substitution $X_j \to (\frac{a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}})^{\frac{1}{\lambda_j}}$ $(j = 1,2,\ldots,m)$ in (28) and (29), respectively, we obtain the desired inequalities (36) and (37). □

By the same method as in Theorem 2.11, and using Lemma 2.10, we obtain the following sharpened and generalized version of inequality (4).

Theorem 2.13 *Let* $a_{rj} > 0$, $\lambda_j > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *let* $m \geq 2$, *and let* $\rho = \min\{\sum_{j=1}^{m} \frac{1}{\lambda_j}, 1\}$. *Then*

$$\begin{aligned}
&\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \\
&\quad \leq n^{1-\rho}\left\{1 - \frac{2}{m(m-1)} \sum_{1\leq i<j\leq m}\left[\sum_{r=2}^{n}\left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2\right\}^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}} \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj} \\
&\quad \leq n^{1-\rho}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}.
\end{aligned} \tag{38}$$

Therefore, from Lemma 2.3 and Theorem 2.13 we get a new refinement and generalization of inequality (4).

Corollary 2.14 *Let* $a_{rj} > 0$, $\lambda_j > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *let* $m \geq 2$, *and let* $\rho = \min\{\sum_{j=1}^{m} \frac{1}{\lambda_j}, 1\}$. *If* $\max\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \geq \frac{m}{2}$, *then*

$$\begin{aligned}
&\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \\
&\quad \leq n^{1-\rho}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj} - \frac{n^{1-\rho}\prod_{j=1}^{m} a_{1j}}{(m-1)\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}} \sum_{1\leq i<j\leq m}\left[\sum_{r=2}^{n}\left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2 \\
&\quad \leq n^{1-\rho}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}.
\end{aligned} \tag{39}$$

If $\max\{\lambda_1, \lambda_2, \ldots, \lambda_m\} < \frac{m}{2}$, *then*

$$\begin{aligned}
&\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \\
&\quad \leq n^{1-\rho}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj} - \frac{2n^{1-\rho}\prod_{j=1}^{m} a_{1j}}{m(m-1)} \sum_{1\leq i<j\leq m}\left[\sum_{r=2}^{n}\left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2 \\
&\quad \leq n^{1-\rho}\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}.
\end{aligned} \tag{40}$$

Remark 2.15 If we set $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geq 1$ in Corollary 2.14, then inequalities (39) and (40) reduce to Wu's inequality ([11, Theorem 1]).

In particular, putting $m = 2$, $\lambda_1 = p$, $\lambda_2 = q$, $a_{r1} = a_r$, $a_{r2} = b_r$ ($r = 1, 2, \ldots, n$) in Theorem 2.13, we obtain a new refinement and generalization of inequality (2).

Corollary 2.16 *Let* $a_r > 0$, $b_r > 0$ $(r = 1, 2, \ldots, n)$, *let* $p, q > 0$, $\rho = \min\{\frac{1}{p} + \frac{1}{q}, 1\}$, *and let* $a_1^p - \sum_{r=2}^n a_r^p > 0$, $b_1^q - \sum_{r=2}^n b_r^q > 0$. *Then*

$$\left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{r=2}^n b_r^q\right)^{\frac{1}{q}} \\ \leq n^{1-\rho} \left\{1 - \left[\sum_{r=2}^n \left(\frac{a_r^p}{a_1^p} - \frac{b_r^q}{b_1^q}\right)\right]^2\right\}^{\frac{1}{\max\{p,q\}}} a_1 b_1 - \sum_{r=2}^n a_r b_r. \tag{41}$$

Similarly, putting $m = 2$, $\lambda_1 = p$, $\lambda_2 = q$, $a_{r1} = a_r$, $a_{r2} = b_r$ $(r = 1, 2, \ldots, n)$ in Theorem 2.12 and Theorem 2.11, respectively, we obtain a new refinement and generalization of inequality (3).

Corollary 2.17 *Let* $a_r > 0$, $b_r > 0$ $(r = 1, 2, \ldots, n)$, *let* $p < 0$, $q \neq 0$, $\alpha = \max\{\frac{1}{p} + \frac{1}{q}, 1\}$, *and let* $a_1^p - \sum_{r=2}^n a_r^p > 0$, $b_1^q - \sum_{r=2}^n b_r^q > 0$. *Then*

$$\left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{r=2}^n b_r^q\right)^{\frac{1}{q}} \\ \geq n^{1-\alpha} \left\{1 - \left[\sum_{r=2}^n \left(\frac{a_r^p}{a_1^p} - \frac{b_r^q}{b_1^q}\right)\right]^2\right\}^{\frac{1}{\min\{p,q\}}} a_1 b_1 - \sum_{r=2}^n a_r b_r. \tag{42}$$

From Lemma 2.3 and Theorem 2.11 we obtain the following refinement of inequality (5).

