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Barnes-type Daehee of the second kind and poly-Cauchy of the second kind mixed-type polynomials

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Abstract
In this paper, we introduce the mixed-type polynomials: Barnes-type Daehee polynomials of the second kind and poly-Cauchy polynomials of the second kind. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.
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1 Introduction

In this paper, we consider the polynomials $\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r)$ called the Barnes-type Daehee of the second kind and poly-Cauchy of the second kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^x=\sum_{n=0}^{\infty}\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r)\frac{t^n}{n!}, \tag{1}$$

where $a_1,\ldots,a_r\neq 0$. Here, $\operatorname{Lif}_k(x)$ ($k\in\mathbb{Z}$) is the polyfactorial function [1] defined by

$$\operatorname{Lif}_k(x)=\sum_{m=0}^{\infty}\frac{x^m}{m!(m+1)^k}.$$

When $x=0$, $\widehat{D}_n^{(k)}(a_1,\ldots,a_r)=\widehat{D}_n^{(k)}(0|a_1,\ldots,a_r)$ is called the Barnes-type Daehee of the second kind and poly-Cauchy of the second kind mixed-type number.

Recall that the Barnes-type Daehee polynomials of the second kind, denoted by $\widehat{D}_n(x|a_1,\ldots,a_r)$, are given by the generating function to be

$$\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)(1+t)^x=\sum_{n=0}^{\infty}\widehat{D}_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}.$$

If $a_1=\cdots=a_r=1$, then $\widehat{D}_n^{(r)}(x)=\widehat{D}_n(x|\underbrace{1,\ldots,1}_{r})$ are the Daehee polynomials of the second kind of order r. Daehee polynomials were defined by the second author [2] and were investigated in [3, 4].

The poly-Cauchy polynomials of the second kind, denoted by $\hat{c}_n^{(k)}(x)$ [5, 6], are given by the generating function as follows:

$$\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^x = \sum_{n=0}^{\infty} \hat{c}_n^{(k)}(x)\frac{t^n}{n!}.$$

In this paper, we introduce the mixed-type polynomials: Barnes-type Daehee polynomials of the second kind and poly-Cauchy polynomials of the second kind. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\bigl\langle f(t)|x^n\bigr\rangle = a_n \quad (n \geq 0). \tag{3}$$

In particular,

$$\bigl\langle t^k|x^n\bigr\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a *delta series*; if $O(f(t)) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$ for $n,k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\bigl\langle f(t)g(t)|p(x)\bigr\rangle = \bigl\langle f(t)|g(t)p(x)\bigr\rangle = \bigl\langle g(t)|f(t)p(x)\bigr\rangle \tag{5}$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) | x^k \rangle \frac{t^k}{k!},$$
$$p(x) = \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!} \tag{6}$$

[7, Theorem 2.2.5]. Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x+y). \tag{7}$$

Sheffer sequences are characterized in the generating function [7, Theorem 2.3.4].

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim (g(t), f(t))$, we have the following equations [7, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t) s_n(x) = n s_{n-1}(x) \quad (n \geq 0), \tag{8}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle x^j, \tag{9}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{10}$$

where $p_n(x) = g(t) s_n(x)$.

Assume that $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then the transfer formula [7, Corollary 3.8.2] is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have [7, p.132]

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle. \tag{11}$$

3 Main results

From definition (1), $\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r)$ is the Sheffer sequence for the pair

$$g(t)=\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\frac{1}{\mathrm{Lif}_k(-t)}\quad\text{and}\quad f(t)=e^t-1.$$

So,

$$\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r)\sim\left(\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\frac{1}{\mathrm{Lif}_k(-t)},e^t-1\right). \tag{12}$$

3.1 Explicit expressions

Recall that Barnes' multiple Bernoulli polynomials $B_n(x|a_1,\ldots,a_r)$ are defined by the generating function as follows:

$$\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}e^{xt}=\sum_{n=0}^{\infty}B_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}, \tag{13}$$

where $a_1,\ldots,a_r\neq 0$ [8, 9]. Let $(n)_j=n(n-1)\cdots(n-j+1)$ $(j\geq 1)$ with $(n)_0=1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n=\sum_{m=0}^{n}S_1(n,m)x^m.$$

Theorem 1

$$\begin{aligned}
\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r)&=\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}B_l(x+a_1+\cdots+a_r|a_1,\ldots,a_r) &(14)\\
&=\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,j)\hat{c}_i^{(k)}\widehat{D}_{l-i}(a_1,\ldots,a_r)x^j &(15)\\
&=\sum_{l=0}^{n}\binom{n}{l}\widehat{D}_{n-l}(a_1,\ldots,a_r)\hat{c}_l^{(k)}(x) &(16)\\
&=\sum_{l=0}^{n}\binom{n}{l}\hat{c}_{n-l}^{(k)}\widehat{D}_l(x|a_1,\ldots,a_r). &(17)
\end{aligned}$$

