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Strong convergence theorems for solutions of fixed point and variational inequality problems

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Abstract
The purpose of this paper is to investigate viscosity approximation methods for finding a common element in the set of fixed points of a strict pseudocontraction and in the set of solutions of a generalized variational inequality in the framework of Banach spaces.

Keywords: sunny nonexpansive retraction; inverse-strongly accretive mapping; nonexpansive mapping; variational inequality

1 Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let P_C be the metric projection of H onto C. Recall that a mapping $A : C \to H$ is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0 \quad \forall x, y \in C.$$

Recall that a mapping $A : C \to H$ is said to be inverse-strongly monotone iff there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2 \quad \forall x, y \in C.$$

For such a case, A is said to be α-inverse-strongly monotone.

Recall that the classical variational inequality problem, denoted by $\mathrm{VI}(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \quad \forall v \in C. \tag{1.1}$$

It is clear that variational inequality problem (1.1) is equivalent to a fixed point problem. u is a solution of the above inequality iff it is a fixed point of the mapping $P_C(I - rA)$, where I is the identity and r is some positive real number.

Variational inequality problems have emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, and network. Recently, many authors studied the solutions of inequality (1.1) based on iterative methods; see [1–17] and the references therein.

Let $S: C \to C$ be a mapping. In this paper, we denote by $F(S)$ the set of fixed points of the mapping S.

Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$

Recall that S is said to be a strict pseudocontraction iff there exits a positive constant λ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \lambda \big\|(I - S)x - (I - S)y\big\|^2 \quad \forall x, y \in C.$$

It is clear that the class of strict pseudocontractions includes the class of nonexpansive mappings as a special case.

Recently, many authors have investigated the problems of finding a common element in the set of solution of variational inequalities for an inverse-strongly monotone mapping and in the set of fixed points of nonexpansive mappings or strict pseudocontractions; see [18–25] and the references therein. However, most of the results are in the framework of Hilbert spaces. In this paper, we investigate a common element problem in the framework of Banach spaces. A strong convergence theorem for common solutions to fixed point problems of strict pseudocontractions and solution problems of variational inequality (1.1) is established in uniformly convex and 2-uniformly smooth Banach spaces. The results presented in this paper improve and extend the corresponding results announced by Iiduka and Takahashi [5] and Hao [26].

2 Preliminaries

Let C be a nonempty closed and convex subset of a Banach space E. Let E^* be the dual space of E, and let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q: E \to 2^{E^*}$ is defined by

$$J_q(x) = \big\{ f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \big\}$$

for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2} J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$, the identity mapping. The normalized duality mapping J has the following properties:

(1) if E is smooth, then J is single-valued;
(2) if E is strictly convex, then it is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;
(3) if E is reflexive, then J is surjective;
(4) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Hilbert spaces are 2-uniformly convex, while L^p is $\max\{p, 2\}$-uniformly convex for every $p > 1$.

A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. The norm of E is said to be Fréchet differentiable if, for any $x \in U$, the limit (2.1) is attained uniformly for all $y \in U$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}\bigl(\|x+y\| + \|x-y\|\bigr) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\right\},$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0}\frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

We remark that all Hilbert spaces, L_p (or l_p) spaces ($p \geq 2$) and the Sobolev spaces W_m^p ($p \geq 2$) are 2-uniformly smooth, while L_p (or l_p) and W_m^p spaces ($1 < p \leq 2$) are p-uniformly smooth. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Recall that a mapping S is said to be λ-strictly pseudocontractive iff there exist a constant $\lambda \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\bigl\langle Sx - Sy, j(x-y)\bigr\rangle \leq \|x-y\|^2 - \lambda\bigl\|(I-S)x-(I-S)y\bigr\|^2 \quad \forall x, y \in C. \tag{2.2}$$

It is clear that (2.2) is equivalent to the following:

$$\bigl\langle (I-S)x-(I-S)y, j(x-y)\bigr\rangle \geq \lambda\bigl\|(I-S)x-(I-S)y\bigr\|^2 \quad \forall x, y \in C. \tag{2.3}$$

Next, we assume that E is a smooth Banach space. Let C be a nonempty closed convex subset of E. Recall that an operator A of C into E is said to be accretive iff

$$\bigl\langle Ax - Ay, J(x-y)\bigr\rangle \geq 0 \quad \forall x, y \in C.$$

An accretive operator A is said to be m-accretive if the range of $I + rA$ is E for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone.

