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Existence of iterative roots for the sickle-like functions

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Abstract

The problem of iterative roots for strictly monotone self-mappings has been well solved. Most of known results concerning existence of iterative roots for a continuous function were given under the assumption that the function has finitely many non-monotonic points. When a function has infinitely many non-monotonic points, the problem of the existence of its iterative roots will become more complicated. In this paper, we study the existence of iterative roots for the sickle-like functions, as a special class of non-monotonic functions, each of which has not only one isolated non-monotonic point but also infinitely many non-isolated non-monotonic points.

MSC: 37E05; 39B12

Keywords: non-monotonic point; sickle-like function; fixed point; iterative root

1 Introduction

Let $I := [0,1]$. For any integer $n \geq 0$, consider a mapping $F : I \to I$. An *iterative root* of order n of F is a mapping $f : I \to I$ such that

$$f^n(x) = F(x), \quad \forall x \in I,$$

where f^n denotes the nth iterate of the mapping $f : I \to I$, *i.e.* $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$ for all $x \in I$ inductively. By studying the iterative roots, people can find the missing information in the iterative process. Meanwhile, being a weak version of the problem of embedding a function into a flow or into a semi-flow, the existence of iterative roots of a given mapping is a basic problem in both the theory of functional equations and the theorem of dynamical systems. The problem of finding iterative roots for a given function is still alive since the work of Babbage [1, 2] at the beginning of the 19th century, more and more attention has been turned to this problem (see *e.g.* [3–8] and references therein). Plentiful results have been obtained for continuous and strictly monotonic mappings on intervals. In the monographs [9, 10], Kuczma, Choczewski and Ger gave a complete description of iterative roots of continuous and strictly monotonic self-mappings on a given interval.

An interior point x_0 of I is called a *monotonic point* of mapping $F : I \to I$ if F is strictly monotonic in a neighborhood of x_0; otherwise, x_0 is referred to as a *non-monotonic point* or simply a *fort* of F (see [11, 12]). Consequently, the function is strictly monotonic on I if and only if it has no non-monotonic points in the interior of I. A function having finitely many non-monotonic points is called a strictly piecewise monotonic function or simply called a PM function (see [11]). Each of the non-monotonic points of a PM function is an isolated non-monotonic point. It seems that it was JZ Zhang and L Yang who first, in 1983, started to study iterative roots of PM functions explicitly in the paper [11]. They introduced the concept of *characteristic interval* for PM functions, and studied the existence of iterative roots of PM functions which have the characteristic interval. Later, Blokh, Coven, Misiurewicz, Nitecki and WN Zhang established some new results for iterative roots of PM functions (see [12, 13]). Recently, there are some advances obtained for iterative roots. For example see in [14–18].

When a function has infinitely many non-monotonic points, the study of existence of its iterative roots will become more difficult. One of the typical cases that the function has infinitely many non-monotonic points is that at least there exists one nontrivial subinterval (*i.e.*, not singleton) on which the function is constant (see *e.g.* [19–21]). In 1992, the author in the paper [22] proved the existence of iterative roots of a class of self-mappings possessing infinitely many non-monotonic points. Later, TX Sun and HJ Xi discussed the iterative roots of a class of self-mappings with a constant on two subintervals (see [23]).

However, in both paper [22] and paper [23], each of non-monotonic points of the function is non-isolated. The main purpose of this paper is to study the existence of continuous iterative roots for a class of functions, each of which has not only infinitely many non-isolated non-monotonic points but also one isolated non-monotonic point.

Assume that $a, b \in (0,1)$, $a < b$ and $F \in C(I,I)$, the set of all continuous self-mappings on I. Then F is called a *sickle-like function* if one of the following conditions is fulfilled: (C1) F is constant on $[0,a]$, and F is strictly decreasing on $[a,b]$ but strictly increasing on $[b,1]$; (C2) F is constant on $[0,a]$, and F is strictly increasing on $[a,b]$ but strictly decreasing on $[b,1]$; (C3) F is constant on $[b,1]$, and F is strictly increasing on $[0,a]$ but strictly decreasing on $[a,b]$; (C4) F is constant on $[b,1]$, and F is strictly decreasing on $[0,a]$ but strictly increasing on $[a,b]$. If F satisfies (C3) (resp. (C4)), then H satisfies (C1) (resp. (C2)), where H is defined by $H(x) := h^{-1} \circ F \circ h(x)$ for all $x \in I$, and $h : I \to I$ is defined by

$$h(x) := \begin{cases} \frac{b-1}{a}x + 1, & \forall x \in [0,a), \\ b + a - x, & \forall x \in [a,b), \\ \frac{a}{b-1}(x-1), & \forall x \in [b,1]. \end{cases}$$

Hence, it suffices to confine ourselves to discuss F satisfying (C1) or (C2). For this purpose, let S_1 (resp. S_2) denote the set of all those sickle-like functions satisfying (C1) (resp. (C2)) (see Figures 1 and 2), and $S := S_1 \cup S_2$. If $F \in S$, then b is an isolated non-monotonic point of F but every point belonging to $[0,a]$ is a non-isolated non-monotonic point of F.

The paper is organized as follows: at first some important properties of iterative roots of sickle-like functions will be given in Section 2. Then in Sections 3 and 4 we will discuss the existence of iterative roots of $F \in S_1$ and $F \in S_2$, respectively. Throughout this paper, n stands for a positive integer and $F|_E$ represents the restriction of F on E for a set $E \subset I$.

2 Preliminaries

Lemma 2.1 *Suppose that f is an iterative root of order $n \geq 2$ of $F \in S$ on I. Then*

(i) $f([0,a]) = \{f(a)\}$ *if there exists* $x_0 \in [0,a]$ *such that* $f(x_0) > a$;

(ii) $f([0,a]) \subset [0,a]$ *if there exists* $x_0 \in [0,a]$ *such that* $f(x_0) \leq a$;

(iii) *there exists* $a' \in [0,a)$ *such that* $f([a',a]) = \{a\}$ *iff* $f(a) = a$.

Figure 1 $F \in S_1$.

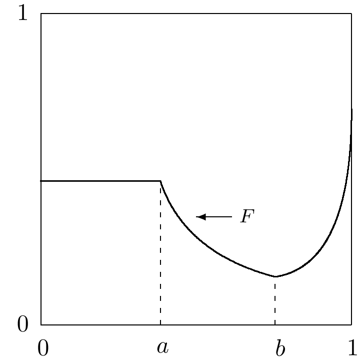
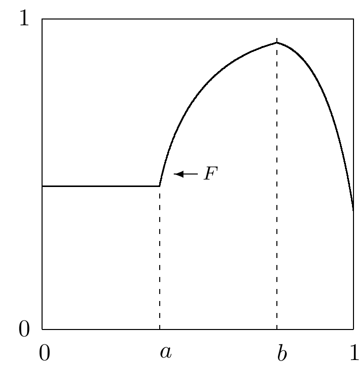


Figure 2 $F \in S_2$.



Proof For result (i), we only prove that $f([0,a]) = \{f(x_0)\}$. By reduction to absurdity, suppose that $f([0,a]) \neq \{f(x_0)\}$. Then there exists $x_* \in [0,a]$ such that $f(x_*) \neq f(x_0)$. By the assumption, we have

$$\begin{aligned}&\big(f\big([0,a]\big) \cap [a,b]\big) \cup \big(f\big([0,a]\big) \cap [b,1]\big)\\ &\quad = f\big([0,a]\big) \cap [a,1] \supset f\big([x_0;x_*]\big) \cap [a,1] \supset \big[f(x_0);f(x_*)\big] \cap [a,1] \neq \emptyset,\end{aligned}$$

where $[x_0;x_*]$ denotes either $[x_0,x_*]$ or $[x_*,x_0]$. It follows that at least one of $f([0,a]) \cap [a,b]$ and $f([0,a]) \cap [b,1]$ is a nonempty closed interval (not a singleton). Without loss of generality, we may assume that there are $t_1, t_2 \in [a,b]$ such that $t_1 < t_2$ and

$$f\big([0,a]\big) \cap [a,b] = [t_1,t_2].$$

Thus, there are $t_1^*, t_2^* \in [0,a]$ such that $t_1^* \neq t_2^*$, $f(t_1^*) = t_1$ and $f(t_2^*) = t_2$. On the other hand, noting that $F = f \circ f^{n-1}$ is strictly monotonic on $[a,b]$, we find that f^{n-1} is also strictly monotonic on $[a,b]$, which yields

$$F\big(t_1^*\big) = f^n\big(t_1^*\big) = f^{n-1}(t_1) \neq f^{n-1}(t_2) = f^n\big(t_2^*\big) = F\big(t_2^*\big),$$

contrary to the fact that F is constant on $[0,a]$. This contradiction completes the proof of result (i).

