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Strong convergence theorems of the Halpern-Mann's mixed iteration for a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping in Banach spaces

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Abstract

In this paper, we introduce a class of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping to modify the Halpern-Mann-type iteration algorithm for a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping, which has the strong convergence under a limit condition only in the framework of Banach spaces. Our results are applied to study the approximation problem of solution to a system of equilibrium problems. Also, the results presented in the paper improve and extend the corresponding results of Chang *et al.* (Appl. Math. Comput. 218:7864-7870, 2012) and others.

Keywords: totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping; iterative sequence; Halpern and Mann-type iteration algorithm; nonexpansive retraction; generalized projection

1 Introduction and preliminaries

A Banach space X is said to be strictly convex if $\|\frac{x+y}{2}\| \leq 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\} \subset X$ with $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 0$.

The norm of Banach space X is said to be Gâteaux differentiable, if, for each $x, y \in S(x)$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists, where $S(x) = \{x : \|x\| = 1, x \in X\}$. In this case, X is said to be smooth. The norm of Banach space X is said to be Fréchet differentiable, if, for each $x \in S(x)$, the limit (1.1) is attained uniformly for $y \in S(x)$ and the norm is uniformly Fréchet differentiable if the limit (1.1) is attained uniformly for $x, y \in S(x)$. In this case, X is said to be uniformly smooth.

Let D be a nonempty closed subset of a real Banach space X. A mapping $T : D \to D$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of a nonself multi-valued mapping $T : D \to X$ if $p \in Tp$. The set of fixed points of T is represented by $F(T)$.

A subset D of X is said to be retract of X, if there exists a continuous mapping $P : X \to D$ such that $Px = x$, for all $x \in X$. It is well known that every nonempty, closed, convex subset of a uniformly convex Banach space X is a retract of X. A mapping $P : X \to D$ is said to be a retraction, if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A mapping $P : X \to D$ is said to be a nonexpansive retraction, if it is nonexpansive and it is a retraction from X to D.

Assume that X is a real Banach space with the dual X^*, D is a nonempty, closed, convex subset of X. We also denote by J the normalized duality mapping from X to 2^{X^*} which is defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Next we assume that X is a smooth, strictly convex and reflexive Banach space and D is a nonempty, closed, convex subset of X. In the sequel, we always use $\phi : X \times X \to R^+$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad x, y \in X. \tag{1.2}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \tag{1.3}$$

$$\phi(y, x) = \phi(y, z) + \phi(z, x) + 2\langle z - y, Jx - Jz \rangle, \quad x, y, z \in X, \tag{1.4}$$

and

$$\phi\big(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)\big) \le \lambda\phi(x, y) + (1 - \lambda)\phi(x, z) \tag{1.5}$$

for all $\lambda \in [0, 1]$ and $x, y, z \in X$.

Following Alber [2], the generalized projection $\Pi_D : X \to D$ is defined by

$$\Pi_D(x) = \arg\inf_{y \in D} \phi(y, x), \quad \forall x \in X. \tag{1.6}$$

Lemma 1.1 (see [3]) *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\{x_n\}$ and $\{y_n\}$ is bounded, if $\phi(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$.*

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping.

In the sequel, we denote the strong convergence and weak convergence of the sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Lemma 1.2 (see [2]) *Let X be a smooth, strictly convex, and reflexive Banach space and D be a nonempty, closed, convex subset of X. Then the following conclusions hold*:

(a) $\phi(x, y) = 0$ *if and only if* $x = y$;
(b) $\phi(x, \Pi_D y) + \phi(\Pi_D y, y) \le \phi(x, y)$, $\forall x, y \in D$;
(c) *if* $x \in X$ *and* $z \in D$, *then* $z = \Pi_D x$ *if and only if* $\langle z - y, Jx - Jz \rangle \ge 0$, $\forall y \in D$.

Remark 1.1 (see [4]) Let Π_D be the generalized projection from a smooth, reflexive and strictly convex Banach space X onto a nonempty, closed, convex subset D of X. Then Π_D is a closed and quasi-ϕ-nonexpansive from X onto D.

Remark 1.2 (see [4]) If H is a real Hilbert space, then $\phi(x,y) = \|x-y\|^2$, and Π_D is the metric projection of H onto D.