Proof Since

$$\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\frac{1}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r)\sim\left(1,e^t-1\right) \tag{18}$$

and

$$(x)_n\sim\left(1,e^t-1\right), \tag{19}$$

we have

$$\begin{aligned}
\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\mathrm{Lif}_k(-t)(x)_n \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\mathrm{Lif}_k(-t)x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\sum_{l=0}^{m}\frac{(-1)^l t^l}{l!(l+1)^k}x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\sum_{l=0}^{m}\frac{(-1)^l (m)_l}{l!(l+1)^k}x^{m-l} \\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l (m)_l}{l!(l+1)^k}\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)x^{m-l} \\
&= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^l\binom{m}{l}}{(l+1)^k}B_{m-l}(x+a_1+\cdots+a_r|a_1,\ldots,a_r) \\
&= \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}B_l(x+a_1+\cdots+a_r|a_1,\ldots,a_r).
\end{aligned}$$

So, we get (14).

By (9) with (12), we get

$$\begin{aligned}
&\left\langle g\bigl(\bar{f}(t)\bigr)^{-1}\bar{f}(t)^j|x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^j\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\middle| \sum_{l=0}^{\infty}\frac{j!}{(l+j)!}S_1(l+j,j)t^{l+j}x^n\right\rangle \\
&= \sum_{l=0}^{n-j}\frac{j!}{(l+j)!}S_1(l+j,j)(n)_{l+j}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\middle| x^{n-l-j}\right\rangle \\
&= \sum_{l=0}^{n-j} j!\binom{n}{l+j}S_1(l+j,j)\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\middle| \sum_{i=0}^{n-l-j}\hat{c}_i^{(k)}\frac{t^i}{i!}x^{n-l-j}\right\rangle \\
&= \sum_{l=0}^{n-j} j!\binom{n}{l+j}S_1(l+j,j)\sum_{i=0}^{n-l-j}\hat{c}_i^{(k)}\frac{(n-l-j)_i}{i!}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\middle| x^{n-l-j-i}\right\rangle \\
&= \sum_{l=0}^{n-j} j!\binom{n}{l+j}S_1(l+j,j)\sum_{i=0}^{n-l-j}\hat{c}_i^{(k)}\frac{(n-l-j)_i}{i!}\left\langle \sum_{m=0}^{\infty}\widehat{D}_m(a_1,\ldots,a_r)\frac{t^m}{m!}\middle| x^{n-l-j-i}\right\rangle \\
&= \sum_{l=0}^{n-j}\sum_{i=0}^{n-l-j} j!\binom{n}{l+j}\binom{n-l-j}{i}S_1(l+j,j)\hat{c}_i^{(k)}\widehat{D}_{n-l-j-i}(a_1,\ldots,a_r) \\
&= \sum_{l=0}^{n-j}\sum_{i=0}^{l} j!\binom{n}{l}\binom{l}{i}S_1(n-l,j)\hat{c}_i^{(k)}\widehat{D}_{l-i}(a_1,\ldots,a_r).
\end{aligned}$$

Thus, we obtain

$$\widehat{D}_n^{(k)}(x|a_1,\dots,a_r)=\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,j)\hat{c}_i^{(k)}\widehat{D}_{l-i}(a_1,\dots,a_r)x^j,$$

which is identity (15).

Next,

$$\begin{aligned}
\widehat{D}_n^{(k)}(y|a_1,\dots,a_r)&=\left\langle\sum_{i=0}^{\infty}\widehat{D}_i^{(k)}(y|a_1,\dots,a_r)\frac{t^i}{i!}\middle|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\middle|\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^yx^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\middle|\sum_{l=0}^{n}\hat{c}_l^{(k)}(y)\frac{t^l}{l!}x^n\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}\hat{c}_l^{(k)}(y)\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\middle|x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}\hat{c}_l^{(k)}(y)\left\langle\sum_{i=0}^{\infty}\widehat{D}_i(a_1,\dots,a_r)\frac{t^i}{i!}\middle|x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}\hat{c}_l^{(k)}(y)\widehat{D}_{n-l}(a_1,\dots,a_r).
\end{aligned}$$

Thus, we obtain (16).