For result (ii), we will give a proof by contradiction. Suppose that there exists $t_3 \in [0,a)$ such that $f(t_3) > a$. It follows from the result (i), proved just now, that $f([0,a]) = \{f(a)\}$, showing that $f(x_0) = f(a) = f(t_3) > a$, contrary to the assumption that $f(x_0) \leq a$. This contradiction shows us that $f([0,a]) \subset [0,a]$, and thus the result (ii) is proved.

For result (iii), it becomes obvious by the result (ii) that $f([0,a]) \subset [0,a]$, which shows that

$$f^n([0,a]) = F([0,a]) = \{a\}.$$

It follows that there exists an integer $n' \geq 0$ such that $f^{n'}([0,a]) \supsetneqq \{a\}$ but $f^{n'+1}([0,a]) = \{a\}$. Using the continuity of $f^{n'}$ and $f^{n'}([0,a]) \subset [0,a]$, there exists $a' \in [0,a)$ such that $f^{n'}([0,a]) = [a',a]$. Thus,

$$f([a',a]) = f^{n'+1}([0,a]) = \{a\},$$

and the result (iii) is proved. This completes the proof of Lemma 2.1. □

Lemma 2.2 *Suppose that f is an iterative root of order $n \geq 2$ of $F \in S$ on I. Then*
(i) *$f([0,a]) \subset [0,a]$ if $F(a) < a$, and*
(ii) *either $f([a,1]) \subset [a,b]$ or $f([a,1]) \subset [b,1]$ if $F(a) \geq a$.*

Proof First of all we claim that if $f(a) \geq a$, then

$$f([a,1]) \subset [a,1]. \tag{2.1}$$

Indeed, if $f(a) \geq a$, then it follows from the results (i) and (iii) of Lemma 2.1 that there exists $a' \in [0,a)$ such that

$$f([a',a]) = \{f(a)\}. \tag{2.2}$$

Suppose that there is an $x' \in (a,1]$ such that $f(x') < a$. Let $a'' := \max\{a', f(x')\}$. Then we have

$$[a'',a] = [a',a] \cap [f(x'),f(a)] \subset [a',a] \cap f([a,x']). \tag{2.3}$$

Hence, according to the continuity of f, there exist $s,t \in [a,x']$ such that $s < t$ and $f([s,t]) = [a'',a]$. It follows from (2.2) and (2.3) that

$$F([s,t]) = f^{n-1}(f([s,t])) = f^{n-1}([a'',a]) \subset f^{n-1}([a',a]) = \{f^{n-1}(a)\},$$

i.e. F is constant on $[s,t] \subset [a,1]$, contradicting to the assumption on F. Thus the claimed (2.1) holds.

For result (i), suppose for an indirect proof that there exists $t^* \in [0,a]$ such that $f(t^*) > a$. It follows from the result (i) of Lemma 2.1 that $f([0,a]) = \{f(a)\}$, which shows that $f(a) = f(t^*) > a$. Thus (2.1) holds, implying that $F(a) = f^n(a) \in [a,1]$, contradicting to the assumption and the result (i) is proved.

For result (ii), firstly we claim that

$$f(a) \geq a. \tag{2.4}$$

In fact, if $f(a) < a$, then there exists $\xi \in [0,a)$ such that $f(\xi) = \xi$, which implies that $F(a) = F(\xi) = f^n(\xi) = \xi < a$, a contradiction and thus the claimed (2.4) holds. It follows that (2.1) holds, which shows that $f|_{[a,1]}$ is a non-monotonic self-mapping since $F|_{[a,1]}$ is non-monotonic. Note that both $F|_{[a,b]}$ and $F|_{[b,1]}$ are strictly monotonic, implying that $f|_{[a,b]}$ and $f|_{[b,1]}$ are both strictly monotonic. Thus, b is the unique non-monotonic point of $f|_{[a,1]}$.

Secondly we claim that

$$f\bigl((a,b)\bigr) \cap \{b\} = \emptyset \quad \text{and} \quad f\bigl((b,1)\bigr) \cap \{b\} = \emptyset. \tag{2.5}$$

Suppose, on the contrary, that $f((a,b)) \cap \{b\} \neq \emptyset$. Then there exists $t \in (a,b)$ such that $f(t) = b$. Take any neighborhood $U \subset (a,b)$ of t. Since f is strictly monotonic in U, $f(U)$ is a neighborhood of $f(t)$, *i.e.* $f(U)$ is a neighborhood of b. Thus f is not strictly monotonic in $f(U)$ as b is a non-monotonic point of f. In other words, f^2 is not strictly monotonic in U. This contradicts the fact that f^2 is strictly monotonic on $[a,b]$ since $F = f^{n-2} \circ f^2$ is strictly increasing on $[a,b]$. This contradiction shows that the first result of (2.5) holds. Similarly, we can deduce that the second result of (2.5) holds. It follows from (2.5) that either $f([a,b]) \subset [a,b]$ or $f([a,b]) \subset [b,1]$, and either $f([b,1]) \subset [a,b]$ or $f([b,1]) \subset [b,1]$. If $f([a,b]) \subset [a,b]$, then $f([b,1]) \subset [a,b]$. In fact, otherwise, if $f([b,1]) \subset [b,1]$, then $f(b) = b$. Thus $f|_{[a,b]}$ and $f|_{[b,1]}$ are both strictly increasing, which together with (2.1) guarantee that $f|_{[a,1]}$ is a strictly increasing self-mapping. It implies that $F|_{[a,1]} = (f|_{[a,1]})^n$ is also strictly increasing, a contradiction. Thus we have

$$f\bigl([a,1]\bigr) = f\bigl([a,b]\bigr) \cup f\bigl([b,1]\bigr) \subset [a,b].$$

Similarly, if $f([a,b]) \subset [b,1]$, then $f([b,1]) \subset [b,1]$. It implies that

$$f\bigl([a,1]\bigr) = f\bigl([a,b]\bigr) \cup f\bigl([b,1]\bigr) \subset [b,1].$$

Hence, the proof of Lemma 2.2 is completed. □

Lemma 2.3 *Suppose that $F \in S$ with $F(a) \geq a$. Then the following hold*:

(i) *Either $F(I) \subset [a,b]$ or $F(I) \subset [b,1]$ if F has iterative roots of order $n \geq 2$ on I*;

(ii) *F has iterative roots of order $n \geq 2$ on I if and only if $F|_{[a,1]}$ has iterative roots of order $n \geq 2$ on $[a,1]$.*

Proof For result (i), suppose that f is an iterative root of order $n \geq 2$ of F on I. It follows from Lemma 2.2 that either $f([a,1]) \subset [a,b]$ or $f([a,1]) \subset [b,1]$. If $f([a,1]) \subset [a,b]$, then $f([a,b]) \subset [a,b]$, which shows, since $F([0,a]) = \{F(a)\}$, that

$$F(I) = F\bigl([a,1]\bigr) = f^{n-1}\bigl(f\bigl([a,1]\bigr)\bigr) \subset f^{n-1}\bigl([a,b]\bigr) \subset [a,b].$$

If $f([a,1]) \subset [b,1]$, then $f([b,1]) \subset [b,1]$. It follows that

$$F(I) = F\bigl([a,1]\bigr) = f^{n-1}\bigl(f\bigl([a,1]\bigr)\bigr) \subset f^{n-1}\bigl([b,1]\bigr) \subset [b,1].$$

Thus the result (i) is proved.

For result (ii), firstly we prove the sufficiency. Suppose that f_0 is an iterative root of order $n \geq 2$ of $F|_{[a,1]}$ on $[a,1]$. Then, obviously, the function f defined by

$$f(x) := \begin{cases} f_0(a), & \forall x \in [0,a], \\ f_0(x), & \forall x \in (a,1], \end{cases}$$

is an iterative root of order $n \geq 2$ of F on I.

Conversely, let f be an iterative roots of order $n \geq 2$ of F on I. Making use of Lemma 2.2, one obtains that $f|_{[a,1]}$ is a self-mapping. Thus, $F|_{[a,1]} = (f|_{[a,1]})^n$, *i.e.* $f|_{[a,1]}$ is an iterative roots of order $n \geq 2$ of $F|_{[a,1]}$ on $[a,1]$, and necessity is proved. This completes the proof of Lemma 2.3. □

We end this section with Lemma 2.4, which gives some basic results concerning the existence of iterative roots for strictly monotonic self-mappings.

