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Fixed points for generalized (α, ψ)-contractions on generalized metric spaces

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Abstract

In this paper, we introduce some generalized (α, ψ)-contractive mappings in the setting of generalized metric spaces and, based on the very recent paper (Kirk and Shahzad in Fixed Point Theory Appl. 2013:129, 2013), we omit the Hausdorff hypothesis to prove some fixed point results involving such mappings. Some consequences on existing fixed point theorems are also derived.

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1 Introduction and preliminaries

The topology of generalized metrics has some disadvantages:

(P1) A generalized metric does not need to be continuous.

(P2) A convergent sequence in generalized metric spaces does not need to be Cauchy.

(P3) A generalized metric space does not need to be Hausdorff, and hence the uniqueness of limits cannot be guaranteed.

Very recently, Samet *et al.* [1] suggested a very interesting class of mappings, known as α-ψ contractive mappings, to investigate the existence and uniqueness of a fixed point. Several well-known fixed point theorems, including the Banach mapping principle were concluded as consequences of the main result of this interesting paper. The techniques used in this paper have been studied and improved by a number of authors; see *e.g.* [2–7].

In this paper, we investigate the existence and uniqueness of fixed points of (α, ψ)-contractive mappings in the setting of generalized metric spaces by caring the problems (P1)-(P3) mentioned above. In the literature, notice that there are distinct notions that are called 'a generalized metric'. In the sequel, when we mention a 'generalized metric' we mean the metric introduced by Branciari [8] who introduced the concept of a generalized metric space by replacing the triangle inequality by a more general inequality. As such, any metric space is a generalized metric space but the converse is not true [8]. He proved the Banach fixed point theorem in such a space. For more details, the reader can refer to [5, 9–24].

First, we recollect some fundamental definitions and notations and basic results that will be used throughout this paper.

$\mathbb{N}$ and $\mathbb{R}^+$ denote the set of positive integers and the set of nonnegative reals, respectively. Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ψ is upper semicontinuous;
(ii) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, for all $t > 0$;
(iii) $\psi(t) < t$, for any $t > 0$.

In the following, we recall the notion of a generalized metric space.

Definition 1 [8] Let X be a nonempty set and let $d : X \times X \longrightarrow [0,\infty]$ satisfy the following conditions, for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y.

$$\begin{aligned} &\text{(GMS1)} \quad d(x,y) = 0 \text{ if and only if } x = y, \\ &\text{(GMS2)} \quad d(x,y) = d(y,x), \\ &\text{(GMS3)} \quad d(x,y) \le d(x,u) + d(u,v) + d(v,y). \end{aligned} \tag{1.1}$$

Then the map d is called a generalized metric and abbreviated as GMS. Here, the pair (X,d) is called a generalized metric space.

In the above definition, if d satisfies only (GMS1) and (GMS2), then it is called a semimetric (see *e.g.* [25]).

The concepts of convergence, Cauchy sequence, completeness, and continuity on a GMS are defined below.

Definition 2

(1) A sequence $\{x_n\}$ in a GMS (X,d) is GMS convergent to a limit x if and only if $d(x_n,x) \to 0$ as $n \to \infty$.
(2) A sequence $\{x_n\}$ in a GMS (X,d) is GMS Cauchy if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$, for all $n > m > N(\varepsilon)$.
(3) A GMS (X,d) is called complete if every GMS Cauchy sequence in X is GMS convergent.
(4) A mapping $T : (X,d) \to (X,d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n,x) \to 0$ as $n \to \infty$, we have $d(Tx_n, Tx) \to 0$ as $n \to \infty$.

The following assumption was suggested by Wilson [25] to replace the triangle inequality with the weakened condition.

(W) For each pair of (distinct) points u, v, there is a number $r_{u,v} > 0$ such that for every $z \in X$,

$$r_{u,v} < d(u,z) + d(z,v).$$

Proposition 3 [26] *In a semimetric space, the assumption* (W) *is equivalent to the assertion that limits are unique*.

Proposition 4 [26] *Suppose that* $\{x_n\}$ *is a Cauchy sequence in a GMS* (X,d) *with* $\lim_{n\to\infty} d(x_n,u) = 0$, *where* $u \in X$. *Then* $\lim_{n\to\infty} d(x_n,z) = d(u,z)$, *for all* $z \in X$. *In particular, the sequence* $\{x_n\}$ *does not converge to* z *if* $z \neq u$.