Definition 1.1 Let $P : X \to D$ be the nonexpansive retraction.

(1) A nonself multi-valued mapping $T : D \to X$ is said to be quasi-ϕ-nonexpansive, if $F(T) \neq \Phi$, and

$$\phi(p,z_n) \leq \phi(p,x), \quad \forall x \in D, p \in F(T), z_n \in T(PT)^{n-1}x, \forall n \geq 1; \tag{1.7}$$

(2) A nonself multi-valued mapping $T : D \to X$ is said to be quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \Phi$ and there exists a real sequence $k_n \subset [1,+\infty)$, $k_n \to 1$ (as $n \to \infty$) such that

$$\phi(p,z_n) \leq k_n\phi(p,x), \quad \forall x \in D, p \in F(T), z_n \in T(PT)^{n-1}x, \forall n \geq 1; \tag{1.8}$$

(3) A nonself multi-valued mapping $T : D \to X$ is said to be totally quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \Phi$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$, with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that

$$\begin{aligned} &\phi(p,z_n) \leq \phi(p,x) + \nu_n\zeta\bigl[\phi(p,x)\bigr] + \mu_n, \\ &\quad \forall x \in D, p \in F(T), z_n \in T(PT)^{n-1}x, \forall n \geq 1. \end{aligned} \tag{1.9}$$

Remark 1.3 From the definitions, it is obvious that a quasi-ϕ-nonexpansive nonself multi-valued mapping is a quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping, and a quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping is a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping, but the converse is not true.

Now, we give an example of totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping.

Example 1.1 (see [4]) Let D be a unit ball in a real Hilbert space l^2 and let $T : D \to l^2$ be a nonself multi-valued mapping defined by

$$T : (x_1, x_2, \ldots) \to \bigl(0, x_1^2, a_2x_2, a_3x_3, \ldots\bigr) \in l^2, \quad \forall (x_1, x_2, \ldots) \in D,$$

where $\{a_i\}$ is a sequence in $(0,1)$ such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

It is proved in [5] that

(i) $\|Tx - Ty\| \leq 2\|x-y\|$, $\forall x, y \in D$;

(ii) $\|T^n x - T^n y\| \leq 2\prod_{j=2}^{n} a_j$, $\forall x, y \in D$, $n \geq 2$.

Let $\sqrt{k_1} = 2$, $\sqrt{k_n} = 2\prod_{j=2}^{n} a_j$, $n \geq 2$. Then $\lim_{n\to\infty} k_n = 1$. Letting $\nu_n = k_n - 1$ $(n \geq 2)$, $\zeta(t) = t$ $(t \geq 0)$ and $\{\mu_n\}$ be a nonnegative real sequence with $\mu_n \to 0$, then from (i) and (ii) we have

$$\|T^n x - T^n y\|^2 \leq \|x-y\|^2 + \nu_n \zeta\left(\|x-y\|^2\right) + \mu_n, \quad \forall x,y \in D.$$

Since D is a unit ball in a real Hilbert space l^2, it follows from Remark 1.2 that $\phi(x,y) = \|x-y\|^2$, $\forall x,y \in D$. The inequality above can be written as

$$\phi\left(T^n x, T^n y\right) \leq \phi(x,y) + \nu_n \zeta\left(\phi(x,y)\right) + \mu_n, \quad \forall x,y \in D.$$

Again since $0 \in D$ and $0 \in F(T)$, this implies that $F(T) \neq \Phi$. From the inequality above, we get

$$\phi(p,z_n) \leq \phi(p,x) + \nu_n \zeta\left(\phi(p,x)\right) + \mu_n, \quad \forall p \in F(T), x \in D, z_n \in T(PT)^{n-1}x,$$

where P is the nonexpansive retraction. This shows that the mapping T defined above is a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping.

Lemma 1.3 *Let X be a smooth, strictly convex and reflexive Banach space and D be a nonempty, closed, convex subset of X. Let $T: D \to X$ be a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping with $\mu_1 = 0$. Then $F(T)$ is a closed and convex subset of D.*

Proof Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to p$. Since T is a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping, we have

$$\phi(x_n,z) \leq \phi(x_n,p) + \nu_1 \zeta\left(\phi(x_n,p)\right), \quad z \in Tp, \forall n \in N.$$

Therefore,

$$\phi(p,z) = \lim_{n\to\infty} \phi(x_n,z) \leq \lim_{n\to\infty} \phi(x_n,p) + \nu_1 \zeta\left(\phi(x_n,p)\right) = \phi(p,p) = 0.$$

By Lemma 1.2, we obtain $p = z \in Tp$. So we have $p \in F(T)$. This implies that $F(T)$ is closed.