Lemma 2.4 (see [11, Theorems 7 and 10] and [12, pp.119 and 125]) *Let $m, M \in (0,1)$, $m < M$ and $F \in C(I,I)$. Then the following statements are valid*:

(i) *Suppose that F is strictly increasing. Then F has infinitely many strictly increasing iterative roots f of order $n \geq 2$ on I such that $f([m,M]) \subset [F(0),F(1)]$*;
(ii) *Suppose that F is strictly decreasing. Then F has neither strictly increasing iterative roots of order $n \geq 2$ nor strictly decreasing iterative roots of even order $n \geq 2$ on I, F has infinitely many strictly decreasing iterative roots of odd order $n \geq 3$ on I if and only if either $F(I) \subset (0,1)$ or $F(I) = I$. Moreover, if either $F(I) \subset (0,1)$ or $F(I) = I$, then F has infinitely many strictly decreasing iterative roots f of odd order $n \geq 3$ on I such that $f([m,M]) \subset [F(1),F(0)]$.*

3 Iterative roots of $F \in S_1$

In this section, we discuss the existence of iterative roots of $F \in S_1$. It follows from Lemma 2.3 that F has no iterative roots of order $n \geq 2$ if $F(a) \geq a$, $F(I) \not\subset [b,1]$ and $F(I) \not\subset [a,b]$. Thus, we only consider the cases that $F(I) \subset [b,1]$ and $F(I) \subset [a,b]$ if $F(a) \geq a$. Thus we start our discussion with Theorem 3.1.

Theorem 3.1 *Suppose that $F \in S_1$ with $F(I) \subset [b,1]$. Then the following hold*:

(i) *If either $F(1) = 1$ or $F(a) < 1$, then F has infinitely many iterative roots of order $n \geq 2$ on I*;
(ii) *If $F(b) = b$ and $F(a) = 1 > F(1)$, then F has no iterative roots of order $n \geq 2$ on I*;
(iii) *If $F(b) > b$ and $F(a) = 1 > F(1)$, then F has no iterative roots of order $n \geq 3$ on I.*

Proof For result (i), let $m := F(b)$, $M := \max\{F(a),F(1)\}$. Then $[m,M]$ is the range of F on I. If $F(1) < 1$, then, by the assumption, we have $F(a) < 1$, and thus $M < 1$. If $F(1) = 1$, then it is obvious that $M = 1$. With the aid of Lemma 2.4, we obtain that $F|_{[b,1]}$ has infinitely many strictly increasing iterative roots f_* of order $n \geq 2$ on $[b,1]$ such that

$$f_*\big([m,M]\big) \subset \big[F(b),F(1)\big],$$

i.e. f_* maps $[m, M]$ into $[F(b), F(1)]$. Define the function $f : I \to I$ by

$$f(x) := \begin{cases} F|_{[b,1]}^{-1} \circ f_* \circ F(x), & \forall x \in [0, b), \\ f_*(x), & \forall x \in [b, 1]. \end{cases} \tag{3.1}$$

Note that

$$F|_{[b,1]}^{-1} \circ f_* \circ F(b) = F|_{[b,1]}^{-1} \circ F|_{[b,1]} \circ f_*(b) = f_*(b),$$

implying that f defined by (3.1) is continuous. Since

$$\begin{aligned} f^n(x) &= f_*^{n-1} \circ F|_{[b,1]}^{-1} \circ f_* \circ F(x) \\ &= f_*^{n-1} \circ \left(f_*^n\right)^{-1} \circ f_* \circ F(x) = F(x), \quad \forall x \in [0, b), \\ f^n(x) &= f_*^n(x) = F(x), \quad \forall x \in [b, 1], \end{aligned}$$

we find that the function f defined by (3.1) is an iterative root of order $n \geq 2$ of F on I, and the result (i) is proved.

For result (ii), suppose, by an indirect proof, that f is an iterative root of order $n \geq 2$ of F on I. Since $F(a) > a$ and $F(I) \subset [b, 1]$, we have, on account of Lemma 2.2, $f([b, 1]) \subset [b, 1]$, which shows that $f|_{[b,1]}$ is a strictly monotonic iterative root of order $n \geq 2$ of $F|_{[b,1]}$. The fact that $b = F(b) < F(1) < 1$ guarantees that $f|_{[b,1]}$ is strictly increasing. In fact, if $f|_{[b,1]}$ is strictly decreasing, then at least one of $f(b) = 1$ and $f(1) = b$ cannot hold. Without loss of generality, we may assume that $b \leq f(1) < f(b) < 1$. Consequently,

$$F(b) = f^{n-1}\big(f(b)\big) \in f^{n-1}\big([b, 1)\big) \subset f^{n-2}\big((b, 1)\big) \subset (b, 1).$$

This contradiction shows that $f|_{[b,1]}$ is strictly increasing. Hence, $f(b) = b$ and $f(1) < 1$ since $F(b) = b$ and $F(1) < 1$. Noting by Lemma 2.2 again that $f([a, 1]) \subset [b, 1]$, we infer that $f^{n-1}(a) \in [b, 1]$ which shows that

$$F(a) = f\big(f^{n-1}(a)\big) \leq f(1) < 1. \tag{3.2}$$

This contradiction completes the proof of result (ii).

For result (iii), assume by indirect proof that f is an iterative root of order $n \geq 3$ of F on I. Similarly, $f|_{[b,1]}$ is a strictly monotonic iterative root of order $n \geq 3$ of $F|_{[b,1]}$. Lemma 2.2 and the fact $F(1) < 1$ force that

$$f\big([a, 1]\big) \subset [b, 1] \quad \text{and} \quad f(1) < 1. \tag{3.3}$$

If $f(b) < 1$, then we deduce, by the monotonicity of $f|_{[b,1]}$, that

$$f(x) < 1, \quad \forall x \in [b, 1],$$

which with (3.3) shows that (3.2) holds, a contradiction.

Figure 3 F_1.

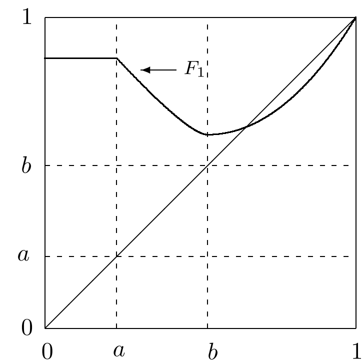
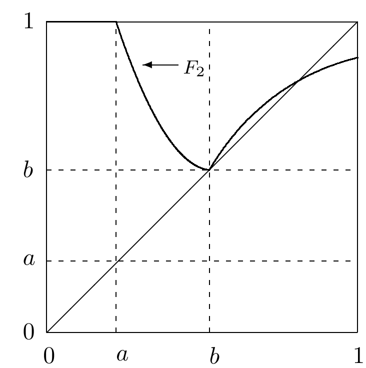


Figure 4 F_2.



If $f(b) = 1$, then $f|_{[b,1]}$ is strictly decreasing. Note that $F(1) < 1$, implying that $f(1) > b$. It follows from (3.3) that

$$\begin{aligned} F(a) &= f^{n-1}(f(a)) \in f^{n-1}([b,1]) = f^{n-2}(f([b,1])) \subset f^{n-2}((b,1]) \\ &= f^{n-3}(f((b,1])) \subset f^{n-3}((b,1)) \subset (b,1), \end{aligned}$$

contradicting to the assumption. Thus, the result (iii) is proved, and this completes the proof of Theorem 3.1. □

Example 3.1 Consider $F_1 : I \to I$ and $F_2 : I \to I$ given by Figures 3 and 4, respectively.

It follows from Theorem 3.1 that F_1 has iterative roots of order $n \geq 2$ on I but F_2 has no iterative roots of order $n \geq 2$ on I.

Theorem 3.2 *Suppose that $F \in S_1$ with $F(I) \subset [a,b]$. Then the following statements are valid*:

(i) *F has no iterative roots of even order $n \geq 2$ on I*;
(ii) *F has infinitely many iterative roots of odd order $n \geq 3$ on I if and only if either $F([a,b]) = [a,b]$ or $F(I) \subset (a,b)$.*

Proof For result (i), suppose, for an indirect proof, that f is an iterative root of even order $n \geq 2$ of F on I. By Lemma 2.2, we see that

$$f([a,b]) \subset f([a,1]) \subset [a,b],$$

which implies that $(f|_{[a,b]})^n = F|_{[a,b]}$, *i.e.* $f|_{[a,b]}$ is an iterative root of even order $n \geq 2$ of $F|_{[a,b]}$. However, as a strictly decreasing function, we have, according to Lemma 2.4, that $F|_{[a,b]}$ has no iterative roots of even order. This contradiction completes the proof of result (i).