Finally, we obtain that

$$\begin{aligned}
\widehat{D}_n^{(k)}(y|a_1,\dots,a_r)&=\left\langle\sum_{i=0}^{\infty}\widehat{D}_i^{(k)}(y|a_1,\dots,a_r)\frac{t^i}{i!}\middle|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^y\middle|x^n\right\rangle\\
&=\left\langle\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)\middle|\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)(1+t)^yx^n\right\rangle\\
&=\left\langle\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)\middle|\sum_{l=0}^{n}\widehat{D}_l(y|a_1,\dots,a_r)\frac{t^l}{l!}x^n\right\rangle\\
&=\sum_{l=0}^{n}\widehat{D}_l(y|a_1,\dots,a_r)\binom{n}{l}\bigl\langle\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)|x^{n-l}\bigr\rangle\\
&=\sum_{l=0}^{n}\widehat{D}_l(y|a_1,\dots,a_r)\binom{n}{l}\left\langle\sum_{i=0}^{\infty}\hat{c}_i^{(k)}\frac{t^i}{i!}\middle|x^{n-l}\right\rangle\\
&=\sum_{l=0}^{n}\binom{n}{l}\widehat{D}_l(y|a_1,\dots,a_r)\hat{c}_{n-l}^{(k)}.
\end{aligned}$$

Thus, we get identity (17). □

3.2 Sheffer identity

Theorem 2

$$\widehat{D}_n^{(k)}(x+y|a_1,\ldots,a_r) = \sum_{j=0}^{n}\binom{n}{j}\widehat{D}_j^{(k)}(x|a_1,\ldots,a_r)(y)_{n-j}. \tag{20}$$

Proof By (12) with

$$\begin{aligned} p_n(x) &= \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\frac{1}{\mathrm{Lif}_k(-t)}\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r) \\ &= (x)_n \sim \left(1, e^t-1\right), \end{aligned}$$

using (10), we have (20). □

3.3 Difference relations

Theorem 3

$$\widehat{D}_n^{(k)}(x+1|a_1,\ldots,a_r) - \widehat{D}_n^{(k)}(x|a_1,\ldots,a_r) = n\widehat{D}_{n-1}^{(k)}(x|a_1,\ldots,a_r). \tag{21}$$

Proof By (8) with (12), we get

$$\left(e^t-1\right)\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r) = n\widehat{D}_{n-1}^{(k)}(x|a_1,\ldots,a_r).$$

By (7), we have (21). □

3.4 Recurrence

Theorem 4

$$\begin{aligned} \widehat{D}_{n+1}^{(k)}(x|a_1,\ldots,a_r) &= x\widehat{D}_n^{(k)}(x-1|a_1,\ldots,a_r) \\ &\quad - \sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\frac{(-1)^{l-i}\binom{m+1}{l}\binom{l}{i}}{(m+1)(l-i+1)^k}S_1(n,m) \\ &\quad \times (-a_j)^{m+1-l}B_{m+1-l}B_i(x+a_1+\cdots+a_r-1|a_1,\ldots,a_r) \\ &\quad + \sum_{j=1}^{r}a_j\sum_{m=0}^{n}\sum_{i=0}^{m}\frac{(-1)^{m-i}\binom{m}{i}}{(m-i+1)^k}S_1(n,m) \\ &\quad \times B_i(x+a_1+\cdots+a_r-1|a_1,\ldots,a_r) \\ &\quad - \sum_{m=0}^{n}\sum_{l=0}^{m+1}\frac{(-1)^{m+1-l}\binom{m}{l}}{(m+2-l)^k}S_1(n,m) \\ &\quad \times B_l(x+a_1+\cdots+a_r-1|a_1,\ldots,a_r), \end{aligned} \tag{22}$$

where B_n is the nth ordinary Bernoulli number.

Proof By applying

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \tag{23}$$

[7, Corollary 3.7.2] with (12), we get

$$\widehat{D}_{n+1}^{(k)}(x|a_1,\dots,a_r) = x\widehat{D}_n^{(k)}(x-1|a_1,\dots,a_r) - e^{-t}\frac{g'(t)}{g(t)}\widehat{D}_n^{(k)}(x|a_1,\dots,a_r).$$