Let $p,q \in F(T)$ and $t \in (0,1)$, and put $w = tp + (1-t)q$. We prove that $w \in F(T)$. Indeed, in view of the definition of ϕ, let $\{u_n\}$ be a sequence generated by $u_1 \in Tw, u_2 \in T(PT)w, u_3 \in T(PT)^2 w, \ldots, u_n \in T(PT)^{n-1}w \subset TPu_{n-1}$, we have

$$\begin{aligned}\phi(w,u_n) &= \|w\|^2 - 2\langle w, Ju_n\rangle + \|u_n\|^2 \\ &= \|w\|^2 - 2\left\langle tp + (1-t)q, Ju_n\right\rangle + \|u_n\|^2 \\ &= \|w\|^2 + t\phi(p,u_n) + (1-t)\phi(q,u_n) - t\|p\|^2 - (1-t)\|q\|^2. \end{aligned} \tag{1.10}$$

Since

$$\begin{aligned}&t\phi(p,u_n) + (1-t)\phi(q,u_n) \\ &\quad \leq t\left[\phi(p,w) + \nu_n\zeta\left[\phi(p,w)\right] + \mu_n\right] + (1-t)\left[\phi(q,w) + \nu_n\zeta\left[\phi(q,w)\right] + \mu_n\right]\end{aligned}$$

$$\begin{aligned}
&= t\{\|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 + \nu_n\zeta[\phi(p,w)] + \mu_n\} \\
&\quad + (1-t)\{\|q\|^2 - 2\langle q, Jw\rangle + \|w\|^2 + \nu_n\zeta[\phi(q,w)] + \mu_n\} \\
&= t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2 + t\nu_n\zeta[\phi(p,w)] + (1-t)\nu_n\zeta[\phi(q,w)] + \mu_n. \qquad (1.11)
\end{aligned}$$

Substituting (1.11) into (1.10) and simplifying it, we have

$$\phi(w,u_n) \le t\nu_n\zeta[\phi(p,w)] + (1-t)\nu_n\zeta[\phi(q,w)] + \mu_n \to 0 \quad (\text{as } n \to \infty).$$

Hence, $u_n \to w$ holds, which yields $u_{n+1} \to w$. Since TP is closed and $u_{n+1} \in T(PT)^n w \subset TPu_n$, we have $w \in TPw$. It follows from $w \in D$ that $w \in Tw$, *i.e.*, $w \in F(T)$. This implies that $F(T)$ is convex. This completes the proof of Lemma 1.3. □

Definition 1.2 (see [1]) A nonself mapping $T: D \to X$ is said to be uniformly L-Lipschitz continuous, if there exists a constant $L > 0$, such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le L\|x-y\|, \quad \forall x, y \in D, \forall n \ge 1. \qquad (1.12)$$

Definition 1.3 A nonself multi-valued mapping $T: D \to X$ is said to be uniformly L-Lipschitz continuous, if there exists a constant $L > 0$, such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \le L\|x-y\|, \quad \forall x, y \in D, \forall n \ge 1, \qquad (1.13)$$

where $d(\cdot,\cdot)$ is Hausdorff metric.

Strong and weak convergence of asymptotically nonexpansive self or nonself mappings, relatively nonexpansive, quasi-ϕ-nonexpansive and quasi-ϕ-asymptotically nonexpansive self or nonself mappings have been considered extensively by several authors in the setting of Hilbert or Banach spaces (see [1–4, 6–24]). In recent years, by hybrid projection methods, strong and weak convergence problems for totally quasi-ϕ quasi-ϕ-asymptotically nonexpansive nonself and multi-valued mapping, respectively, was also studied by Kim *et al.* (see [6, 7]), Li *et al.* (see [8]), Chang *et al.* (see [9]) and Yang *et al.* (see [10]).