For result (ii), firstly we prove the sufficiency. Let $m := F(b)$ and $M := \max\{F(a), F(1)\}$. Then $[m, M]$ is the range of F on I, and either $a = m < M = b$ or $a < m < M < b$. It follows from Lemma 2.4 that $F|_{[a,b]}$ has infinitely many strictly decreasing iterative root g_* of odd order $n \geq 3$ such that $g_*([m, M]) \subset [F(b), F(a)]$. Now we define the function $g : I \to I$ by

$$g(x) := \begin{cases} g_*(x), & \forall x \in [a,b], \\ F|_{[a,b]}^{-1} \circ g_* \circ F(x), & \forall x \in I \setminus [a,b]. \end{cases} \tag{3.4}$$

It is easy to see that $g : I \to I$ defined by (3.4) is an iterative root of odd order $n \geq 3$ of F on I and the sufficiency is proved.

In what follows we prove necessity. Suppose that f is an iterative root of odd order n of F on I. It follows from Lemma 2.2 that

$$f\big([a,b]\big) \subset f\big([a,1]\big) \subset [a,b]. \tag{3.5}$$

Thus $f|_{[a,b]}$ is a strictly decreasing iterative root of odd order $n \geq 3$ of $F|_{[a,b]}$. By Lemma 2.4 again, we obtain either $F([a,b]) = [a,b]$ or $F([a,b]) \subset (a,b)$.

If $F([a,b]) \subset (a,b)$, which shows that at least one of $f(a) = b$ and $f(b) = a$ does not hold. Without loss of generality we may assume that $a < f(b) < f(a) \leq b$. It follows from (3.5) and the monotonicity of $f|_{[a,b]}$ that

$$\begin{aligned} F(1) &= f^{n-1}\big(f(1)\big) \in f^{n-1}\big([a,b]\big) = f^{n-2}\big(f\big([a,b]\big)\big) \\ &\subset f^{n-2}\big((a,b]\big) = f^{n-3}\big(f\big((a,b]\big)\big) \subset f^{n-3}\big((a,b)\big) \subset (a,b). \end{aligned}$$

Thus, $F(I) = [F(b), \max\{F(a), F(1)\}] \subset (a,b)$. The necessity is proved and the proof of Theorem 3.2 is completed. □

Example 3.2 The functions $F_3 : I \to I$ and $F_4 : I \to I$ are given by Figures 5 and 6, respectively.

By Theorem 3.2, we find that F_3 has exactly iterative roots of odd order $n \geq 3$ on I but F_4 has no iterative roots of order $n \geq 2$ on I.

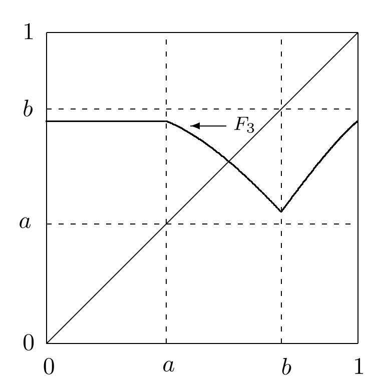


Figure 5 F_3.

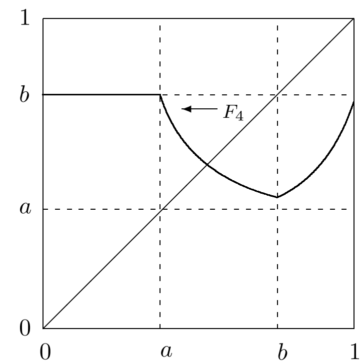


Figure 6 F_4.

Theorem 3.3 *Suppose that $F \in S_1$ with $F(a) < a$. Then F has infinitely many iterative roots of order $n \geq 2$.*

Proof Choose points $x_1, \dots, x_{2n-3}$ from $(F(a), a)$, arbitrarily, such that

$$x_0 := F(a) < x_1 < \cdots < x_{2n-3} < x_{2n-2} := a. \tag{3.6}$$

Put $I_k := (x_{2k-2}, x_{2k-1}]$, $I_k^* := (x_{2k-1}, x_{2k}]$ for integer $k \geq 1$. Further, we define $f_1 : I_1 \to [F(b), x_0)$ is a decreasing homeomorphism and $f_1^* : I_1^* \to (F(b), x_0]$ is an increasing homeomorphism, arbitrarily. If $n > 2$, then we give the increasing homeomorphisms $f_k : I_k \to I_{k-1}$ and $f_k^* : I_k^* \to I_{k-1}^*$ for $k = 2, \dots, n-1$, arbitrarily. Now define the function g_1 on $[0, a]$ by

$$g_1(x) := \begin{cases} x_0, & \forall x \in [0, x_0], \\ f_k(x), & \forall x \in I_k, k = 1, \dots, n-1, \\ f_k^*(x), & \forall x \in I_k^*, k = 1, \dots, n-1, \end{cases} \tag{3.7}$$

which is continuous on $[0, a]$. Note that

$$g_1^k(x) \in \bigl[F(b), x_0\bigr] \subset [0, x_0], \quad \forall x \in I_k \cup I_k^*, k = 1, \dots, n-1,$$

implying that $g_1^n([0, a]) = \{x_0\}$. In what follows we discuss several cases:

In the case that $F(1) \leq F(a)$: Let the functions g_2 and g_3 be defined, respectively, by

$$g_2(x) := f_{n-1}^{-1} \circ f_{n-2}^{-1} \circ \cdots \circ f_1^{-1} \circ F(x), \quad \forall x \in (a, b] \quad \text{and}$$
$$g_3(x) := \bigl(f_{n-1}^*\bigr)^{-1} \circ \bigl(f_{n-2}^*\bigr)^{-1} \circ \cdots \circ \bigl(f_1^*\bigr)^{-1} \circ F(x), \quad \forall x \in (b, 1].$$

Since $f_{n-1}^{-1} \circ f_{n-2}^{-1} \circ \cdots \circ f_1^{-1} : [F(b), x_0) \to I_{n-1}$ is a decreasing homeomorphism, $(f_{n-1}^*)^{-1} \circ (f_{n-2}^*)^{-1} \circ \cdots \circ (f_1^*)^{-1} : (F(b), x_0] \to I_{n-1}^*$ is an increasing homeomorphism and $F((b,1]) \subset (F(b), x_0]$, we find that $g_2 : (a, b] \to I_{n-1}$ is an increasing homeomorphism, and $g_3 : (b, 1] \to I_{n-1}^*$ is a strictly increasing function. Moreover,

$$g_1(a) = x_{2n-4} = \lim_{x \to a+} g_2(x) \quad \text{and} \quad g_2(b) = x_{2n-3} = \lim_{x \to b+} g_3(x).$$

Thus the mapping $g : I \to I$ defined by

$$g(x) := \begin{cases} g_1(x), & \forall x \in [0,a], \\ g_2(x), & \forall x \in (a,b], \\ g_3(x), & \forall x \in (b,1], \end{cases}$$

is an iterative root of order $n \geq 2$ of F on I.

In the case that $F(1) > F(a)$ and F has no fixed points on $[b,1]$: Define

$$x_{2n-1} := b, \qquad x_{2n+k} := F|_{[b,1]}^{-1}(x_k) \tag{3.8}$$

inductively for all those integer $k \geq 0$ for which the recurrence procedure (3.8) is performable. Note that F has no fixed points on $[b,1]$, implying that there is an integer $k_0 \geq 0$ such that

$$x_{k_0} < F(1) \leq x_{k_0+1}. \tag{3.9}$$

In fact, if $x_k < F(1)$ for all integers $k \geq 0$, then, by (3.6), (3.8), and the monotonicity of $F|_{[b,1]}$, we infer that the sequence $\{x_k\}$ is infinite and strictly increasing. Therefore, by (3.8), $\lim_{k\to+\infty} x_k (\in [b,1])$ is a fixed point of F on $[b,1]$, a contradiction. It follows from (3.9) that x_{2n+k_0} exists and $x_{2n+k_0} < 1$. Let $x_{2n+k_0+1} := 1$. Without loss of generality we may assume that k_0 is odd, and we let $k_0 := 2k_0' + 1$ for some integer $k_0' \geq 0$. For $k = n, n+1, \ldots, n+k_0'+1$, define f_k on I_k and f_k^* on I_k^* inductively by

$$f_k(x) := f_{k-1}^{-1} \circ f_{k-2}^{-1} \circ \cdots \circ f_{k-(n-1)}^{-1} \circ F(x), \quad \forall x \in I_k \quad \text{and} \tag{3.10}$$

$$f_k^*(x) := \left(f_{k-1}^*\right)^{-1} \circ \left(f_{k-2}^*\right)^{-1} \circ \cdots \circ \left(f_{k-(n-1)}^*\right)^{-1} \circ F(x), \quad \forall x \in I_k^*, \tag{3.11}$$