Now,

$$\begin{aligned}
\frac{g'(t)}{g(t)} &= \bigl(\ln g(t)\bigr)' = \left(\sum_{j=1}^{r}\ln\bigl(e^{a_jt}-1\bigr) - r\ln t - \left(\sum_{j=1}^{r}a_j\right)t - \ln \mathrm{Lif}_k(-t)\right)' \\
&= \sum_{j=1}^{r}\frac{a_je^{a_jt}}{e^{a_jt}-1} - \frac{r}{t} - \sum_{j=1}^{r}a_j + \frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)} \\
&= \frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{t\prod_{j=1}^{r}(e^{a_jt}-1)} - \sum_{j=1}^{r}a_j + \frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{\prod_{j=1}^{r}(e^{a_jt}-1)} \\
&\quad = \frac{\frac{1}{2}(\sum_{j=1}^{r}a_1\cdots a_{j-1}a_j^2a_{j+1}\cdots a_r)t^{r+1}+\cdots}{(a_1\cdots a_r)t^r+\cdots} \\
&\quad = \frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots
\end{aligned}$$

is a series with order ≥ 1. Since

$$\begin{aligned}
\widehat{D}_n^{(k)}(x|a_1,\dots,a_r) &= \prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\mathrm{Lif}_k(-t)(x)_n \\
&= \sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\mathrm{Lif}_k(-t)x^m,
\end{aligned}$$

we have

$$\begin{aligned}
\frac{g'(t)}{g(t)}\widehat{D}_n^{(k)}(x|a_1,\dots,a_r) &= \sum_{m=0}^{n}S_1(n,m)\,\mathrm{Lif}_k(-t)\left(\prod_{j=1}^{r}\frac{te^{a_jt}}{e^{a_jt}-1}\right)\frac{g'(t)}{g(t)}x^m \\
&= \sum_{m=0}^{n}S_1(n,m)\,\mathrm{Lif}_k(-t)\left(\prod_{j=1}^{r}\frac{te^{a_jt}}{e^{a_jt}-1}\right) \\
&\quad \times \frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{t\prod_{j=1}^{r}(e^{a_jt}-1)}x^m \\
&\quad - \sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\,\mathrm{Lif}_k(-t)\left(\prod_{j=1}^{r}\frac{te^{a_jt}}{e^{a_jt}-1}\right)x^m \\
&\quad + \sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{te^{a_jt}}{e^{a_jt}-1}\right)\mathrm{Lif}_k'(-t)x^m. \qquad (24)
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_i t}-1)(a_j t e^{a_j t}-e^{a_j t}+1)}{t\prod_{j=1}^{r}(e^{a_j t}-1)}x^m \\
&= \frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_i t}-1)(a_j t e^{a_j t}-e^{a_j t}+1)}{\prod_{j=1}^{r}(e^{a_j t}-1)}\frac{x^{m+1}}{m+1} \\
&= \frac{1}{m+1}\sum_{j=1}^{r}\left(\frac{a_j t e^{a_j t}}{e^{a_j t}-1}-1\right)x^{m+1} \\
&= \frac{1}{m+1}\sum_{j=1}^{r}\left(\sum_{l=0}^{\infty}\frac{(-a_j)^l B_l}{l!}t^l-1\right)x^{m+1} \\
&= \frac{1}{m+1}\sum_{j=1}^{r}\sum_{l=0}^{m}\binom{m+1}{l}(-a_j)^{m+1-l}B_{m+1-l}x^l,
\end{aligned}$$

the first term in (24) is

$$\begin{aligned}
&\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}\sum_{j=1}^{r}\sum_{l=0}^{m}\binom{m+1}{l}(-a_j)^{m+1-l}B_{m+1-l}\operatorname{Lif}_k(-t)\left(\prod_{j=1}^{r}\frac{te^{a_j t}}{e^{a_j t}-1}\right)x^l \\
&= \sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}\sum_{j=1}^{r}\sum_{l=0}^{m}\binom{m+1}{l}(-a_j)^{m+1-l}B_{m+1-l} \\
&\quad\times\sum_{i=0}^{l}\frac{(-1)^i t^i}{i!(i+1)^k}B_l(x+a_1+\cdots+a_r|a_1,\ldots,a_r) \\
&= \sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\frac{(-1)^{l-i}\binom{m+1}{l}\binom{l}{i}}{(m+1)(l-i+1)^k}S_1(n,m)(-a_j)^{m+1-l}B_{m+1-l} \\
&\quad\times B_i(x+a_1+\cdots+a_r|a_1,\ldots,a_r).
\end{aligned}$$

Since

$$\operatorname{Lif}_{k-1}(-t)-\operatorname{Lif}_k(-t)=\left(\frac{1}{2^k}-\frac{1}{2^{k-1}}\right)t+\cdots, \tag{25}$$

the second term in (24) is

$$\begin{aligned}
&\sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\operatorname{Lif}_k(-t)B_m(x+a_1+\cdots+a_r|a_1,\ldots,a_r) \\
&= \sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\sum_{i=0}^{m}\frac{(-1)^i t^i}{i!(i+1)^k}B_m(x+a_1+\cdots+a_r|a_1,\ldots,a_r) \\
&= \sum_{j=1}^{r}a_j\sum_{m=0}^{n}\sum_{i=0}^{m}\frac{(-1)^{m-i}\binom{m}{i}}{(m-i+1)^k}S_1(n,m)B_i(x+a_1+\cdots+a_r|a_1,\ldots,a_r).
\end{aligned}$$