Inspired by specialists above, the purpose of this paper is to modify the Halpern-Mann's mixed type iteration algorithm for a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping, which has the strong convergence under a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper improve and extend the corresponding results of Chang *et al.* [1, 11–13], Hao *et al.* [14], Guo *et al.* [15], Yildirim *et al.* [16], Thianwan [17], Nilsrakoo *et al.* [18], Pathak *et al.* [19], Qin *et al.* [20], Su *et al.* [21], Wang [22, 23], Yang *et al.* [24] and others.

2 Main results

Theorem 2.1 *Let X be a real uniformly smooth and uniformly convex Banach space, D be a nonempty, closed, convex subset of X. Let $P: X \to D$ be the nonexpansive retraction. Let*

$T : D \to X$ *be a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping with sequence* $\{\nu_n\}$, $\{\mu_n\}$ ($\mu_1 = 0$), *with* $\nu_n, \mu_n \to 0$ (*as* $n \to \infty$) *and a strictly increasing continuous function* $\zeta : R^+ \to R^+$ *with* $\zeta(0) = 0$ *such that* T *is uniformly L-Lipschitz continuous. Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *and* $\{\beta_n\}$ *be a sequence in* $(0,1)$ *satisfying the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \lim_{n\to\infty} \inf \beta_n \le \lim_{n\to\infty} \sup \beta_n < 1$.

Let $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 \in X \quad \text{is arbitrary}; \qquad D_1 = D, \\ y_n = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)Jz_n)], \quad z_n \in T(PT)^{n-1}x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{2.1}$$

where $\xi_n = \nu_n \sup_{p\in F(T)} \zeta(\phi(p, x_n)) + \mu_n$, $\Pi_{D_{n+1}}$ *is the generalized projection of* X *onto* D_{n+1}. *If* $F(T) \neq \emptyset$, *then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)} x_1$.

Proof (I) *First, we prove that* D_n *are closed and convex subsets in* D.

In fact, by Lemma 1.3, $F(T)$ is closed and convex in D. By the assumption, $D_1 = D$ is closed and convex. Suppose that D_n is closed and convex for some $n \ge 1$. In view of the definition of ϕ, we have

$$\begin{aligned} D_{n+1} &= \bigl\{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\bigr\} \\ &= \bigl\{z \in D : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\bigr\} \cap D_n \\ &= \bigl\{z \in D : 2\alpha_n \langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_n\rangle \\ &\qquad \le \alpha_n \|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2\bigr\} \cap D_n. \end{aligned}$$

This shows that D_{n+1} is closed and convex. The conclusions are proved.

(II) *Next, we prove that* $F(T) \subset D_n$, *for all* $n \ge 1$.

It is obvious that $F(T) \subset D_1$. Suppose that $F(T) \subset D_n$, $w_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n)$ and $z_n \in T(PT)^{n-1}x_n$. Hence, for any $u \in F(T) \subset D_n$, by (1.5), we have

$$\begin{aligned} \phi(u, y_n) &= \phi\bigl(u, J^{-1}\bigl(\alpha_n Jx_1 + (1-\alpha_n)Jw_n\bigr)\bigr) \\ &\le \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, w_n), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \phi(u, w_n) &= \phi\bigl(u, J^{-1}\bigl(\beta_n Jx_n + (1-\beta_n)Jz_n\bigr)\bigr) \\ &\le \beta_n \phi(u, x_n) + (1-\beta_n)\phi(u, z_n) \\ &\le \beta_n \phi(u, x_n) + (1-\beta_n)\bigl\{\phi(u, x_n) + \nu_n \zeta\bigl[\phi(u, x_n)\bigr] + \mu_n\bigr\} \\ &= \phi(u, x_n) + (1-\beta_n)\nu_n \zeta\bigl[\phi(u, x_n)\bigr] + (1-\beta_n)\mu_n. \end{aligned} \tag{2.3}$$

Therefore, we have

$$\begin{aligned}\phi(u,y_n) &\le \alpha_n\phi(u,x_1) + (1-\alpha_n)\bigl[\phi(u,x_n) + (1-\beta_n)\nu_n\zeta\bigl[\phi(u,x_n)\bigr] + (1-\beta_n)\mu_n\bigr] \\ &\le \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,x_n) + \nu_n \sup_{p\in F(T)} \zeta\bigl[\phi(p,x_n)\bigr] \\ &= \alpha_n\phi(u,x_1) + (1-\alpha_n)\phi(u,x_n) + \xi_n,\end{aligned} \tag{2.4}$$

where $\xi_n = \nu_n \sup_{p\in F(T)} \zeta(\phi(p,x_n)) + \mu_n$. This shows that $u \in D_{n+1}$ and so $F(T) \subset D_n$. The conclusion is proved.