respectively. Noting that $F|_{I_n} : I_n \to [F(b), x_0)$ is decreasing homeomorphism, we find that $f_n : I_n \to I_{n-1}$ is an increasing homeomorphism. Fix an $k \in \{n, n+1, \ldots, n+k_0'\}$ and suppose that the function $f_i : I_i \to I_{i-1}$ is an increasing homeomorphism for $i = k, k-1, \ldots, k-n+2$. Thus $f_k^{-1} \circ f_{k-1}^{-1} \circ \cdots \circ f_{k-n+2}^{-1} : I_{k-n+1} \to I_k$ is an increasing homeomorphism. Note that $F : I_{k+1} \to I_{k-n+1}$ is an increasing homeomorphism, implying by (3.10) that the function $f_{k+1} : I_{k+1} \to I_k$ is an increasing homeomorphism. By induction, we deduce that the function $f_k : I_k \to I_{k-1}$ defined by (3.10) is an increasing homeomorphism for $k = n, n+1, \ldots, n+k_0'+1$. Similarly, we see that the function $f_k^* : I_k^* \to I_{k-1}^*$ defined by (3.11) is also an increasing homeomorphism for $k = n, n+1, \ldots, n+k_0'$. Since

$$F\left(I_{n+k_0'+1}^*\right) = \left[F(x_{2n+2k_0'+1}), F(1)\right] \subset [x_{2k_0'+1}, x_{2k_0'+2}] = I_{k_0'+1}^*,$$

$f_{n+k_0'+1}^*$ is continuous and strictly increasing on $I_{n+k_0'+1}^*$. Define $f : I \to I$ by

$$f(x) := \begin{cases} g_1(x), & \forall x \in [0,a], \\ f_k(x), & \forall x \in I_k, k = n, n+1, \ldots, n+k_0'+1, \\ f_k^*(x), & \forall x \in I_k^*, k = n, n+1, \ldots, n+k_0'+1. \end{cases} \tag{3.12}$$

It follows from (3.7), (3.10), and (3.11) that $f^n(x) = F(x)$ for all $x \in I$, *i.e.* the mapping $f : I \to I$, defined by (3.12), is an iterative root of order $n \geq 2$ of F on I.

In the case that $F(1) > F(a)$ and F has fixed points on $[b,1]$: Let $\xi := \min\{x : F(x) = x, x \in [b,1]\}$. Then $b < \xi$ since $F(b) < a < b$. There is no loss of generality in assuming that $\xi < 1$. Because $F|_{[\xi,1]}$ is a strictly increasing self-mapping, and in view of Lemma 2.4, $F|_{[\xi,1]}$ has infinitely many strictly increasing iterative roots g_1^* of order $n \geq 2$ such that $g_1^*(\xi) = \xi$. Meanwhile, the infinite sequence $\{x_k\}$ defined by (3.6) and (3.8) is strictly increasing and $\lim_{k\to\infty} x_k = \xi$. Now we define a mapping $f : I \to I$ by

$$f(x) := \begin{cases} g_1(x), & \forall x \in [0,a], \\ f_k(x), & \forall x \in I_k, k = n, n+1, \dots, \\ f_k^*(x), & \forall x \in I_k^*, k = n, n+1, \dots, \\ g_1^*(x), & \forall x \in [\xi,1], \end{cases} \tag{3.13}$$

where f_k on I_k (resp. f_k^* on I_k^*) is defined inductively by (3.10) (resp. (3.11)). As we have just seen in the preceding case, it is easy to check that the mapping $f : I \to I$ defined by (3.13) is an iterative root of order $n \geq 2$ of F on I, and the above discussion completes the proof of Theorem 3.3. □

Example 3.3 Consider the functions $F_5 : I \to I$ and $F_6 : I \to I$ given by Figures 7 and 8, respectively.

We know by Lemma 2.3 that F_5 has no iterative roots of order $n \geq 2$ on I. However, Theorem 3.3 shows that F_6 has iterative roots of order $n \geq 2$ on I.

Figure 7 F_5.

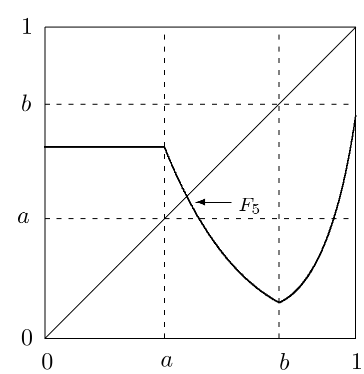
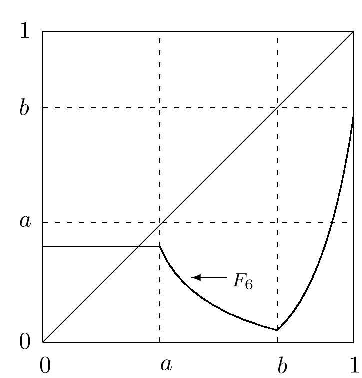


Figure 8 F_6.



4 Iterative roots of $F \in S_2$

In this section we shall discuss the existence of iterative roots of $F \in S_2$. Making use of Lemma 2.3, we find that F has no iterative roots of order $n \geq 2$ if $F(a) \geq a$, $F(I) \not\subset [b,1]$ and $F(I) \not\subset [a,b]$. Thus, we only discuss the two cases $F(I) \subset [a,b]$ and $F(I) \subset [b,1]$ if $F(a) \geq a$. If $F([a,1]) \subset [a,1]$, then, according to Lemma 2.3, it suffices to discuss the existence of iterative roots of $F|_{[a,1]}$. Let $G(x) := g^{-1} \circ F \circ g(x)$ for all $x \in [a,1]$, where $g : [a,1] \to [a,1]$ is defined by

$$g(x) := \begin{cases} \frac{1-b}{a-b}(x-b) + b, & \forall x \in [a,b), \\ \frac{a-b}{1-b}(x-b) + b, & \forall x \in [b,1]. \end{cases}$$

Thus, $G([a,1]) \subset [a,1]$ and G is strictly decreasing on $[a,b]$ but is strictly increasing on $[b,1]$ if $F([a,1]) \subset [a,1]$. By Lemma 2.3 and Theorems 3.1 and 3.2, we obtain immediately Theorems 4.1 and 4.2.

Theorem 4.1 *Suppose that $F \in S_2$ with $F(I) \subset [b,1]$. Then F has no iterative roots of even order $n \geq 2$ on I, and F has infinitely many iterative roots of odd order $n \geq 3$ if and only if either $F(I) \subset (b,1)$ or $F([b,1]) = [b,1]$.*

Theorem 4.2 *Suppose that $F \in S_2$ with $F(I) \subset [a,b]$. Then the following statements are valid*:

(i) *If either $F(a) = a$ or $F(1) > a$, then F has infinitely many iterative roots of order $n \geq 2$ on I;*

(ii) *If $F(b) = b$ and $F(a) > F(1) = a$, then F has no iterative roots of order $n \geq 2$ on I;*

(iii) *If $F(b) < b$ and $F(a) > F(1) = a$, then F has no iterative roots of order $n \geq 3$ on I.*

Theorem 4.3 *Suppose that $F \in S_2$ with $F(a) < a$. Then F has infinitely many iterative roots of order $n \geq 2$ if and only if F satisfies one of the following conditions*:

(i) $F(b) < a$;

(ii) $a \leq F(b) \leq b$ *and* $F(a) \leq F(1)$.

Proof For sufficiency, firstly suppose that F satisfies the condition (i). Choose points $a_0, a_1, \ldots, a_{3n-2}$ from $[F(a), a]$ such that

$$F(a) = a_0 < F(b) = a_1 < a_2 < \cdots < a_{3n-2} = a.$$

Write $A_k := (a_k, a_{k+1}]$ for integers $k \geq 0$. Let $\alpha_1 : A_1 \to A_0$ and $\alpha_3 : A_3 \to (F(1), a_1]$ be both arbitrary increasing homeomorphisms, and let $\alpha_2 : A_2 \to [F(1), a_1)$ be an arbitrary decreasing homeomorphism. If $n > 2$, then we take increasing homeomorphisms $\alpha_k : A_k \to A_{k-3}$ for $k = 4, 5, \ldots, 3n-3$, arbitrarily. Define $\psi_1 : [0,a] \to [0,a]$ by

$$\psi_1(x) := \begin{cases} a_0, & \forall x \in [0, a_1], \\ \alpha_k(x), & \forall x \in A_k, k = 1, 2, \ldots, 3n-3, \end{cases}$$

which clearly is continuous on $[0,a]$ and $\psi_1(x) \in [0, a_1]$ for all $x \in [0, a_4]$. Thus we see that

$$\psi_1^2(x) = a_0, \quad \forall x \in [0, a_4]. \tag{4.1}$$

Take any $x \in [a_4, a]$. Then there exists an $k \in \{4, 5, \dots, 3n-3\}$ such that $x \in A_k$. Writing $k = 3k_1 + k_2$, where $k_1 \in \{1, 2, \dots, n-2\}$ and $k_2 \in \{1, 2, 3\}$, we get

$$\psi_1^{k_1}(x) = \alpha_{k-3(k_1-1)} \circ \cdots \circ \alpha_{k-3} \circ \alpha_k(x) \in A_{k-3k_1} = A_{k_2} \subset [0, a_4]. \tag{4.2}$$