Corollary 2.18 *Let* $a_{rj} > 0$, $\lambda_j < 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *and let* $m \geq 2$. *Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj} \\ - \frac{a_{11} a_{12} \cdots a_{1m}}{(m-1)\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}} \sum_{1 \leq i < j \leq m} \left[\sum_{r=2}^n \left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2. \tag{43}$$

Similarly, from Lemma 2.3 and Theorem 2.12 we obtain the following refinement and generalization of inequality (5).

Corollary 2.19 *Let* $\lambda_m > 0$, $\lambda_j < 0$ $(j = 1, 2, \ldots, m-1)$, *let* $a_{rj} > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$, $r = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, *and let* $\alpha = \max\{\sum_{j=1}^m \frac{1}{\lambda_j}, 1\}$, $m > 2$. *Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geq n^{1-\alpha} \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj} \\ - \frac{a_{11} a_{12} \cdots a_{1m} n^{1-\alpha}}{(m-2)\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}} \sum_{1 \leq i < j \leq m-1} \left[\sum_{r=2}^n \left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2. \tag{44}$$

If we set $\sum_{j=1}^m \frac{1}{\lambda_j} \leq 1$, then from Corollary 2.18 and Corollary 2.19 we obtain the following refinement of inequality (5).

Corollary 2.20 *Let $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2,3,\ldots,m)$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leq 1$, let $a_{rj} > 0$, $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$, $r = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, and let $m > 2$. Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}$$
$$- \frac{a_{11}a_{12}\cdots a_{1m}}{(m-1)\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}} \sum_{1\leq i<j\leq m-1}\left[\sum_{r=2}^{n}\left(\frac{a_{ri}^{\lambda_i}}{a_{1i}^{\lambda_i}} - \frac{a_{rj}^{\lambda_j}}{a_{1j}^{\lambda_j}}\right)\right]^2. \tag{45}$$

3 Application

In this section, we show an application of the inequality newly obtained in Section 2.

Theorem 3.1 *Let $A_j > 0$ $(j = 1,2,\ldots,m)$, let $\lambda_1 > 0$, $\lambda_j < 0$ $(j = 2,3,\ldots,m)$, $\sum_{j=1}^{m} \lambda_j = 1$, $m > 2$, and let $f_j(x)$ $(j = 1,2,\ldots,m)$ be positive integrable functions defined on $[a,b]$ with $A_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\,\mathrm{d}x > 0$. Then*

$$\prod_{j=1}^{m}\left(A_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\,\mathrm{d}x\right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^{m} A_j - \int_a^b \prod_{j=1}^{m} f_j(x)\,\mathrm{d}x$$
$$- \frac{A_1A_2\cdots A_m}{(m-2)\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}} \sum_{1\leq i<j\leq m-1}\left[\int_a^b\left(\frac{f_i^{\lambda_i}(x)}{A_i^{\lambda_i}} - \frac{f_j^{\lambda_j}(x)}{A_j^{\lambda_j}}\right)\mathrm{d}x\right]^2. \tag{46}$$

Proof For any positive integers n, we choose an equidistant partition of $[a,b]$ as

$$a < a + \frac{b-a}{n} < \cdots < a + \frac{b-a}{n}k < \cdots < a + \frac{b-a}{n}(n-1) < b,$$
$$x_i = a + \frac{b-a}{n}i, \quad i = 0,1,\ldots,n, \qquad \Delta x_k = \frac{b-a}{n}, \quad k = 1,2,\ldots,n.$$

Noting that $A_j^{\lambda_j} - \int_a^b f_j^{\lambda_j}(x)\,\mathrm{d}x > 0$ $(j = 1,2,\ldots,m)$, we have

$$A_j^{\lambda_j} - \lim_{n\to\infty}\sum_{k=1}^{n} f_j^{\lambda_j}\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n} > 0 \quad (j = 1,2,\ldots,m).$$

Consequently, there exists a positive integer N, such that

$$A_j^{\lambda_j} - \sum_{k=1}^{n} f_j^{\lambda_j}\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n} > 0,$$

for all $n, l > N$ and $j = 1,2,\ldots,m$.