(III) *Now we prove that* $\{x_n\}$ *converges strongly to some point* p^*.

Since $x_n = \Pi_{D_n}x_1$, from Lemma 1.2(c), we have

$$\langle x_n - y, Jx_1 - Jx_n\rangle \ge 0, \quad \forall y \in D_n.$$

Again since $F(T) \subset D_n$, we have

$$\langle x_n - u, Jx_1 - Jx_n\rangle \ge 0, \quad \forall u \in F(T).$$

It follows from Lemma 1.2(b) that, for each $u \in F(T)$ and for each $n \ge 1$,

$$\phi(x_n,x_1) = \phi(\Pi_{D_n}x_1,x_1) \le \phi(u,x_1) - \phi(u,x_n) \le \phi(u,x_1). \tag{2.5}$$

Therefore, $\{\phi(x_n,x_1)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = \Pi_{D_n}x_1$ and $x_{n+1} = \Pi_{D_{n+1}}x_1 \in D_{n+1} \subset D_n$, we have $\phi(x_n,x_1) \le \phi(x_{n+1},x_1)$. This implies that $\{\phi(x_n,x_1)\}$ is nondecreasing. Hence $\lim_{n\to\infty}\phi(x_n,x_1)$ exists.

By the construction of $\{D_n\}$, for any $m \ge n$, we have $D_m \subset D_n$ and $x_m = \Pi_{D_m}x_1 \in D_n$. This shows that

$$\phi(x_m,x_n) = \phi(x_m,\Pi_{D_n}x_1) \le \phi(x_m,x_1) - \phi(x_n,x_1) \to 0 \quad (\text{as } n\to\infty).$$

It follows from Lemma 1.1 that $\lim_{n\to\infty}\|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in D. Since D is complete, without loss of generality, we can assume that $\lim_{n\to\infty} x_n = p^*$ (some point in D).

By the assumption, it is easy to see that

$$\lim_{n\to\infty}\xi_n = \lim_{n\to\infty}\Bigl[\nu_n \sup_{p\in F(T)} \zeta\bigl(\phi(p,x_n)\bigr) + \mu_n\Bigr] = 0. \tag{2.6}$$

(IV) *Now we prove that* $p^* \in F(T)$.

Since $x_{n+1} \in D_{n+1}$, from (2.1) and (2.6), we have

$$\phi(x_{n+1},y_n) \le \alpha_n\phi(x_{n+1},x_1) + (1-\alpha_n)\phi(x_{n+1},x_n) + \xi_n \to 0. \tag{2.7}$$

Since $x_n \to p^*$, it follows from (2.7) and Lemma 1.1 that

$$y_n \to p^*. \tag{2.8}$$

Since $\{x_n\}$ is bounded and T is a totally quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping, we have

$$\phi(p,z_n) \le \phi(p,x_n) + \nu_n\zeta\big[\phi(p,x_n)\big] + \mu_n, \quad \forall x \in D, \forall n, i \ge 1, p \in F(T).$$

This implies that $\{z_n\}$ is also bounded.

By condition (ii), we have

$$\begin{aligned} \|w_n\| &= \big\|J^{-1}\big(\beta_n Jx_n + (1-\beta_n)Jz_n\big)\big\| \\ &\le \beta_n\|x_n\| + (1-\beta_n)\|z_n\| \\ &\le \|x_n\| + \|z_n\|, \end{aligned}$$

this implies that $\{w_n\}$ is also bounded.