Recall that an operator A of C into E is said to be α-inverse strongly accretive iff there exits a real constant $\alpha > 0$ such that

$$\bigl\langle Ax - Ay, J(x-y)\bigr\rangle \geq \alpha\|Ax - Ay\|^2 \quad \forall x, y \in C.$$

Evidently, the definition of an inverse-strongly accretive operator is based on that of an inverse-strongly monotone operator.

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if

$$Q\bigl(Qx + t(x - Qx)\bigr) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 2.1 [27] *Let E be a smooth Banach space, and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction, and let J be the normalized duality mapping on E. Then the following are equivalent*:
(1) Q_C *is sunny and nonexpansive*;
(2) $\|Q_C x - Q_C y\|^2 \leq \langle x - y, J(Q_C x - Q_C y)\rangle$ $\forall x, y \in E$;
(3) $\langle x - Q_C x, J(y - Q_C x)\rangle \leq 0$ $\forall x \in E, y \in C$.

Proposition 2.2 [28] *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Recently, Aoyama *et al.* [29] considered the following generalized variational inequality problem.

Let C be a nonempty closed convex subset of E, and let A be an accretive operator of C into E. Find a point $u \in C$ such that

$$\langle Au, J(v - u)\rangle \geq 0 \quad \forall v \in C. \tag{2.4}$$

Next, we use $\mathrm{BVI}(C, A)$ to denote the set of solutions of variational inequality problem (2.4).

Aoyama *et al.* [29] proved that variational inequality (2.4) is equivalent to a fixed point problem. The element $u \in C$ is a solution of variational inequality (2.4) iff $u \in C$ is a fixed point of the mapping $Q_C(I - rA)$, where $r > 0$ is a constant and Q_C is a sunny nonexpansive retraction from E onto C.

The following lemmas also play an important role in this paper.

Lemma 2.3 [30] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n + e_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{e_n\}$ and $\{\delta_n\}$ are sequences such that
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\sum_{n=1}^{\infty} e_n < \infty$;
(3) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.4 [31] *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2 \quad \forall x, y \in E.$$

Lemma 2.5 [29] *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$\mathrm{BVI}(C,A) = F\big(Q_C(I - \lambda A)\big).$$

Lemma 2.6 [32] *Let C be a closed convex subset of a real strictly convex Banach space E and $S_i : C \to C$ $(i = 1,2)$ be two nonexpansive mappings such that $F = F(S_1) \cap F(S_2) \neq \emptyset$. Define $Sx = \delta S_1 x + (1 - \delta)S_2 x$, where $\delta \in (0,1)$. Then $S : C \to C$ is a nonexpansive mapping with $F(S) = F \neq \emptyset$.*

Lemma 2.7 [33] *Let C be a nonempty subset of a real 2-uniformly smooth Banach space E, and let $T : C \to C$ be a κ-strict pseudocontraction. For $\alpha \in (0,1)$, we define $T_\alpha x = (1 - \alpha)x + \alpha Tx$ for every $x \in C$. Then, as $\alpha \in (0, \frac{\kappa}{K^2}]$, T_α is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 2.8 [34] *Let E be a real uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with a fixed point, and let $f : C \to C$ be a contraction. For each $t \in (0,1)$, let z_t be the unique solution of the equation $x = tf(x) + (1-t)Tx$. Then $\{z_t\}$ converges to a fixed point of T as $t \to 0$ and $Q(f) = s\text{-}\lim_{t\to 0} z_t$ defines the unique sunny nonexpansive retraction from C onto $F(T)$.*