In virtue of (4.1) and (4.2), we obtain

$$\psi_1^n(x) = a_0, \quad \forall x \in [0, a], \tag{4.3}$$

i.e. ψ_1 is an iterative root of order $n \geq 2$ of $F|_{[0,a]}$. Define the functions ψ_2 on $(a, b]$ and ψ_3 on $(b, 1]$ by

$$\begin{aligned} &\psi_2(x) := \alpha_{3n-5}^{-1} \circ \cdots \alpha_4^{-1} \circ \alpha_1^{-1} \circ F(x), \quad \forall x \in (a, b] \quad \text{and} \\ &\psi_3(x) := \alpha_{3n-4}^{-1} \circ \cdots \alpha_5^{-1} \circ \alpha_2^{-1} \circ F(x), \quad \forall x \in (b, 1], \end{aligned}$$

respectively. Note that $F((a, b]) = \alpha_1(A_1)$ and $F((b, 1]) = \alpha_2(A_2)$, implying that $\psi_2 : (a, b] \to A_{3n-5}$ and $\psi_3 : (b, 1] \to A_{3n-4}$ are both increasing homeomorphisms. Moreover,

$$\psi_1(a) = a_{3n-5} = \lim_{x \to a+} \psi_2(x) \quad \text{and} \quad \psi_2(b) = a_{3n-4} = \lim_{x \to b+} \psi_3(x).$$

Now we define ψ on I by

$$\psi(x) := \begin{cases} \psi_1(x), & \forall x \in [0, a], \\ \psi_2(x), & \forall x \in (a, b], \\ \psi_3(x), & \forall x \in (b, 1]. \end{cases} \tag{4.4}$$

It is natural that

$$\begin{aligned} &\psi^n(x) = \alpha_1 \circ \alpha_4 \circ \cdots \circ \alpha_{3n-5} \circ \psi_2(x) = F(x), \quad \forall x \in (a, b] \quad \text{and} \\ &\psi^n(x) = \alpha_2 \circ \alpha_5 \circ \cdots \circ \alpha_{3n-4} \circ \psi_3(x) = F(x), \quad \forall x \in (b, 1], \end{aligned}$$

which together with (4.3) imply that the function $\psi : I \to I$ defined by (4.4) is an iterative root of order $n \geq 2$ of F on I.

Secondly assume that F satisfies the condition (ii), *i.e.* $a \leq F(b) \leq b$ and $F(a) \leq F(1)$. Choose points $x_1, \dots, x_{n-1}$ in $(F(a), a)$, arbitrarily, such that

$$F(a) = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = a. \tag{4.5}$$

Let $X_k := (x_k, x_{k+1}]$ for integers $k \geq 0$. We give the increasing homeomorphisms $\beta_k : X_k \to X_{k-1}$ for $k = 1, \dots, n-1$, arbitrarily. Now put

$$x_{n+k} := F^{-1}(x_k), \quad x_{n+k} \in [a, b] \tag{4.6}$$

(which is equivalent to $x_k = F(x_{n+k})$) for all those positive integers k for which the recurrence procedure is performable. Next we shall discuss the two distinguished cases:

In the case that F has no fixed points on $[a,b]$: Then the condition $F(a) < a$ implies that $F(b) < b$. Moreover, there exists a positive integer k_0 such that

$$x_{k_0-1} < F(b) \le x_{k_0}.$$

In fact, otherwise, note that $x_k < F(b)$ for any integer $k \ge n$, implying, by the fact that $F(a) < a \le x_k$ and (4.6), that x_{k+1} exists, which shows that x_k exists for all integers $k \ge 1$ and the sequence $\{x_k\}$ is strictly increasing. It follows from (4.6) that $\lim_{k\to+\infty} x_k$ is a fixed point of F on $[a,b]$, a contradiction. The fact $x_1 < F(b)$ yields $k_0 \ge 2$. There is no loss of generality in assuming that $k_0 \ge 3$. Let $x_{n+k_0} := b$. Define the functions β_k on X_k inductively for $k = n, n+1, \dots, n+k_0-1$ by

$$\beta_k(x) := \beta_{k-1}^{-1} \circ \beta_{k-2}^{-1} \circ \cdots \circ \beta_{k-n+1}^{-1} \circ F(x), \quad \forall x \in X_k. \tag{4.7}$$

Since $F|_{X_n} : X_n \to X_0$ and $\beta_{n-1}^{-1} \circ \beta_{n-2}^{-1} \circ \cdots \circ \beta_1^{-1} : X_0 \to X_{n-1}$ are both increasing homeomorphisms, $\beta_n : X_n \to X_{n-1}$ defined by (4.7) is an increasing homeomorphism. Similarly, by induction, we find that $\beta_k : X_k \to X_{k-1}$ defined by (4.7) is an increasing homeomorphism for $k = n, n+1, \dots, n+k_0-2$. Note that $\beta_{n+k_0-2}^{-1} \circ \beta_{n+k_0-3}^{-1} \circ \cdots \circ \beta_{k_0}^{-1} : X_{k_0-1} \to X_{n+k_0-2}$ is an increasing homeomorphism, and we have

$$F(X_{n+k_0-1}) = \big(x_{k_0-1}, F(b)\big] \subset (x_{k_0-1}, x_{k_0}] = X_{k_0-1},$$

implying that $\beta_{n+k_0-1} : X_{n+k_0-1} \to X_{n+k_0-2}$ is a strictly increasing function. Define ϕ_1 on $[x_1, b]$ by

$$\phi_1(x) := \begin{cases} x_0, & x = x_1, \\ \beta_k(x), & \forall x \in X_k, k = 1, 2, \dots, n+k_0-1. \end{cases}$$

It is natural that ϕ_1 is continuous and strictly increasing on $[x_1, b]$. Moreover,

$$\big[F(1), F(b)\big) \subset \big[F(a), x_{k_0}\big) = [x_0, x_{k_0}) = \phi_1\big([x_1, x_{k_0+1})\big).$$

Put $J_0 := [F(1), F(b))$ and define $J_k := \phi_1^{-1}(J_{k-1})$ inductively for $k = 1, 2, \dots, n-1$. Then $J_k \subset [x_k, x_{k+k_0})$ for $k = 0, 1, \dots, n-1$. Let the function ϕ_2 on $(b, 1]$ be defined by

$$\phi_2(x) := \phi_1|_{J_{n-1}}^{-1} \circ \phi_1|_{J_{n-2}}^{-1} \circ \cdots \circ \phi_1|_{J_1}^{-1} \circ F(x), \quad \text{for } x \in (b, 1].$$

Because $F : (b, 1] \to J_0$ is a decreasing homeomorphism and $\phi_1|_{J_k}^{-1} : J_{k-1} \to J_k$ is an increasing homeomorphism for $k = 1, 2, \dots, n-1$, we infer that $\phi_2 : (b, 1] \to J_{n-1}$ is a decreasing homeomorphism. By the definition of ϕ_1, we possess that $\phi_1|_{X_k}^{-1} (= \beta_k^{-1}) : X_{k-1} \to X_k$ is an increasing homeomorphism for $k = 1, 2, \dots, n+k_0-1$. On the other hand, noting that $F(b) \in (x_{k_0-1}, x_{k_0}]$ and $F|_{[b,1]}$ is strictly decreasing, we have $b_* \in (b, 1)$ such that

$$F\big((b, b_*)\big) = \big(F(b_*), F(b)\big) \subset J_0 \cap (x_{k_0-1}, x_{k_0}) \subset J_0 \cap X_{k_0-1}.$$

Thus

$$\begin{aligned}\phi_2(x) &:= \phi_1|_{J_{n-1}}^{-1} \circ \phi_1|_{J_{n-2}}^{-1} \circ \cdots \circ \phi_1|_{J_1}^{-1} \circ F(x) \\ &= \phi_1|_{X_{n+k_0-2}}^{-1} \circ \phi_1|_{X_{n+k_0-3}}^{-1} \circ \cdots \circ \phi_1|_{X_{k_0}}^{-1} \circ F(x) \\ &= \beta_{n+k_0-2}^{-1} \circ \beta_{n+k_0-3}^{-1} \circ \cdots \circ \beta_{k_0}^{-1} \circ F(x), \quad \forall x \in (b, b_*).\end{aligned}$$

This shows that

$$\lim_{x\to b+} \phi_2(x) = \beta_{n+k_0-2}^{-1} \circ \beta_{n+k_0-3}^{-1} \circ \cdots \circ \beta_{k_0}^{-1} \circ F(b) = \phi_1(b). \tag{4.8}$$

Define $\phi : I \to I$ by

$$\phi(x) := \begin{cases} x_0, & \forall x \in [0, x_1), \\ \phi_1(x), & \forall x \in [x_1, b], \\ \phi_2(x), & \forall x \in (b, 1], \end{cases} \tag{4.9}$$

which is continuous since $\lim_{x\to x_1-} \phi(x) = x_0 = \phi_1(x_1)$ and (4.8). By the definition of ϕ_1 and ϕ_2, we see that $\phi : I \to I$ defined by (4.9) is an iterative root of order $n \geq 2$ of F on I.