In view of $\alpha_n \to 0$, from (2.1), we have

$$\lim_{n\to\infty}\|Jy_n - Jw_n\| = \lim_{n\to\infty}\alpha_n\|Jx_1 - Jw_n\| = 0. \tag{2.9}$$

Since J^{-1} is uniformly continuous on each bounded subset of X^*, it follows from (2.8) and (2.9) that

$$w_n \to p^*. \tag{2.10}$$

Since J is uniformly continuous on each bounded subset of X, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty}\big\|Jw_n - Jp^*\big\| \\ &= \lim_{n\to\infty}\big\|\big(\beta_n Jx_n + (1-\beta_n)Jz_n\big) - Jp^*\big\| \\ &= \lim_{n\to\infty}\big\|\beta_n(Jx_n - Jp^*) + (1-\beta_n)\big(Jz_n - Jp^*\big)\big\| \\ &= \lim_{n\to\infty}(1-\beta_n)\big\|Jz_n - Jp^*\big\|. \end{aligned} \tag{2.11}$$

By condition (ii), we have

$$\lim_{n\to\infty}\big\|Jz_n - Jp^*\big\| = 0.$$

Since J is uniformly continuous, this shows that

$$\lim_{n\to\infty} z_n = p^*. \tag{2.12}$$

Again by the assumptions that $T : D \to X$ be uniformly L-Lipschitz continuous, thus we have

$$\begin{aligned} & d\big(T(PT)^n x_n, T(PT)^{n-1}x_n\big) \\ &\quad \le d\big(T(PT)^n x_n, T(PT)^n x_{n+1}\big) + d\big(T(PT)^n x_{n+1}, x_{n+1}\big) \\ &\qquad + \|x_{n+1} - x_n\| + d\big(x_n, T(PT)^{n-1}x_n\big) \\ &\quad \le (L+1)\|x_{n+1} - x_n\| + d\big(T(PT)^n x_{n+1}, x_{n+1}\big) + d\big(x_n, T(PT)^{n-1}x_n\big). \end{aligned} \tag{2.13}$$

We get $\lim_{n\to\infty} d(T(PT)^n x_n, T(PT)^{n-1}x_n) = 0$, since $\lim_{n\to\infty} z_n = p^*$ and $\lim_{n\to\infty} x_n = p^*$.

In view of the continuity of TP, it yields $p^* \in TPp^*$. We have $p^* \in C$, which implies that $p^* \in Tp^*$. We have $p^* \in F(T)$.

(V) *Finally, we prove that* $p^* = \Pi_{F(T)}x_1$ *and so* $x_n \to \Pi_{F(T)}x_1 = p^*$.

Let $w = \Pi_{F(T)}x_1$. Since $w \in F(T) \subset D_n$ and $x_n = \Pi_{D_n}x_1$, we have $\phi(x_n, x_1) \le \phi(w, x_1)$. This implies that

$$\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \le \phi(w, x_1), \tag{2.14}$$

which yields $p^* = w = \Pi_{F(T)}x_1$. Therefore, $x_n \to \Pi_{F(T)}x_1$. The proof of Theorem 3.1 is completed. □

By Remark 1.3, the following corollary is obtained.

Corollary 2.1 *Let X, D, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem* 2.1. *Let $T : D \to X$ be a quasi-ϕ-asymptotically nonexpansive nonself multi-valued mapping with sequence $k_n \subset [1, +\infty)$, $k_n \to 1$, $T : D \to X$ be uniformly L-Lipschitz continuous.*

Suppose $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in X \quad \textit{is arbitrary}; \qquad D_1 = D, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n)(\beta_n J x_n + (1-\beta_n) J z_n)] \quad (i \ge 1), z_n \in T(PT)^{n-1}x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{2.15}$$

where $\xi_n = (k_n - 1)\sup_{p\in F(T)} \phi(p, x_n)$, $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}. If $F(T) \ne \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

Corollary 2.2 *Let X, D, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem* 2.1. *Let $T : D \to X$ be a quasi-ϕ-nonexpansive nonself multi-valued mapping, $T : D \to X$ be uniformly L-Lipschitz continuous.*

Suppose $\{x_n\}$ is a sequence generated by

$$\begin{cases} x_1 \in X \quad \textit{is arbitrary}; \qquad D_1 = D, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n)(\beta_n J x_n + (1-\beta_n) J z_n)] \quad (i \ge 1), z_n \in T(PT)^{n-1}x_n, \\ D_{n+1} = \{z \in D_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{D_{n+1}} x_1 \quad (n = 1, 2, \ldots), \end{cases} \tag{2.16}$$

where $\xi_n = (k_n - 1)\sup_{p\in F(T)} \phi(p, x_n)$, $\Pi_{D_{n+1}}$ is the generalized projection of X onto D_{n+1}. If $F(T) \ne \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_1$.