By using Theorem 2.12, for any $n > N$, the following inequality holds:

$$\prod_{j=1}^{m}\left[A_j^{\lambda_j} - \sum_{k=1}^{n} f_j^{\lambda_j}\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{\lambda_j}}$$
$$\geq \prod_{j=1}^{m} A_j^{\lambda_j} - \sum_{k=1}^{n}\left[\prod_{j=1}^{m} f_j\left(a + \frac{k(b-a)}{n}\right)\right]\left(\frac{b-a}{n}\right)^{\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\cdots+\frac{1}{\lambda_m}}$$

$$
\begin{aligned}
&- \frac{A_1 A_2 \cdots A_m}{(m-2)\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}} \sum_{1\le i<j\le m} \Biggl\{ \sum_{k=1}^{n} \biggl[\frac{1}{A_i^{\lambda_i}} f_i^{\lambda_i}\biggl(a + \frac{k(b-a)}{n}\biggr)\frac{b-a}{n} \\
&- \frac{1}{A_j^{\lambda_j}} f_j^{\lambda_j}\biggl(a + \frac{k(b-a)}{n}\biggr)\frac{b-a}{n}\biggr]\Biggr\}^2. \qquad (47)
\end{aligned}
$$

Since

$$
\sum_{j=1}^{m} \frac{1}{\lambda_j} = 1,
$$

we have

$$
\begin{aligned}
&\prod_{j=1}^{m} \biggl[A_j^{\lambda_j} - \sum_{k=1}^{n} f_j^{\lambda_j}\biggl(a + \frac{k(b-a)}{n}\biggr)\frac{b-a}{n}\biggr]^{\frac{1}{\lambda_j}} \\
&\quad \ge \prod_{j=1}^{m} A_j^{\lambda_j} - \sum_{k=1}^{n} \biggl[\prod_{j=1}^{m} f_j\biggl(a + \frac{k(b-a)}{n}\biggr)\biggr]\biggl(\frac{b-a}{n}\biggr) \\
&\quad - \frac{A_1 A_2 \cdots A_m}{(m-2)\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}} \sum_{1\le i<j\le m} \Biggl\{ \sum_{k=1}^{n} \biggl[\frac{1}{A_i^{\lambda_i}} f_i^{\lambda_i}\biggl(a + \frac{k(b-a)}{n}\biggr)\frac{b-a}{n} \\
&\quad - \frac{1}{A_j^{\lambda_j}} f_j^{\lambda_j}\biggl(a + \frac{k(b-a)}{n}\biggr)\frac{b-a}{n}\biggr]\Biggr\}^2. \qquad (48)
\end{aligned}
$$

Noting that $f_j(x)$ $(j = 1, 2, \ldots, m)$ are positive Riemann integrable functions on $[a, b]$, we know that $\prod_{j=1}^{m} f_j(x)$ and $f_j^{\lambda_j}(x)$ are also integrable on $[a, b]$. Letting $n \to \infty$ on both sides of inequality (48), we get the desired inequality (46). The proof of Theorem 3.1 is completed. □

Remark 3.2 Obviously, inequality (46) is sharper than inequality (6).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Aczél, J: Some general methods in the theory of functional equations in one variable. New applications of functional equations. Usp. Mat. Nauk **11**(3), 3-68 (1956) (in Russian)
2. Farid, G, Pečarić, J, Ur Rehman, A: On refinements of Aczél's, Popoviciu, Bellman's inequalities and related results. J. Inequal. Appl. **2010**, Article ID 579567 (2010)
3. Popoviciu, T: On an inequality. Gaz. Mat. Fiz., Ser. A **11**(64), 451-461 (1959) (in Romanian)

4. Tian, J: A sharpened and generalized version of Aczél-Vasić-Pečarić inequality and its application. J. Inequal. Appl. **2013**, Article ID 497 (2013)
5. Tian, J-F: Reversed version of a generalized Aczél's inequality and its application. J. Inequal. Appl. **2012**, Article ID 202 (2012)
6. Tian, J-F, Wang, S: Refinements of generalized Aczél's inequality and Bellman's inequality and their applications. J. Appl. Math. **2013**, Article ID 645263 (2013)
7. Vasić, PM, Pečarić, JE: On Hölder and some related inequalities. Rev. Anal. Numér. Théor. Approx. **25**, 95-103 (1982)
8. Vasić, PM, Pečarić, JE: On the Jensen inequality for monotone functions. An. Univ. Timiş., Ser. ştiinţe Mat. **17**(1), 95-104 (1979)
9. Wu, S, Debnath, L: Generalizations of Aczél's inequality and Popoviciu's inequality. Indian J. Pure Appl. Math. **36**(2), 49-62 (2005)
10. Beckenbach, EF, Bellman, R: Inequalities. Springer, Berlin (1983)
11. Wu, S: Some improvements of Aczél's inequality and Popoviciu's inequality. Comput. Math. Appl. **56**(5), 1196-1205 (2008)

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