Recall that Samet *et al.* [1] introduced the following concepts.

Definition 5 [1] For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is α-admissible if, for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1. \tag{1.2}$$

Some interesting examples of such mappings were given in [1]. The notion of a α-ψ contractive mapping is also defined in the following way.

Definition 6 [1] Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and a certain ψ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi\big(d(x, y)\big), \quad \text{for all } x, y \in X. \tag{1.3}$$

It is obvious that any contractive mapping that is a mapping satisfying the Banach contraction is a α-ψ contractive mapping with $\alpha(x, y) = 1$, for all $x, y \in X$ and $\psi(t) = kt$, $k \in (0, 1)$.

Very recently, Karapınar [5] gave the analog of the notion of a α-ψ contractive mapping, in the context of generalized metric spaces as follows.

Definition 7 Let (X, d) be a generalized metric space and $T : X \to X$ be a given mapping. We say that T is a α-ψ contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and a certain ψ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi\big(d(x, y)\big), \quad \text{for all } x, y \in X. \tag{1.4}$$

Karapınar [5] also stated the following fixed point theorems.

Theorem 8 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a α-ψ contractive mapping. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$*;
(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 9 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a α-ψ contractive mapping. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$*;
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$, for all n.*

Then there exists a $u \in X$ such that $Tu = u$.

For the uniqueness, we need an additional condition:

(U) For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x, y) \geq 1$, where $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 10 *Adding condition* (U) *to the hypotheses of Theorem* 8 (*resp. Theorem* 9), *we find that u is the unique fixed point of* T.

As an alternative condition for the uniqueness of a fixed point of a α-ψ contractive mapping, we shall consider the following hypothesis.

(H) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Theorem 11 *Adding conditions* (H) *and* (W) *to the hypotheses of Theorem* 8 (*resp. Theorem* 9), *we find that u is the unique fixed point of* T.

Corollary 12 *Adding condition* (H) *to the hypotheses of Theorem* 8 (*resp. Theorem* 9) *and assuming that* (X,d) *is Hausdorff, we find that u is the unique fixed point of* T.

In this paper, we give some generalized (α,ψ)-contractive mappings. Using such mappings, we prove some fixed point results in the setting of generalized metric spaces. Our results have been found without the assumption that the GMS space (X,d) is Hausdorff. Note that this statement has been observed first in [26]. Some consequences on existing fixed point theorems are provided.

2 Main results

First, we give in the following definitions a generalization of the notion of α-ψ contractive mappings in the context of a generalized metric space.

Definition 13 Let (X,d) be a generalized metric space and $T : X \to X$ be a given mapping. We say that T is a generalized (α,ψ)-contractive mapping of type I if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(M(x,y)\bigr), \quad \text{for all } x,y \in X, \tag{2.1}$$

where

$$M(x,y) = \max\bigl\{d(x,y), d(x,Tx), d(y,Ty)\bigr\}. \tag{2.2}$$

Definition 14 Let (X,d) be a generalized metric space and $T : X \to X$ be a given mapping. We say that T is a generalized (α,ψ)-contractive mapping of type II if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi\bigl(N(x,y)\bigr), \quad \text{for all } x,y \in X, \tag{2.3}$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}\right\}. \tag{2.4}$$

Now, we state our first fixed point result.

Theorem 15 *Let* (X,d) *be a complete generalized metric space and* $T : X \to X$ *be a generalized* α-ψ *contractive mapping of type I. Suppose that*

(i) *T is α-admissible*;
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$*;
(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. We define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \geq 0$. Suppose that $x_{n_0} = x_{n_0+1}$ for some n_0. So the proof is completed since $u = x_{n_0} = x_{n_0+1} = Tx_{n_0} = Tu$. Consequently, throughout the proof, we assume that

$$x_n \neq x_{n+1}, \quad \text{for all } n. \tag{2.5}$$

Observe that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \quad \Rightarrow \quad \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1,$$

since T is α-admissible. By repeating the process above, we derive

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n = 0, 1, \ldots. \tag{2.6}$$