3 Main results

Theorem 3.1 *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, and let C be a nonempty, closed and convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let $A : C \to E$ be an α-inverse strongly accretive mapping. Let $S : C \to C$ be a λ-strict pseudocontraction with a fixed point. Assume that $F := F(S) \cap \mathrm{BVI}(C,A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Suppose that $x_1 = x \in C$ and that $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n\big[\mu S_t x_n + (1 - \mu)Q_C(x_n - \lambda A x_n)\big] + \gamma_n Q_C e_n, \quad n \geq 1,$$

where $S_t = (1-t)x + tSx$, $t \in (0, \frac{\lambda}{K^2}]$, $f : C \to C$ is a κ-contractive mapping, $\{e_n\}$ is a bounded computational error in E, $\lambda \in (0, \alpha/K^2]$ and $\mu \in (0,1)$. Assume that the following restrictions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(b) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\{x_n\}$ converges strongly to $x = Q_F f(x)$, where Q_F is a sunny nonexpansive retraction from C onto F.

Proof Fixing $x^* \in F$, we find that $x^* = Q_C(x^* - \lambda A x^*)$ and $Sx^* = x^*$. It follows from Lemma 2.7 that $S_t x^* = x^*$. Put $y_n = Q_C(x_n - \lambda A x_n)$. In view of Lemma 2.4, we find that

$$\begin{aligned}\|y_n - x^*\|^2 &\leq \big\|(x_n - x^*) - \lambda\big(Ax_n - Ax^*\big)\big\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\lambda\big\langle Ax_n - Ax^*, J\big(x_n - x^*\big)\big\rangle \\ &\quad + 2K^2\lambda^2\|Ax_n - Ax^*\|^2\end{aligned}$$

$$\begin{aligned}
&\le \|x_n - x^*\|^2 - 2\lambda\alpha\|Ax_n - Ax^*\|^2 + 2K^2\lambda^2\|Ax_n - Ax^*\|^2 \\
&= \|x_n - x^*\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax_n - Ax^*\|^2.
\end{aligned}$$

Since $\lambda \in (0, \alpha/K^2]$, we have that

$$\|y_n - x^*\| \le \|x_n - x^*\|.$$

This implies that $Q_C(I - \lambda A)$ is a nonexpansive mapping. Hence, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\quad = \|\alpha_n f(x_n) + \beta_n[\mu S x_n + (1-\mu)Q_C(x_n - \lambda A x_n)] + \gamma_n Q_C e_n - x^*\| \\
&\quad \le \alpha_n\|f(x_n) - x^*\| + \beta_n\|\mu S x_n + (1-\mu)Q_C(x_n - \lambda A x_n) - x^*\| \\
&\qquad + \gamma_n\|Q_C e_n - x^*\| \\
&\quad \le \alpha_n\|f(x_n) - x^*\| + \beta_n\mu\|x_n - x^*\| + \beta_n(1-\mu)\|x_n - x^*\| \\
&\qquad + \gamma_n\|Q_C e_n - x^*\| \\
&\quad \le \alpha_n\kappa\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + \beta_n\|x_n - x^*\| \\
&\qquad + \gamma_n\|e_n - x^*\| \\
&\quad \le (1 - \alpha_n(1-\kappa))\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + \gamma_n\|e_n - x^*\| \\
&\quad \le \max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\kappa}\right\} + \gamma_n\|e_n - x^*\|,
\end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Define

$$t_n = \mu S_t x_n + (1-\mu)Q_C(x_n - \lambda A x_n).$$

It follows that

$$\begin{aligned}
&\|t_n - t_{n-1}\| \\
&\quad = \|\mu S_t x_n + (1-\mu)Q_C(I - \lambda A)x_n - [\mu S_t x_{n-1} + (1-\mu)Q_C(I - \lambda A)x_{n-1}]\| \\
&\quad \le \mu\|S_t x_n - S_t x_{n-1}\| + (1-\mu)\|Q_C(I - \lambda A)x_n - Q_C(I - \lambda A)x_{n-1}\| \\
&\quad \le \mu\|x_n - x_{n-1}\| + (1-\mu)\|x_n - x_{n-1}\| \\
&\quad = \|x_n - x_{n-1}\|. \qquad (3.1)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\qquad + \beta_n\|t_n - t_{n-1}\| + |\beta_n - \beta_{n-1}|\|t_{n-1}\| + \gamma_n\|Q_C e_n - Q_C e_{n-1}\| \\
&\qquad + |\gamma_n - \gamma_{n-1}|\|Q_C e_n\|. \qquad (3.2)
\end{aligned}$$