In the case that F has fixed points on $[a, b]$: Let $\xi = \min\{x : F(x) = x, x \in [a, b]\}$. Then x_k is well defined inductively by (4.6) for all integers $k \geq n$ and $\lim_{k\to\infty} x_k = \xi$. Without loss of generality, assume that $\xi < b$. Let $\beta_k : X_k \to X_{k+1}$ be defined by (4.7) for all integers $k \geq n$. If $x \in (a, \xi)$, then there exists X_k such that $x \in X_k$, which implies

$$x_{k-1} = \beta_k(x_k) < \beta_k(x) \leq \beta_k(x_{k+1}) = x_k.$$

Noting that $x \to \xi$ is equivalent to $k \to \infty$, we get the conclusion that

$$\lim_{k\to\infty} x_{k-1} \leq \lim_{x\to\xi-0} \beta_k(x) \leq \lim_{k\to\infty} x_k.$$

Hence,

$$\lim_{x\to\xi-0} \beta_k(x) = \xi. \tag{4.10}$$

On the other hand, since $F|_{[\xi,b]}$ is a strictly increasing self-mapping and $F(\xi) = \xi$, we obtain, in view of Lemma 2.4, that $F|_{[\xi,b]}$ has infinitely many strictly increasing iterative roots β_* of order $n \geq 2$ on $[\xi, b]$ such that

$$\beta_*(\xi) = \xi. \tag{4.11}$$

Define the function ϕ_3 on $(x_1, b]$ by

$$\phi_3(x) := \begin{cases} \beta_k(x), & \forall x \in X_k, k = 1, 2, \ldots, \\ \beta_*(x), & \forall x \in [\xi, b]. \end{cases} \tag{4.12}$$

By the definition of β_k and β_*, (4.10) and (4.11), we see that the function ϕ_3 on $(x_1, b]$ defined by (4.12) is continuous and strictly increasing. Let $\mathcal{J}_0 := [F(1), F(b))$. $\mathcal{J}_k := \phi_3^{-1}(\mathcal{J}_{k-1})$

is well defined inductively for $k = 1, 2, \dots, n-1$. Thus $\phi_3|_{\mathcal{J}_k}^{-1} : \mathcal{J}_{k-1} \to \mathcal{J}_k$ is an increasing homeomorphism for $k = 1, 2, \dots, n-1$. Then the function ϕ_4 on $(b, 1]$ given by

$$\phi_4(x) := \phi_3|_{\mathcal{J}_{n-1}}^{-1} \circ \phi_3|_{\mathcal{J}_{n-2}}^{-1} \circ \cdots \circ \phi_3|_{\mathcal{J}_1}^{-1} \circ F(x), \quad \forall x \in (b, 1], \tag{4.13}$$

is a decreasing homeomorphism from $(b, 1]$ to $\mathcal{J}_{n-1}$. By the definition of ϕ_3 and β_*, we deduce that, for $k = 1, 2, \dots, n-1$,

$$\phi_3|_{\mathcal{J}_k}^{-1}(y) = \phi_3|_{(\xi,b)}^{-1}(y) = \beta_*^{-1}(y), \quad \forall y \in \beta_*\big((\xi, b)\big) \cap \mathcal{J}_{k-1}. \tag{4.14}$$

Choose $b^* \in (b, 1)$ satisfying

$$F\big((b, b^*)\big) \subset \big(\xi, F(b)\big) \cap \mathcal{J}_0 = \beta_*^n\big((\xi, b)\big) \cap \mathcal{J}_0.$$

Consequently, for $k = 1, 2, \dots, n-1$,

$$\begin{aligned} \beta_*^{-k+1} \circ F(x) &= \phi_3^{-k+1} \circ F(x) \in \phi_3^{-k+1}\big(\big(\xi, F(b)\big) \cap \mathcal{J}_0\big) \\ &\subset \beta_*^{n-k+1}\big((\xi, b)\big) \cap \mathcal{J}_{k-1} \subset \beta_*\big((\xi, b)\big) \cap \mathcal{J}_{k-1}, \quad \forall x \in \big(b, b^*\big). \end{aligned}$$

It follows from (4.14) that

$$\phi_3|_{\mathcal{J}_k}^{-1}\big(\beta_*^{-k+1} \circ F(x)\big) = \beta_*^{-k} \circ F(x), \quad \forall x \in \big(b, b^*\big), k = 1, 2, \dots, n-1,$$

which jointly with (4.13) guarantees that

$$\phi_4(x) = \beta_*^{-n+1} \circ F(x), \quad \forall x \in \big(b, b^*\big),$$

showing that

$$\lim_{x \to b+} \phi_4(x) = \lim_{x \to b+} \beta_*^{-n+1} \circ F(x) = \beta_*^{-n+1} \circ F(b) = \beta_*(b) = \phi_3(b).$$

In addition, $\lim_{x \to x_1+} \phi_3(x) = x_0$. Consequently, the mapping $\tilde{\phi} : I \to I$ defined by

$$\tilde{\phi}(x) := \begin{cases} x_0, & \forall x \in [0, x_1], \\ \phi_3(x), & \forall x \in (x_1, b], \\ \phi_4(x), & \forall x \in (b, 1], \end{cases} \tag{4.15}$$

is continuous. According to (4.12) and (4.13), the mapping $\tilde{\phi} : I \to I$ defined by (4.15) is an iterative root of order $n \geq 2$ of F on I. Consequently, the sufficiency is proved.

To prove necessity, suppose that f is an iterative root of order $n \geq 2$ of F on I. It follows from Lemma 2.2 that

$$f\big([0, a]\big) \subset [0, a], \tag{4.16}$$

which implies that f has the fixed points on $[0, a]$. Note that the fixed point of f is bound to the fixed point of F and $F(a)$ is the unique fixed point of F on $[0, a]$, implying that $F(a)$

is the unique fixed point of f on $[0,a]$. It follows that

$$f(a) < a, \qquad f\big(F(a)\big) = F(a). \tag{4.17}$$

Since $f^n([0,a]) = F([0,a]) = \{F(a)\}$ and $f^0([0,a]) = [0,a]$, there exists $t_0 \in \{0,1,\dots,n-1\}$ such that $f^{t_0}([0,a]) \neq \{F(a)\}$ but $f^{t_0+1}([0,a]) = \{F(a)\}$. By the continuity of f^{t_0}, there are $p,q \in [0,a]$ such that $p < q$ and $[p,q] = f^{t_0}([0,a])$. Thus the definition of t_0 and (4.17) yields

$$F(a) \in [p,q] \quad \text{and} \quad f(x) = F(a), \quad \forall x \in [p,q]. \tag{4.18}$$

We claim that

$$f\big([0,a]\big) \cap f\big((a,b]\big) = \emptyset. \tag{4.19}$$

Otherwise, if there exist $x' \in [0,a]$ and $x'' \in (a,b]$ such that $f(x') = f(x'')$, then $F(x') = f^n(x') = f^n(x'') = F(x'')$, which contradicts to the fact that $F(x') = F(a) < F(x'')$. Therefore, as claimed (4.19) holds. On the other hand, $f|_{[a,b]}$ and $f|_{[b,1]}$ are both strictly monotonic since $F|_{[a,b]}$ and $F|_{[b,1]}$ are both strictly monotonic. In what follows we shall distinguish the following several cases:

In the case that $f(a) < F(a)$: Suppose for indirect proof that $F(b) \geq a$. Then there exists $b' \in (a,b]$ such that $F(b') = a$ since $F(a) < a$. On the other hand, we have, in virtue of (4.17) and (4.19),

$$\big[f(a), F(a)\big] \cap f\big((a,b]\big) \subset f\big(\big[F(a),a\big]\big) \cap f\big((a,b]\big) \subset f\big([0,a]\big) \cap f\big((a,b]\big) = \emptyset.$$

It follows that $f|_{[a,b]}$ is strictly decreasing. Thus

$$f(a) > f(x), \quad \forall x \in (a,b]. \tag{4.20}$$

By (4.17) and (4.20), we get $f(b') < f(a) < a$. Thus

$$f(a) = f\big(F(b')\big) = f\big(f^n(b')\big) = f^n\big(f(b')\big) = F\big(f(b')\big) = F(a),$$

contradicting to the assumption that $f(a) < F(a)$. This contradiction shows that $F(b) < a$ holds, *i.e.* F satisfies the condition (i).