By using the same technique above, we get

$$\alpha(x_0, x_2) = \alpha\left(x_0, T^2x_0\right) \geq 1 \quad \Rightarrow \quad \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1.$$

The expression above yields

$$\alpha(x_n, x_{n+2}) \geq 1, \quad \text{for all } n = 0, 1, \ldots. \tag{2.7}$$

Step 1: We shall prove

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.8}$$

Combining (2.1) and (2.6), we find that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n) \leq \psi\left(M(x_{n-1}, x_n)\right), \tag{2.9}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\right\} \\ &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\} \\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}. \end{aligned} \tag{2.10}$$

If for some n, $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ ($\neq 0$), then the inequality (2.9) turns into

$$d(x_n, x_{n+1}) \leq \psi\left(M(x_{n-1}, x_n)\right) = \psi\left(d(x_n, x_{n+1})\right) < d(x_n, x_{n+1}),$$

which is a contradiction. Hence, $M(x_{n-1},x_n) = d(x_{n-1},x_n)$, for all $n \in \mathbb{N}$, and (2.9) becomes

$$d(x_n,x_{n+1}) \leq \psi\big(d(x_{n-1},x_n)\big), \quad \text{for all } n \in \mathbb{N}. \tag{2.11}$$

This yields

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n), \quad \text{for all } n \in \mathbb{N}. \tag{2.12}$$

By (2.11), we find that

$$d(x_n,x_{n+1}) \leq \psi^n\big(d(x_0,x_1)\big), \quad \text{for all } n \in \mathbb{N}. \tag{2.13}$$

By the property of ψ, it is evident that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

Step 2: We shall prove

$$\lim_{n\to\infty} d(x_n,x_{n+2}) = 0. \tag{2.14}$$

Regarding (2.1) and (2.7), we deduce that

$$d(x_n,x_{n+2}) = d(Tx_{n-1},Tx_{n+1}) \leq \alpha(x_{n-1},x_{n+1})d(Tx_{n-1},Tx_{n+1}) \leq \psi\big(M(x_{n-1},x_{n+1})\big), \tag{2.15}$$

for all $n \geq 1$, where

$$\begin{aligned} M(x_{n-1},x_{n+1}) &= \max\big\{d(x_{n-1},x_{n+1}), d(x_{n-1},Tx_{n-1}), d(x_{n+1},Tx_{n+1})\big\} \\ &= \max\big\{d(x_{n-1},x_{n+1}), d(x_{n-1},x_n), d(x_{n+1},x_{n+2})\big\}. \end{aligned} \tag{2.16}$$

By (2.12), we have

$$M(x_{n-1},x_{n+1}) = \max\big\{d(x_{n-1},x_{n+1}), d(x_{n-1},x_n)\big\}.$$

Take $a_n = d(x_n,x_{n+2})$ and $b_n = d(x_n,x_{n+1})$. Thus, from (2.16)

$$a_n = d(x_n,x_{n+2}) \leq \psi\big(M(x_{n-1},x_{n+1})\big) = \psi\big(\max\{a_{n-1},b_{n-1}\}\big), \quad \text{for all } n \in \mathbb{N}. \tag{2.17}$$

Again, by (2.12)

$$b_n \leq b_{n-1} \leq \max\{a_{n-1},b_{n-1}\}.$$

Therefore

$$\max\{a_n,b_n\} \leq \max\{a_{n-1},b_{n-1}\}, \quad \text{for all } n \in \mathbb{N}.$$

Then the sequence $\{\max\{a_n, b_n\}\}$ is monotone nonincreasing, so it converges to some $t \geq 0$. Assume that $t > 0$. Now, by (2.8)

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} \max\{a_n, b_n\} = \lim_{n\to\infty} \max\{a_n, b_n\} = t.$$

Taking $n \to \infty$ in (2.17)

$$t = \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \psi\left(\max\{a_{n-1}, b_{n-1}\}\right) \leq \psi\left(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}\right) = \psi(t) < t,$$

which is a contradiction, that is, (2.14) is proved.