Substituting (3.1) into (3.2), we see that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \bigl(1 - \alpha_n(1-\kappa)\bigr)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\bigl\|f(x_{n-1})\bigr\| \\
&\qquad + |\beta_n - \beta_{n-1}|\|t_{n-1}\| + \gamma_n\|Q_C e_n - Q_C e_{n-1}\| \\
&\qquad + |\gamma_n - \gamma_{n-1}|\|Q_C e_n\| \\
&\quad \le \bigl(1 - \alpha_n(1-\kappa)\bigr)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\bigl(\bigl\|f(x_{n-1})\bigr\| + \|t_{n-1}\|\bigr) \\
&\qquad + |\gamma_n - \gamma_{n-1}|\bigl(\|t_{n-1}\| + \|Q_C e_n\|\bigr) + \gamma_n\|Q_C e_n - Q_C e_{n-1}\|.
\end{aligned}$$

In view of Lemma 2.3, we find from the restrictions (a) and (b) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.3}$$

Note that

$$\begin{aligned}
\|x_n - t_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - t_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\bigl\|f(x_n) - t_n\bigr\| + \gamma_n\|Q_C e_n - t_n\|.
\end{aligned}$$

Using (3.3), we find from the restrictions (a) and (b) that

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.4}$$

Define a mapping V by

$$Vx = \mu S_t x + (1-\mu)Q_C(I - \lambda A)x \quad \forall x \in C.$$

Using Lemma 2.6, we see that the mapping V is a nonexpansive mapping with

$$F(V) = F(S_t) \cap F\bigl(Q_C(I - \lambda A)\bigr) = F(S_t) \cap \mathrm{BVI}(C,A) = F(S) \cap \mathrm{BVI}(C,A) = F.$$

From (3.4), we see that

$$\lim_{n\to\infty}\|x_n - Vx_n\| = 0. \tag{3.5}$$

Next, we show that

$$\limsup_{n\to\infty}\bigl\langle f(x) - x, J(x_n - x)\bigr\rangle \le 0, \tag{3.6}$$

where $x = Q_F f(x)$, and Q_F is a sunny nonexpansive retraction from C onto F, the strong limit of the sequence z_t defined by

$$z_t = tf(z_t) + (1-t)Vz_t.$$

It follows that

$$\|z_t - x_n\| = \big\|(1-t)(Vz_t - x_n) + t\big(f(z_t) - x_n\big)\big\|.$$

For any $t \in (0,1)$, we see that

$$\begin{aligned}
\|z_t - x_n\|^2 &\le (1-t)^2\|Vz_t - x_n\|^2 + 2t\big\langle f(z_t) - x_n, j(z_t - x_n)\big\rangle \\
&\le (1-t)^2\big(\|Vz_t - Vx_n\|^2 + \|Vx_n - x_n\|^2 \\
&\quad + 2\|Vz_t - Vx_n\|\|Vx_n - x_n\|\big) + 2t\big\langle f(z_t) - z_t, j(z_t - x_n)\big\rangle \\
&\quad + 2t\big\langle z_t - x_n, j(z_t - x_n)\big\rangle \\
&\le (1-t)^2\|z_t - x_n\|^2 + \lambda_n(t) + 2t\big\langle f(z_t) - z_t, j(z_t - x_n)\big\rangle \\
&\quad + 2t\|z_t - x_n\|^2,
\end{aligned} \tag{3.7}$$

where

$$\lambda_n(t) = \|Vx_n - x_n\|^2 + 2\|z_t - x_n\|\|Vx_n - x_n\|.$$

It follows from (3.7) that

$$\big\langle z_t - f(z_t), j(z_t - x_n)\big\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}\lambda_n(t).$$