The third term in (24) is

$$\begin{aligned}
&\sum_{m=0}^{n} S_1(n,m)\frac{\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)}{t}B_m(x+a_1+\cdots+a_r|a_1,\ldots,a_r) \\
&\quad=\sum_{m=0}^{n} S_1(n,m)\bigl(\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)\bigr)\frac{B_{m+1}(x+a_1+\cdots+a_r|a_1,\ldots,a_r)}{m+1} \\
&\quad=\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}\left(\sum_{l=0}^{m+1}\frac{(-1)^l t^l}{l!(l+1)^{k-1}}B_{m+1}(x+a_1+\cdots+a_r|a_1,\ldots,a_r)\right. \\
&\qquad\left.-\sum_{l=0}^{m+1}\frac{(-1)^l t^l}{l!(l+1)^{k}}B_{m+1}(x+a_1+\cdots+a_r|a_1,\ldots,a_r)\right) \\
&\quad=\sum_{m=0}^{n}\sum_{l=0}^{m+1}\frac{(-1)^{m+1-l}\binom{m}{l}}{(m+2-l)^k}S_1(n,m)B_l(x+a_1+\cdots+a_r|a_1,\ldots,a_r).
\end{aligned}$$

Thus we have identity (22). □

3.5 Differentiation

Theorem 5

$$\frac{d}{dx}\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{D}_l^{(k)}(x|a_1,\ldots,a_r). \tag{26}$$

Proof We shall use

$$\frac{d}{dx}s_n(x)=\sum_{l=0}^{n-1}\binom{n}{l}\bigl\langle\bar{f}(t)|x^{n-l}\bigr\rangle s_l(x)$$

(*cf.* [7, Theorem 2.3.12]). Since

$$\begin{aligned}
\bigl\langle\bar{f}(t)|x^{n-l}\bigr\rangle&=\bigl\langle\ln(1+t)|x^{n-l}\bigr\rangle \\
&=\left\langle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\middle|x^{n-l}\right\rangle \\
&=(-1)^{n-l-1}(n-l-1)!,
\end{aligned}$$

with (12), we have

$$\begin{aligned}
\frac{d}{dx}\widehat{D}_n^{(k)}(x|a_1,\ldots,a_r)&=\sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!\widehat{D}_l^{(k)}(x|a_1,\ldots,a_r) \\
&=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{D}_l^{(k)}(x|a_1,\ldots,a_r),
\end{aligned}$$

which is identity (26). □

3.6 More relations

The classical Cauchy numbers c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

(see *e.g.* [1, 10]).

Theorem 6

$$\begin{aligned}
\widehat{D}_n^{(k)}(x|a_1,\dots,a_r) &= \left(x + \sum_{j=1}^{r} a_j\right)\widehat{D}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l} c_l \widehat{D}_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{r-1}{n}\sum_{l=0}^{n}\binom{n}{l} c_l \widehat{D}_{n-l}^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l} a_j c_l \widehat{D}_{n-l}^{(k)}(x-1|a_1,\dots,a_r,a_j).
\end{aligned} \tag{27}$$

Proof For $n \geq 1$, we have

$$\begin{aligned}
\widehat{D}_n^{(k)}(y|a_1,\dots,a_r) &= \left\langle \sum_{l=0}^{\infty} \widehat{D}_l^{(k)}(y|a_1,\dots,a_r)\frac{t^l}{l!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right) \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \partial_t \left(\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right) \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^y\right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\right) \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^y \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right) \bigl(\partial_t \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigr)(1+t)^y \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right) \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\partial_t(1+t)^y\bigr) \middle| x^{n-1} \right\rangle.
\end{aligned}$$

The third term is

$$\begin{aligned}
&y\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right) \mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
&\quad = yD_{n-1}^{(k)}(y-1|a_1,\dots,a_r).
\end{aligned}$$