Step 3: We shall prove that

$$x_n \neq x_m, \quad \text{for all } n \neq m. \tag{2.18}$$

We argue by contradiction. Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Since $d(x_p, x_{p+1}) > 0$, for each $p \in \mathbb{N}$, without loss of generality, we may assume that $m > n + 1$. Consider now

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, Tx_n) &= d(x_m, Tx_m) \\ &= d(Tx_{m-1}, Tx_m) \leq \alpha(x_{m-1}, x_m) d(Tx_{m-1}, Tx_m) \\ &\leq \psi\big(M(x_{m-1}, x_m)\big), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} M(x_{m-1}, x_m) &= \max\big\{d(x_{m-1}, x_m), d(x_{m-1}, Tx_{m-1}), d(x_m, Tx_m)\big\} \\ &= \max\big\{d(x_{m-1}, x_m), d(x_{m-1}, x_m), d(x_m, x_{m+1})\big\} \\ &= \max\big\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\big\}. \end{aligned} \tag{2.20}$$

If $M(x_{m-1}, x_m) = d(x_{m-1}, x_m)$, then from (2.19) we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, Tx_n) &= d(x_m, Tx_m) \\ &= d(x_m, x_{m+1}) \leq \alpha(x_m, x_{m+1}) d(Tx_{m-1}, Tx_m) \\ &\leq \psi\big(M(x_{m-1}, x_m)\big) = \psi\big(d(x_{m-1}, x_m)\big) \\ &\leq \psi^{m-n}\big(d(x_n, x_{n+1})\big). \end{aligned} \tag{2.21}$$

If $M(x_{m-1}, x_m) = d(x_m, x_{m+1})$, the inequality (2.19) becomes

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n, Tx_n) &= d(x_m, Tx_m) \\ &= d(Tx_{m-1}, Tx_m) \leq \alpha(x_{m-1}, x_m) d(Tx_{m-1}, Tx_m) \\ &\leq \psi\big(M(x_{m-1}, x_m)\big) = \psi\big(d(x_m, x_{m+1})\big) \\ &\leq \psi^{m-n+1}\big(d(x_n, x_{n+1})\big). \end{aligned} \tag{2.22}$$

Due to a property of ψ, the inequalities (2.21) and (2.22) together yield

$$d(x_n, x_{n+1}) \leq \psi^{m-n}\big(d(x_n, x_{n+1})\big) < d(x_n, x_{n+1}) \tag{2.23}$$

and

$$d(x_n, x_{n+1}) \leq \psi^{m-n+1}\big(d(x_n, x_{n+1})\big) < d(x_n, x_{n+1}), \tag{2.24}$$

respectively. In each case, there is a contradiction.

Step 4: We shall prove that $\{x_n\}$ is a Cauchy sequence, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \quad \text{for all } k \in \mathbb{N}. \tag{2.25}$$

The cases $k = 1$ and $k = 2$ are proved, respectively, by (2.8) and (2.14). Now, take $k \geq 3$ arbitrary. It is sufficient to examine two cases.

Case (I): Suppose that $k = 2m+1$ where $m \geq 1$. Then, by using step 3 and the quadrilateral inequality together with (2.13), we find

$$\begin{aligned} d(x_n, x_{n+k}) = d(x_n, x_{n+2m+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+2m}, x_{n+2m+1}) \\ &\leq \sum_{p=n}^{n+2m} \psi^p\big(d(x_0, x_1)\big) \\ &\leq \sum_{p=n}^{+\infty} \psi^p\big(d(x_0, x_1)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.26}$$

Case (II): Suppose that $k = 2m$ where $m \geq 2$. Again, by applying the quadrilateral inequality and step 3 together with (2.13), we find

$$\begin{aligned} d(x_n, x_{n+k}) = d(x_n, x_{n+2m}) &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{n+2m-1}, x_{n+2m}) \\ &\leq d(x_n, x_{n+2}) + \sum_{p=n+2}^{n+2m-1} \psi^p\big(d(x_0, x_1)\big) \\ &\leq d(x_n, x_{n+2}) + \sum_{p=n}^{+\infty} \psi^p\big(d(x_0, x_1)\big) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.27}$$

By combining the expressions (2.26) and (2.27), we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0, \quad \text{for all } k \geq 3.$$

We conclude that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{2.28}$$

Since T is continuous, we obtain from (2.28) that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0, \tag{2.29}$$

that is, $\lim_{n\to\infty} x_{n+1} = Tu$. Taking Proposition 4 into account, we conclude that $Tu = u$, that is, u is a fixed point of T. □

The following result is deduced from Theorem 15 due to the obvious inequality $N(x,y) \leq M(x,y)$.