This implies that

$$\limsup_{n\to\infty}\big\langle z_t - f(z_t), j(z_t - x_n)\big\rangle \le \frac{t}{2}\|z_t - x_n\|^2.$$

Since E is 2-uniformly smooth, $j : E \to E^*$ is uniformly continuous on any bounded sets of E, which ensures that the $\limsup_{n\to\infty}$ and $\limsup_{t\to 0}$ are interchangeable, hence

$$\limsup_{n\to\infty}\big\langle f(x) - x, j(x_n - x)\big\rangle \le 0.$$

This shows that (3.6) holds.

Finally, we show that $x_n \to x$ as $n \to \infty$. Note that

$$\begin{aligned}
&\|x_{n+1} - x\|^2 \\
&\quad = \alpha_n\big\langle f(x_n) - x, j(x_{n+1} - x)\big\rangle + \beta_n\big\langle t_n - x, j(x_{n+1} - x)\big\rangle \\
&\qquad + \gamma_n\big\langle Q_C e_n - x, j(x_{n+1} - x)\big\rangle \\
&\quad \le \alpha_n\big\langle f(x_n) - x, j(x_{n+1} - q)\big\rangle + \beta_n\|t_n - x\|\|x_{n+1} - x\| \\
&\qquad + \gamma_n\|Q_C e_n - x\|\|x_{n+1} - x\| \\
&\quad \le \alpha_n\kappa\|x_n - x\|\|x_{n+1} - q\| + \alpha_n\big\langle f(x) - x, j(x_{n+1} - q)\big\rangle \\
&\qquad + \beta_n\|x_n - x\|\|x_{n+1} - x\| + \gamma_n\|Q_C e_n - x\|\|x_{n+1} - x\| \\
&\quad \le \frac{\alpha_n\kappa + \beta_n}{2}\big(\|x_n - x\|^2 + \|x_{n+1} - x\|^2\big) + \alpha_n\big\langle f(x) - x, j(x_{n+1} - x)\big\rangle
\end{aligned}$$

$$\begin{aligned}
&+ \frac{\gamma_n}{2}\left(\|e_n - x\|^2 + \|x_{n+1} - x\|^2\right) \\
&\leq \frac{\alpha_n\kappa + \beta_n}{2}\|x_n - x\|^2 + \frac{1-\alpha_n(1-\kappa)}{2}\|x_{n+1} - x\|^2 \\
&+ \alpha_n\langle f(x) - x, j(x_{n+1} - x)\rangle + \frac{\gamma_n}{2}\|e_n - x\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x\|^2 \leq{}& \left(1 - \alpha_n(1-\kappa)\right)\|x_n - x\|^2 \\
&+ 2\alpha_n\langle f(x) - x, j(x_{n+1} - x)\rangle + \gamma_n\|e_n - x\|^2.
\end{aligned}$$

Using Lemma 2.3, we find from the restrictions (a) and (b) that

$$\lim_{n\to\infty}\|x_n - x\| = 0.$$

This completes the proof. □

Remark 3.2 The framework of the space in Theorem 3.1 can be applicable to L^p, $p \geq 2$.

4 Applications

In this section, we always assume that E is a uniformly convex and 2-uniformly smooth Banach space. Let C be a nonempty, closed and convex subset of E.

First, we consider common fixed points of two strict pseudocontractions.

Theorem 4.1 *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let $T : C \to C$ be an α-strict pseudocontraction. Let $S : C \to C$ be a λ-strict pseudocontraction. Assume that $F := F(S) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$. Suppose that $x_1 = x \in C$ and that $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\left[\mu S_t x_n + (1-\mu)\left((1-\alpha)x_n + \alpha T x_n\right)\right], \quad n \geq 1,$$

where $S_t = (1-t)x + tSx$, $t \in (0, \frac{\lambda}{K^2}]$, $f : C \to C$ is a κ-contractive mapping, $\{e_n\}$ is a bounded computational error in E, $\lambda \in (0, \alpha/K^2]$ and $\mu \in (0,1)$. Assume that the following restrictions are satisfied:

(a) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;
(b) $\sum_{n=1}^{\infty}\gamma_n < \infty$.