By (25), the second term is

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\frac{\mathrm{Lif}_{k-1}(-\ln(1+t))-\mathrm{Lif}_k(-\ln(1+t))}{(1+t)\ln(1+t)}(1+t)^y \middle| x^{n-1}\right\rangle\\
&\quad=\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\frac{\mathrm{Lif}_{k-1}(-\ln(1+t))-\mathrm{Lif}_k(-\ln(1+t))}{t}\right.\\
&\qquad\left.\times(1+t)^{y-1}\middle|\frac{t}{\ln(1+t)}x^{n-1}\right\rangle\\
&\quad=\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\left(\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_{k-1}\bigl(-\ln(1+t)\bigr)(1+t)^{y-1}\middle|x^{n-l}\right\rangle\right.\\
&\qquad\left.-\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_{k}\bigl(-\ln(1+t)\bigr)(1+t)^{y-1}\middle|x^{n-l}\right\rangle\right)\\
&\quad=\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\bigl(\widehat{D}_{n-l}^{(k-1)}(y-1|a_1,\ldots,a_r)-\widehat{D}_{n-l}^{(k)}(y-1|a_1,\ldots,a_r)\bigr).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\\
&\quad=\frac{1}{1+t}\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\frac{\sum_{j=1}^{r}\bigl(\frac{t}{\ln(1+t)}-\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}\bigr)}{t}\\
&\qquad+\left(\sum_{j=1}^{r}a_j\right)\frac{1}{1+t}\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right),
\end{aligned}$$

with

$$\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)=-\frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots$$

a series with order ≥ 1.

Now, the first term is

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{y-1}\middle|\frac{\sum_{j=1}^{r}\bigl(\frac{t}{\ln(1+t)}-\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}\bigr)}{t}x^{n-1}\right\rangle\\
&\qquad+\sum_{j=1}^{r}a_j\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{y-1}\middle|x^{n-1}\right\rangle\\
&\quad=\left(\sum_{j=1}^{r}a_j\right)\widehat{D}_{n-1}^{(k)}(y-1|a_1,\ldots,a_r)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{r}{n}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\,\middle|\,\frac{t}{\ln(1+t)}x^n\right\rangle \\
&- \frac{1}{n}\sum_{j=1}^{r}a_j\left\langle \frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\big(-\ln(1+t)\big)\right. \\
&\left.\times (1+t)^{y-1}\,\middle|\,\frac{t}{\ln(1+t)}x^n\right\rangle \\
={}& \left(\sum_{j=1}^{r}a_j\right)\widehat{D}_{n-1}^{(k)}(y-1|a_1,\dots,a_r) + \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{D}_{n-l}^{(k)}(y-1|a_1,\dots,a_r) \\
&- \frac{1}{n}\sum_{i=1}^{r}a_j\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{D}_{n-l}^{(k)}(y-1|a_1,\dots,a_r,a_j).
\end{aligned}
$$

Altogether, we obtain

$$
\begin{aligned}
\widehat{D}_n^{(k)}(x|a_1,\dots,a_r) ={}& \left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\
&+ \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\big(\widehat{D}_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r) - \widehat{D}_{n-l}^{(k)}(x-1|a_1,\dots,a_r)\big) \\
&+ \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{D}_{n-l}^{(k)}(x-1|a_1,\dots,a_r) \\
&- \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jc_l\widehat{D}_{n-l}^{(k)}(x-1|a_1,\dots,a_r,a_j),
\end{aligned}
$$

from which identity (27) follows. □

3.7 A relation including the Stirling numbers of the first kind

Theorem 7 *For $n-1 \geq m \geq 1$, we have*

$$
\begin{aligned}
&m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{D}_l^{(k)}(a_1,\dots,a_r) \\
&= m\sum_{j=1}^{r}a_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{D}_l^{(k)}(-1|a_1,\dots,a_r) \\
&\quad + \frac{mr}{n}\sum_{l=0}^{n-m}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,m)c_{l-i}\widehat{D}_i^{(k)}(-1|a_1,\dots,a_r) \\
&\quad - \frac{m}{n}\sum_{l=0}^{n-m}\sum_{i=0}^{l}\sum_{j=1}^{r}\binom{n}{l}\binom{l}{i}S_1(n-l,m)a_jc_{l-i}\widehat{D}_i^{(k)}(-1|a_1,\dots,a_r,a_j) \\
&\quad + \sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(k-1)}(-1|a_1,\dots,a_r) \\
&\quad + (m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(k)}(-1|a_1,\dots,a_r). \qquad (28)
\end{aligned}
$$

Proof We shall compute

$$\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m \middle| x^n\right\rangle$$

in two different ways. On the one hand, it is

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr) \middle| \bigl(\ln(1+t)\bigr)^m x^n\right\rangle\\
&\quad=\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr) \middle| \sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n\right\rangle\\
&\quad=\sum_{l=0}^{n-m} m!\binom{n}{l}S_1(n-l,m)\widehat{D}_l^{(k)}(a_1,\ldots,a_r).
\end{aligned}$$