Theorem 16 *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a generalized α-ψ contractive mapping of type II. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 15 remains true if we replace the continuity hypothesis by the following property:

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$, for all k.

This statement is given as follows.

Theorem 17 *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a generalized α-ψ contractive mapping of type I. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$, for all k.*

Then there exists $u \in X$ such that $Tu = u$.

Proof Following the proof of Theorem 8, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for all $n \geq 0$, is Cauchy and converges to some $u \in X$. In view of Proposition 4,

$$\lim_{k\to\infty} d(x_{n(k)+1}, Tu) = d(u, Tu). \tag{2.30}$$

We shall show that $Tu = u$. Suppose, on the contrary, that $Tu \neq u$, *i.e.*, $d(Tu, u) > 0$. From (2.6) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$, for all k.

By applying (2.1), we get

$$d(x_{n(k)+1}, Tu) \leq \alpha(x_{n(k)}, u) d(Tx_{n(k)}, Tu) \leq \psi\bigl(M(x_{n(k)}, u)\bigr), \tag{2.31}$$

where

$$\begin{aligned} M(x_{n(k)}, u) &= \max\bigl\{d(x_{n(k)}, u), d(x_{n(k)}, Tx_{n(k)}), d(u, Tu)\bigr\} \\ &= \max\bigl\{d(x_{n(k)}, u), d(x_{n(k)}, x_{n(k)+1}), d(u, Tu)\bigr\}. \end{aligned} \tag{2.32}$$

By (2.8) and (2.30), we have

$$\lim_{k\to\infty} M(x_{n(k)}, u) = d(u, Tu). \tag{2.33}$$

Letting $k \to \infty$ in (2.31) and regarding that ψ is upper semicontinuous

$$d(u, Tu) \leq \psi\big(d(u, Tu)\big) < d(u, Tu), \tag{2.34}$$

which is a contradiction. Hence, we find that u is a fixed point of T, that is, $Tu = u$. □

In the following, the hypothesis of upper semicontinuity of ψ is not required. Similar to Theorem 17, for the generalized α-ψ contractive mappings of type II, we have the following.

Theorem 18 *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a generalized α-ψ contractive mapping of type II. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$, for all k.*

Then there exists $u \in X$ such that $Tu = u$.

Proof Following the proof of Theorem 16 (which is the same as Theorem 15), we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for all $n \geq 0$, is Cauchy and converges to some $u \in X$. Similarly, in view of Proposition 4,

$$\lim_{k\to\infty} d(x_{n(k)+1}, Tu) = d(u, Tu). \tag{2.35}$$

We shall show that $Tu = u$. Suppose, on the contrary, that $Tu \neq u$. From (2.6) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$, for all k. By applying (2.3), for all k, we get

$$d(x_{n(k)+1}, Tu) \leq \alpha(x_{n(k)}, u)d(Tx_{n(k)}, Tu) \leq \psi\big(N(x_{n(k)}, u)\big), \tag{2.36}$$

where

$$N(x_{n(k)}, u) = \max\left\{d(x_{n(k)}, u), \frac{d(x_{n(k)}, Tx_{n(k)}) + d(u, Tu)}{2}\right\}. \tag{2.37}$$

Letting $k \to \infty$ in (2.36), we have

$$\lim_{k\to\infty} N(x_{n(k)}, u) = \frac{d(u, Tu)}{2}. \tag{2.38}$$

From (2.38), for k large enough, we have $N(x_{n(k)}, u) > 0$, which implies that

$$\psi\big(N(x_{n(k)}, u)\big) < N(x_{n(k)}, u).$$

Thus, from (2.36) and (2.38), we have

$$d(u, Tu) \leq \frac{d(u, Tu)}{2},$$

which is a contradiction. Hence, we find that u is a fixed point of T, that is, $Tu = u$. □

Theorem 19 *Adding condition* (U) *to the hypotheses of Theorem* 15 (*resp. Theorem* 17), *we obtain that u is the unique fixed point of T.*