Then $\{x_n\}$ converges strongly to $x = Q_F f(x)$, where Q_F is a sunny nonexpansive retraction from C onto F.

Proof Since $(I - T)$ is an α-inverse strongly accretive mapping, we find from Theorem 3.1 the desired conclusion. □

Closely related to the class of pseudocontractive mappings is the class of accretive mappings. Recall that an operator B with domain $D(B)$ and range $R(B)$ in E is accretive if for each $x_i \in D(B)$ and $y_i \in Bx_i$ $(i = 1, 2)$,

$$\langle y_2 - y_1, J(x_2 - x_1)\rangle \geq 0.$$

An accretive operator B is m-accretive if $R(I + rB) = E$ for each $r > 0$. Next, we assume that B is m-accretive and has a zero (*i.e.*, the inclusion $0 \in B(z)$ is solvable). The set of zeros of B is denoted by Ω. Hence,

$$\Omega = \{z \in D(B) : 0 \in B(z)\} = B^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of B, *i.e.*, $J_r = (I + rB)^{-1}$. Note that if B is m-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = \Omega$ for all $r > 0$.

From the above, we have the following theorem.

Theorem 4.2 *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, and let C be a nonempty, closed and convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, and let $A : C \to E$ be an α-inverse strongly accretive mapping. Let $B : C \to C$ be an m-accretive operator. Assume that $F := A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$. Suppose that $x_1 = x \in C$ and that $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n\big[\mu J_r x_n + (1 - \mu)(x_n - \lambda A x_n)\big] + \gamma_n Q_C e_n, \quad n \geq 1,$$

where $J_r = (I + rB)^{-1}$, $f : C \to C$ is a κ-contractive mapping, $\{e_n\}$ is a bounded computational error in E, $\lambda \in (0, \alpha/K^2]$ and $\mu \in (0, 1)$. Assume that the following restrictions are satisfied:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(b) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\{x_n\}$ converges strongly to $x = Q_F f(x)$, where Q_F is a sunny nonexpansive retraction from C onto F.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this manuscript. All authors read and approved the final manuscript.

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References

1. Zegeye, H, Shahzad, N: Strong convergence theorem for a common point of solution of variational inequality and fixed point problem. Adv. Fixed Point Theory **2**, 374-397 (2012)
2. Cho, SY, Kang, SM: Approximation of common solutions of variational inequalities via strict pseudocontractions. Acta Math. Sci. **32**, 1607-1618 (2012)
3. Qin, X, Su, Y: Approximation of a zero point of accretive operator in Banach spaces. J. Math. Anal. Appl. **329**, 415-424 (2007)
4. Qin, X, Shang, M, Su, Y: Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems. Math. Comput. Model. **48**, 1033-1046 (2008)