3 Application

First, we present an example of a quasi-ϕ-nonexpansive nonself multi-valued mapping.

Example 3.1 (see [4]) Let H be a real Hilbert space, D be a nonempty closed and convex subset of H and $f : D \times D \to R$ be a bifunction satisfying the conditions: (A1) $f(x, x) = 0$,

$\forall x \in D$; (A2) $f(x,y)+f(y,x) \leq 0$, $\forall x,y \in D$; (A3) for each $x,y,z \in D$, $\lim_{t\to 0} f(tz+(1-t)x,y) \leq f(x,y)$; (A4) for each given $x \in D$, the function $y \longmapsto f(x,y)$ is convex and lower semicontinuous. The 'so-called' equilibrium problem for f is to find a $x^* \in D$ such that $f(x^*,y) \geq 0$, $\forall y \in D$. The set of its solutions is denoted by $\mathrm{EP}(f)$.

Let $r > 0$, $x \in H$ and define a mapping $T_r : D \to D \subset H$ as follows:

$$T_r(x) = \left\{ z \in D, f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in D \subset H. \tag{3.1}$$

Then (1) T_r is single-valued, so $z = T_r(x)$; (2) T_r is a relatively nonexpansive nonself mapping, therefore it is a closed quasi-ϕ-nonexpansive nonself mapping; (3) $F(T_r) = \mathrm{EP}(f)$ and $F(T_r)$ is a nonempty and closed convex subset of D; (4) $T_r : D \to D$ is a nonexpansive. Since $F(T_r)$ nonempty, it is a quasi-ϕ-nonexpansive nonself mapping from D to H, where $\phi(x,y) = \|x-y\|^2$, $x,y \in H$.

In this section we utilize Corollary 2.1 to study a modified Halpern iterative algorithm for a system of equilibrium problems. We have the following result.

Theorem 3.1 *Let H be a real Hilbert space, D be a nonempty closed and convex subset of H, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem* 2.1. *Let $f : D \times D \to R$ be a bifunction satisfying conditions* (A1)-(A4) *as given in Example* 3.1. *Let $T_r : D \to D \subset H$ be mapping defined by* (3.1), *i.e.*,

$$T_r(x) = \left\{ z \in D, f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in D \right\}, \quad \forall x \in D \subset H.$$

Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_1 \in D \quad \textit{is arbitrary}; \qquad D_1 = D, \\ f(u_n,y) + \frac{1}{r}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in D, r > 0, \\ y_n = \alpha_n x_1 + (1-\alpha_n)[\beta_n x_n + (1-\beta_n)u_n], \\ D_{n+1} = \{z \in D_n : \|z-y_{n,i}\|^2 \leq \alpha_n\|z-x_1\|^2 + (1-\alpha_n)\|z-x_n\|^2\}, \\ x_{n+1} = \Pi_{D_{n+1}}x_1 \quad (n = 1,2,\ldots). \end{cases} \tag{3.2}$$

If $F(T_r) \neq \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{F(T_r)}x_1$, which is a common solution of the system of equilibrium problems for f.

Proof In Example 3.1, we have pointed out that $u_n = T_r(x_n)$, $F(T_r) = \mathrm{EP}(f)$ is nonempty and convex, T_r is a quasi-ϕ-nonexpansive nonself mapping. Since $F(T_r)$ is nonempty, and so T_r is a quasi-ϕ-nonexpansive mapping and T_r is uniformly 1-Lipschitzian mapping. Hence (3.1) can be rewritten as follows:

$$\begin{cases} x_1 \in H \quad \text{is arbitrary}; \qquad D_1 = D, \\ y_n = \alpha_n x_1 + (1-\alpha_n)[\beta_n x_n + (1-\beta_n)z_n], \quad z_n \in T_r x_n, \\ D_{n+1} = \{z \in D_n : \|z-y_n\|^2 \leq \alpha_n\|z-x_1\|^2 + (1-\alpha_n)\|z-x_n\|^2\}, \\ x_{n+1} = \Pi_{D_{n+1}}x_1 \quad (n = 1,2,\ldots). \end{cases} \tag{3.3}$$

Therefore, the conclusion of Theorem 3.1 can be obtained from Corollary 2.1. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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