Proof In what follows we shall show that u is a unique fixed point of T. Let v be another fixed point of T with $v \neq u$. By hypothesis (U),

$$1 \leq \alpha(u,v) = \alpha(Tu,Tv).$$

Now, due to (2.1), we have

$$\begin{aligned} d(u,v) &\leq \alpha(u,v)d(u,v) \\ &= \alpha(Tu,Tv)d(Tu,Tv) \\ &\leq \psi\big(M(u,v)\big) \\ &= \psi\big(\max\big\{d(u,v), d(u,Tu), d(v,Tv)\big\}\big) \\ &= \psi\big(d(u,v)\big) \\ &< d(u,v), \end{aligned} \tag{2.39}$$

which is a contradiction. Hence, $u = v$. □

Theorem 20 *Adding condition* (U) *to the hypotheses of Theorem* 16 (*resp. Theorem* 18), *we see that u is the unique fixed point of T.*

Proof The proof is an analog of the proof of Theorem 19. Suppose, on the contrary, that v is another fixed point of T with $v \neq u$. It is evident that $1 \leq \alpha(u,v) = \alpha(Tu,Tv)$.

Now, due to (2.3), we have

$$\begin{aligned} d(u,v) &\leq \alpha(u,v)d(u,v) \\ &= \alpha(Tu,Tv)d(Tu,Tv) \\ &\leq \psi\big(N(u,v)\big) \\ &= \psi\left(\max\left\{d(u,v), \frac{d(u,Tu)+d(v,Tv)}{2}\right\}\right) \\ &= \psi\big(d(u,v)\big) \\ &< d(u,v), \end{aligned} \tag{2.40}$$

which is a contradiction. Hence, $u = v$. □

As alternative conditions for the uniqueness of a fixed point of a generalized α-ψ contractive mapping, we shall consider the following hypotheses:

(H1) For all $x, y \in \operatorname{Fix}(T)$, there exists z in X such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

(H2) Let $x, y \in \operatorname{Fix}(T)$. If there exists $\{z_n\}$ in X such that $\alpha(x,z_n) \geq 1$ and $\alpha(y,z_n) \geq 1$, then

$$d(z_n, z_{n+1}) \leq \inf\big\{d(x,z_n), d(y,z_n)\big\}, \quad \text{for all } n \in \mathbb{N}.$$

Theorem 21 *Adding conditions* (H1), (H2), *and* (W) *to the hypotheses of Theorem* 15 (*resp. Theorem* 17), *we find that u is the unique fixed point of T.*

Proof Suppose that v is another fixed point of T, so $d(u,v) > 0$. From (H1), there exists $z \in X$ such that

$$\alpha(u,z) \geq 1 \quad \text{and} \quad \alpha(v,z) \geq 1. \tag{2.41}$$

Since T is α-admissible, from (2.41), we have

$$\alpha\big(u, T^n z\big) \geq 1 \quad \text{and} \quad \alpha\big(v, T^n z\big) \geq 1, \quad \text{for all } n. \tag{2.42}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$, for all $n \geq 0$ and $z_0 = z$. From (2.42), for all n, we have

$$d(u, z_{n+1}) = d(Tu, Tz_n) \leq \alpha(u, z_n) d(Tu, Tz_n) \leq \psi\big(M(u, z_n)\big), \tag{2.43}$$

where

$$M(u, z_n) = \max\big\{d(u, z_n), d(u, Tu), d(z_n, Tz_n)\big\} = \max\big\{d(u, z_n), d(z_n, z_{n+1})\big\}. \tag{2.44}$$

By (H2), we get

$$M(u, z_n) = d(u, z_n), \quad \text{for all } n.$$

Iteratively, by using the inequality (2.43), we get

$$d(u, z_n) \leq \psi^n\big(d(u, z_0)\big), \tag{2.45}$$

for all n. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(z_n, u) = 0. \tag{2.46}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(z_n, v) = 0. \tag{2.47}$$

Regarding (W), there exists $r_{u,v} > 0$ such that, for all n,

$$r_{u,v} < d(u, z_n) + d(v, z_n).$$

From (2.46) and (2.47), by passing $n \to \infty$, it follows that $r_{u,v} = 0$, which is a contradiction. Thus, we proved that u is the unique fixed point of T. □