5. Iiduka, H, Takahashi, W: Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings. Nonlinear Anal. **61**, 341-350 (2005)
6. Lv, S, Wu, C: Convergence of iterative algorithms for a generalized variational inequality and a nonexpansive mapping. Eng. Math. Lett. **1**, 44-57 (2012)
7. Wu, C: Strong convergence theorems for common solutions of variational inequality and fixed point problems. Adv. Fixed Point Theory **4**, 229-244 (2014)
8. Wu, C, Liu, A: Strong convergence of a hybrid projection iterative algorithm for common solutions of operator equations and of inclusion problems. Fixed Point Theory Appl. **2012**, Article ID 90 (2012)
9. Wu, C: Wiener-Hope equations methods for generalized variational inequalities. J. Nonlinear Funct. Anal. **2013**, Article ID 3 (2013)
10. Qin, X, Chang, SS, Cho, YJ: Iterative methods for generalized equilibrium problems and fixed point problems with applications. Nonlinear Anal. **11**, 2963-2972 (2010)
11. Bnouhachem, A: On LQP alternating direction method for solving variational inequality problems with separable structure. J. Inequal. Appl. **2014**, Article ID 80 (2014)
12. Qin, X, Cho, SY, Wang, L: A regularization method for treating zero points of the sum of two monotone operators. Fixed Point Theory Appl. **2014**, Article ID 75 (2014)
13. Chen, JH: Iterations for equilibrium and fixed point problems. J. Nonlinear Funct. Anal. **2013**, Article ID 4 (2013)
14. Guan, WB: An iterative method for variational inequality problems. J. Inequal. Appl. **2013**, Article ID 574 (2013)
15. Qin, X, Cho, SY, Kang, SM: Convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mappings with applications. J. Comput. Appl. Math. **233**, 231-240 (2009)
16. He, R: Coincidence theorem and existence theorems of solutions for a system of Ky Fan type minimax inequalities in FC-spaces. Adv. Fixed Point Theory **2**, 47-57 (2012)
17. Cho, SY, Qin, X: On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems. Appl. Math. Comput. **235**, 430-438 (2014)
18. Luo, H, Wang, Y: Iterative approximation for the common solutions of an infinite variational inequality system for inverse-strongly accretive mappings. J. Math. Comput. Sci. **2**, 1660-1670 (2012)
19. Wang, ZM, Lou, W: A new iterative algorithm of common solutions to quasi-variational inclusion and fixed point problems. J. Math. Comput. Sci. **3**, 57-72 (2013)
20. Lv, S: Strong convergence of a general iterative algorithm in Hilbert spaces. J. Inequal. Appl. **2013**, Article ID 19 (2013)
21. Cho, SY, Qin, X, Wang, L: Strong convergence of a splitting algorithm for treating monotone operators. Fixed Point Theory Appl. **2014**, Article ID 94 (2014)
22. Hao, Y: On variational inclusion and common fixed point problems in Hilbert spaces with applications. Appl. Math. Comput. **217**, 3000-3010 (2010)
23. Kim, KS, Kim, JK, Lim, WH: Convergence theorems for common solutions of various problems with nonlinear mapping. J. Inequal. Appl. **2014**, Article ID 2 (2014)
24. Wang, G, Sun, S: Hybrid projection algorithms for fixed point and equilibrium problems in a Banach space. Adv. Fixed Point Theory **3**, 578-594 (2013)
25. Cho, SY, Li, W, Kang, SM: Convergence analysis of an iterative algorithm for monotone operators. J. Inequal. Appl. **2013**, Article ID 199 (2013)
26. Hao, Y: Strong convergence of an iterative method for inverse strongly accretive operators. J. Inequal. Appl. **2008**, Article ID 420989 (2008)
27. Reich, S: Asymptotic behavior of contractions in Banach spaces. J. Math. Anal. Appl. **44**, 57-70 (1973)
28. Kitahara, S, Takahashi, W: Image recovery by convex combinations of sunny nonexpansive retractions. Topol. Methods Nonlinear Anal. **2**, 333-342 (1993)
29. Aoyama, K, Iiduka, H, Takahashi, W: Weak convergence of an iterative sequence for accretive operators in Banach spaces. Fixed Point Theory Appl. **2006**, Article ID 35390 (2006)
30. Liu, LS: Ishikawa and Mann iteration process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114-125 (1995)
31. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127-1138 (1991)
32. Bruck, RE: Properties of fixed point sets of nonexpansive mappings in Banach spaces. Trans. Am. Math. Soc. **179**, 251-262 (1973)
33. Zhou, H: Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces. Nonlinear Anal. **69**, 3160-3173 (2008)
34. Qin, X, Cho, SY, Wang, L: Iterative algorithms with errors for zero points of *m*-accretive operators. Fixed Point Theory Appl. **2013**, Article ID 148 (2013)

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