On the other hand, it is

$$\begin{aligned}
&\left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\right) \middle| x^{n-1}\right\rangle\\
&\quad=\left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m \middle| x^{n-1}\right\rangle\\
&\qquad+\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\bigl(\partial_t\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)\bigr)\bigl(\ln(1+t)\bigr)^m \middle| x^{n-1}\right\rangle\\
&\qquad+\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\partial_t\bigl(\ln(1+t)\bigr)^m\bigr) \middle| x^{n-1}\right\rangle.
\end{aligned}\tag{29}$$

The third term of (29) is equal to

$$\begin{aligned}
&m\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{-1} \middle| \bigl(\ln(1+t)\bigr)^{m-1}x^{n-1}\right\rangle\\
&\quad=m\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{-1} \middle|\right.\\
&\qquad\left.\sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)t^{l+m-1}x^{n-1}\right\rangle\\
&\quad=m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(k)}(-1|a_1,\ldots,a_r).
\end{aligned}$$

The second term of (29) is equal to

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\left(\frac{\operatorname{Lif}_{k-1}(-\ln(1+t))-\operatorname{Lif}_{k-1}(-\ln(1+t))}{(1+t)\ln(1+t)}\right)\bigl(\ln(1+t)\bigr)^m \middle| x^{n-1}\right\rangle\\
&\quad=\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_{k-1}\bigl(-\ln(1+t)\bigr)(1+t)^{-1} \middle| \bigl(\ln(1+t)\bigr)^{m-1}x^{n-1}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&-\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{-1}\,\Big|\,\bigl(\ln(1+t)\bigr)^{m-1}x^{n-1}\right\rangle \\
&= (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(k-1)}(-1|a_1,\ldots,a_r) \\
&\quad - (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(k)}(-1|a_1,\ldots,a_r).
\end{aligned}
$$

The first term of (29) is equal to

$$
\begin{aligned}
&\frac{1}{n}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{-1}\right. \\
&\quad\left.\times\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\,\Big|\,\bigl(\ln(1+t)\bigr)^{m}x^{n}\right\rangle \\
&\quad+\sum_{j=1}^{r}a_j\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{-1}\,\Big|\,\bigl(\ln(1+t)\bigr)^{m}x^{n-1}\right\rangle \\
&= m!\sum_{j=1}^{r}a_j\sum_{l=0}^{n-m-1}\binom{n-1}{l+m}S_1(l+m,m)\widehat{D}_{n-l-m-1}^{(k)}(-1|a_1,\ldots,a_r) \\
&\quad+\frac{m!}{n}\sum_{l=0}^{n-m}\binom{n}{l+m}S_1(l+m,m) \\
&\quad\times\left(r\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{-1}\,\Big|\,\frac{t}{\ln(1+t)}x^{n-l-m}\right\rangle\right. \\
&\quad-\sum_{j=1}^{r}a_j\left\langle \frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\right. \\
&\quad\left.\left.\times\,\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{-1}\,\Big|\,\frac{t}{\ln(1+t)}x^{n-l-m}\right\rangle\right) \\
&= m!\sum_{j=1}^{r}a_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{D}_l^{(k)}(-1|a_1,\ldots,a_r) \\
&\quad+\frac{m!}{n}\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\left(r\sum_{i=0}^{l}\binom{l}{i}c_i\widehat{D}_{l-i}^{(k)}(-1|a_1,\ldots,a_r)\right. \\
&\quad\left.-\sum_{j=1}^{r}\sum_{i=0}^{l}\binom{l}{i}a_jc_i\widehat{D}_{l-i}^{(k)}(-1|a_1,\ldots,a_r,a_j)\right).
\end{aligned}
$$

Therefore, we get, for $n-1\geq m\geq 1$,

$$
\begin{aligned}
&m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{D}_l^{(k)}(a_1,\ldots,a_r) \\
&= m!\sum_{j=1}^{r}a_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\widehat{D}_l^{(k)}(-1|a_1,\ldots,a_r)
\end{aligned}
$$

$$
\begin{aligned}
&+ m!\frac{r}{n}\sum_{l=0}^{n-m}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,m)c_i\widehat{D}_{l-i}^{(k)}(-1|a_1,\dots,a_r) \\
&- m!\frac{1}{n}\sum_{l=0}^{n-m}\sum_{i=0}^{l}\sum_{j=1}^{r}\binom{n}{l}\binom{l}{i}S_1(n-l,m)a_jc_i\widehat{D}_{l-i}^{(k)}(-1|a_1,\dots,a_r,a_j) \\
&+ (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(k-1)}(-1|a_1,\dots,a_r) \\
&- (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(k)}(-1|a_1,\dots,a_r) \\
&+ m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l^{(k)}(-1|a_1,\dots,a_r).
\end{aligned}
$$