Theorem 22 *Adding conditions* (H1), (H2) *and* (W) *to the hypotheses of Theorem* 16 (*resp. Theorem* 18), *we find that u is the unique fixed point of T.*

Proof Suppose that v is another fixed point of T and $u \neq v$. From (H1), there exists $z \in X$ such that

$$\alpha(u,z) \geq 1 \quad \text{and} \quad \alpha(v,z) \geq 1. \tag{2.48}$$

Since T is α-admissible, from (2.48), we have

$$\alpha\bigl(u,T^n z\bigr) \geq 1 \quad \text{and} \quad \alpha\bigl(v,T^n z\bigr) \geq 1, \quad \text{for all } n. \tag{2.49}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$, for all $n \geq 0$ and $z_0 = z$. From (2.49), for all n, we have

$$d(u,z_{n+1}) = d(Tu,Tz_n) \leq \alpha(u,z_n)d(Tu,Tz_n) \leq \psi\bigl(N(u,z_n)\bigr), \tag{2.50}$$

where

$$N(u,z_n) = \max\left\{d(u,z_n), \frac{d(u,Tu)+d(z_n,Tz_n)}{2}\right\} = \max\left\{d(u,z_n), \frac{d(z_n,z_{n+1})}{2}\right\}. \tag{2.51}$$

By (H2), we get

$$N(u,z_n) = d(u,z_n), \quad \text{for all } n.$$

Iteratively, by using the inequality (2.50), we get

$$d(u,z_n) \leq \psi^n\bigl(d(u,z_0)\bigr), \tag{2.52}$$

for all n. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(z_n,u) = 0. \tag{2.53}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(z_n,v) = 0. \tag{2.54}$$

Similarly, regarding (W) together with (2.53) and (2.54), it follows that $u = v$. Thus we proved that u is the unique fixed point of T. □

It is well known that the Hausdorff property implies the uniqueness of the limit, so the (W) condition in Theorem 21 (resp. Theorem 22) can be replaced by the Hausdorff property. Then the proof of the following result is clear and hence it is omitted.

Corollary 23 *Adding conditions* (H1) *and* (H2) *to the hypotheses of Theorem* 15 (*resp. Theorem* 17, *Theorem* 16, *Theorem* 18) *and assuming that* (X,d) *is Hausdorff, we find that* u *is the unique fixed point of* T.

3 Consequences

Now, we will show that many existing results in the literature can be deduced easily from our obtained results.

Corollary 24 *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a given continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi\bigl(M(x,y)\bigr),$$

for all $x,y \in X$. Then T has a unique fixed point.

Proof It suffices to take $\alpha(x,y) = 1$ in Theorem 19. □

The following fixed point theorems follow immediately from Corollary 24 by taking $\psi(t) = \lambda t$, where $\lambda \in (0,1)$.

Corollary 25 *Let (X,d) be a complete generalized metric space and $T : X \to X$ be a given continuous mapping. Suppose that there exists a constant $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \leq \lambda M(x,y),$$

for all $x,y \in X$. Then T has a unique fixed point.

Following Corollary 3.8 of Karapınar and Samet [6], we deduce an existing result on generalized metric spaces endowed with a partial order. At first, we need to recall some concepts.

Definition 26 Let $(X,\preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$x,y \in X, \quad x \preceq y \quad \Longrightarrow \quad Tx \preceq Ty.$$

Definition 27 Let $(X,\preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$, for all n.

Definition 28 Let $(X,\preceq)$ be a partially ordered set and d be a generalized metric on X. We say that $(X,\preceq,d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$, for all k.

Corollary 29 *Let $(X,\preceq)$ be a partially ordered set and d be a generalized metric on X such that (X,d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \leq \psi\bigl(M(x,y)\bigr),$$

for all $x,y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) *there exists* $x_0 \in X$ *such that* $x_0 \preceq Tx_0$ *and* $x_0 \preceq T^2x_0$;
(ii) T *is continuous or* $(X, \preceq, d)$ *is regular.*

Then T *has a fixed point.*

Proof Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y \text{ or } x \succeq y, \\ 0 & \text{otherwise.} \end{cases}$$

The rest of proof follows similarly to the proof of Corollary 3.8 of Karapınar and Samet [6]. □

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final version of manuscript.

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