Dividing both sides by $(m-1)!$, we get (28). □

3.8 A relation with the falling factorials

Theorem 8

$$
\widehat{D}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\binom{n}{m}\widehat{D}_{n-m}^{(k)}(a_1,\dots,a_r)(x)_m. \tag{30}
$$

Proof For (12) and (19), assume that $\widehat{D}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (11), we have

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\operatorname{Lif}_k\bigl(-\ln(1+t)\bigr)\middle| t^m x^n\right\rangle \\
&= \binom{n}{m}\widehat{D}_{n-m}^{(k)}(a_1,\dots,a_r).
\end{aligned}
$$

Thus, we get identity (30). □

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$
\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}
$$

(see *e.g.* [11]).

Theorem 9

$$
\begin{aligned}
\widehat{D}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\Biggl(&\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j \\
&\times (1-\lambda)^{-j}S_1(n-j-l,m)\widehat{D}_l^{(k)}(a_1,\dots,a_r)\Biggr)H_m^{(s)}(x|\lambda).
\end{aligned} \tag{31}
$$

Proof For (12) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t - \lambda}{1 - \lambda} \right)^s, t \right), \tag{32}$$

assume that $\widehat{D}_n^{(k)}(x|a_1, \ldots, a_r) = \sum_{m=0}^n C_{n,m} H_m^{(s)}(x|\lambda)$. By (11), similarly to the proof of (28), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!(1-\lambda)^s} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \operatorname{Lif}_k\big(-\ln(1+t)\big) \big(\ln(1+t)\big)^m (1 - \lambda + t)^s \middle| x^n \right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \operatorname{Lif}_k\big(-\ln(1+t)\big) \right. \\
&\quad \times \left. \big(\ln(1+t)\big)^m \middle| \sum_{i=0}^{\min\{s,n\}} \binom{s}{i} (1-\lambda)^{s-i} t^i x^n \right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \sum_{i=0}^{n-m} \binom{s}{i} (1-\lambda)^{s-i} (n)_i \\
&\quad \times \left\langle \prod_{j=1}^r \left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \operatorname{Lif}_k\big(-\ln(1+t)\big) \big(\ln(1+t)\big)^m \middle| x^{n-i} \right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \sum_{i=0}^{n-m} \binom{s}{i} (1-\lambda)^{s-i} (n)_i \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) \widehat{D}_l^{(k)}(a_1, \ldots, a_r) \\
&= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{s}{i} \binom{n-i}{l} (n)_i (1-\lambda)^{-i} S_1(n-i-l, m) \widehat{D}_l^{(k)}(a_1, \ldots, a_r).
\end{aligned}$$

Thus, we get identity (31). □

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^\infty \frac{\mathfrak{B}_n^{(r)}(x)}{n!} t^n$$

(see *e.g.* [7, Section 2.2]). In addition, the Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)} \right)^r = \sum_{n=0}^\infty \frac{\mathfrak{C}_n^{(r)}}{n!} t^n$$

(see *e.g.* [12, (2.1)], [13, (6)]).

Theorem 10

$$\begin{aligned}
&\widehat{D}_n^{(k)}(x|a_1, \ldots, a_r) \\
&\quad = \sum_{m=0}^n \left(\sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) \widehat{D}_l^{(k)}(a_1, \ldots, a_r) \right) \mathfrak{B}_m^{(s)}(x).
\end{aligned} \tag{33}$$

Proof For (12) and

$$\mathfrak{B}_n^{(s)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^s, t \right), \tag{34}$$

assume that $\widehat{D}_n^{(k)}(x|a_1, \ldots, a_r) = \sum_{m=0}^{n} C_{n,m} \mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (28), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \big(\ln(1+t)\big)^m \middle| \left(\frac{t}{\ln(1+t)} \right)^s x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \big(\ln(1+t)\big)^m \middle| \sum_{i=0}^{\infty} \mathfrak{C}_i^{(s)} \frac{t^i}{i!} x^n \right\rangle \\
&= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \left\langle \prod_{j=1}^{r} \left(\frac{\ln(1+t)(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \big(\ln(1+t)\big)^m \middle| x^{n-i} \right\rangle \\
&= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) \widehat{D}_l^{(k)}(a_1, \ldots, a_r) \\
&= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) \widehat{D}_l^{(k)}(a_1, \ldots, a_r).
\end{aligned}$$

Thus, we get identity (33). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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