In the case that $f(a) = F(a)$: Firstly we prove that $f|_{[a,b]}$ is strictly decreasing. In fact, if $f|_{[a,b]}$ is strictly increasing, then, by (4.19), we infer that

$$f\big([0,a]\big) \cap \big(f(a), f(b)\big] = f\big([0,a]\big) \cap f\big((a,b]\big) = \emptyset.$$

This forces $f(x)$ to obey

$$f(x) \leq f(a), \quad \forall x \in [0,a], \tag{4.21}$$

which with (4.16) implies that

$$f^{n-1}(x) = f\big(f^{n-2}(x)\big) \leq f(a), \quad \forall x \in [0,a]. \tag{4.22}$$

Note that $(f(a),f(b)] \cap (f(a),a] \neq \emptyset$, implying that there exists $t' \in (a,b]$ such that $f(t') \leq a$, which with (4.22) results in

$$F(t') = f^{n-1}(f(t')) \leq f(a) = F(a),$$

contradicting to the fact that $F(t') > F(a)$. Thus, $f|_{[a,b]}$ is strictly decreasing. It follows from (4.17) that

$$f(b) < f(a) < a, \tag{4.23}$$

which with (4.16) means that

$$F(b) = f^n(b) \leq a. \tag{4.24}$$

Based on (4.19), we have

$$f([0,a]) \cap [f(b),f(a)) = f([0,a]) \cap f((a,b]) = \emptyset,$$

which with (4.16) shows that

$$f([0,a]) \subset [f(a),a]. \tag{4.25}$$

Moreover, we have

$$f(1) \leq a. \tag{4.26}$$

In fact, if $f(1) > a$, then, by (4.23), $f|_{[b,1]}$ is strictly increasing. Note that $F(a) = f(a) \in (f(b),f(1))$, implying by (4.18) that $[p,q] \cap (f(b),f(1)) \neq \emptyset$. Thus there exist $s,t \in (b,1)$ such that $s < t$ and $f([s,t]) = [p,q] \cap (f(b),f(1))$. It follows thus from (4.18) that

$$F([s,t]) = f^n([s,t]) = f^{n-1} \circ f([s,t]) \subset f^{n-1}([p,q]) = \{f^{n-2}(F(a))\},$$

i.e. F is constant on $[s,t]$. This contradiction shows that the claimed (4.26) holds. Making use of (4.25) and (4.26), one obtains

$$F(1) = f^n(1) = f^{n-1}(f(1)) \in f^{n-1}([0,a]) \subset f^{n-2}([f(a),a]) \subset [f(a),a] = [F(a),a],$$

which with (4.24) guarantees that F satisfies one of conditions (i) and (ii).

In the case that $f(a) > F(a)$: It is natural by (4.17) and (4.19) that

$$[F(a),f(a)] \cap f((a,b]) \subset f([0,a]) \cap f((a,b]) = \emptyset.$$

Thus $f|_{[a,b]}$ is strictly increasing. Due to (4.19), we have $f([0,a]) \cap (f(a),f(b)] = \emptyset$. It follows that (4.21) holds.

If $f(b) \leq a$, then (4.16) and (4.21) imply that $F(b) = f(f^{n-1}(b)) \leq f(a) < a$, *i.e.*, F satisfies the condition (i).

If $f(b) > a$, by the continuity and monotonicity of $f|_{[a,b]}$ and (4.17), there is an $a_{-1} \in (a,b)$ such that $f(a_{-1}) = a$. Let $a_k := f(a_{k-1})$ be well defined inductively for $k = 0,1,\dots,n$. It follows thus from (4.17) that $a_1 < a$. Since $F|_{[a,a_{-1}]}$ $(= f^{n-2} \circ f|_{[a_1,a]} \circ f|_{[a,a_{-1}]})$ is strictly increasing, we see that $f|_{[a_1,a]}$ is strictly monotonic, which together with (4.21) implies that $f|_{[a_1,a]}$ is strictly increasing. Fix a $k \in \{0,1,\dots,n-2\}$, arbitrarily. Assume that $f|_{[a_i,a_{i-1}]}$ is strictly increasing for each $i \in \{0,1,2,\dots,k\}$. Noting that $f((a_{-1},b)) \cap (a,a_{-1}] \neq \emptyset$, there is an $a_* \in (a_{-1},b)$ such that $f(a_*) \in (a,a_{-1}]$. Whence, by the induction hypothesis, we infer that

$$f|_{[a_k,a_{k-1}]} \circ f|_{[a_{k-1},a_{k-2}]} \circ \cdots \circ f|_{[a,a_{-1}]} \circ f|_{[a_{-1},a_*]}$$

is strictly increasing, *i.e.* $f^{k+2}|_{[a_{-1},a_*]}$ is strictly increasing. Because $F|_{[a,a_*]}$ $(= f^{n-k-2} \circ f^{k+2}|_{[a,a_*]})$ is strictly increasing, we see that $f^{k+2}|_{[a,a_*]}$ is strictly monotonic. It follows that $f^{k+2}|_{[a,a_{-1}]}$ is strictly increasing. Thus, by the induction hypothesis,

$$f|_{[a_{k+1},a_k]} \circ f|_{[a_k,a_{k-1}]} \circ \cdots \circ f|_{[a_1,a]} \circ f|_{[a,a_{-1}]}$$

is strictly increasing, which shows that $f|_{[a_{k+1},a_k]}$ is strictly increasing. Consequently, by induction, $f|_{[a_k,a_{k-1}]}$ is strictly increasing for $k = 0,1,\dots,n-1$. Thus

$$F(a) = a_n < a_{n-1} < \cdots < a_2 < a_1 < a.$$

Moreover, $f|_{[a_{n-1},b]}$ is strictly increasing, showing that

$$f(x) > F(a), \quad \forall x \in (a_{n-1},b]. \tag{4.27}$$

Meanwhile, it can be seen that

$$f\bigl([a_n,a_{n-1}]\bigr) = f \circ f\bigl([a_{n-1},a_{n-2}]\bigr) = \cdots = f^n\bigl([a_1,a]\bigr) = F\bigl([a_1,a]\bigr) = \bigl\{F(a)\bigr\}. \tag{4.28}$$

If $f|_{[b,1]}$ is strictly increasing, then it follows from (4.27) and (4.28) that $f|_{[F(a),1]}$ is an increasing self-mapping, which forces that $F|_{[F(a),1]}$ is an increasing self-mapping. This contradiction shows that $f|_{[b,1]}$ must be strictly decreasing. Thus the point b is a non-monotonic point of f. If $f(b) > b$, then there exists $b_1 \in (a,b)$ such that $f(b_1) = b$. Consequently, b_1 is a non-monotonic point of f^2. However, f^2 is strictly monotonic on $[a,b]$ since $f^{n-2} \circ f^2|_{[a,b]}$ $(= F|_{[a,b]})$ is strictly increasing. This contradiction yields

$$f(b) \le b. \tag{4.29}$$

Further, we claim that

$$f(1) \ge F(a). \tag{4.30}$$

In fact, if $f(1) < F(a)$, then, by (4.27), $F(a) \in (f(1),f(b))$ and (4.18) shows that $[p,q] \cap (f(1),f(b)) \neq \emptyset$. As the proof of (4.26), we can obtain a contradiction. It follows from (4.27), (4.28), (4.29), and (4.30) that $f|_{[F(a),1]}$ is a self-mapping, and we have

$$f(x) \le b, \quad \forall x \in \bigl[F(a),1\bigr],$$

Figure 9 F_7.

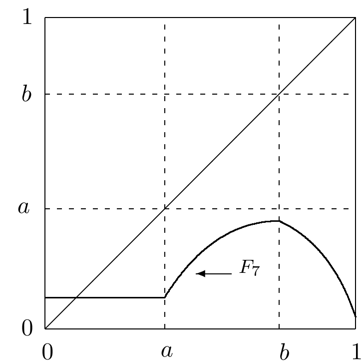


Figure 10 F_8.

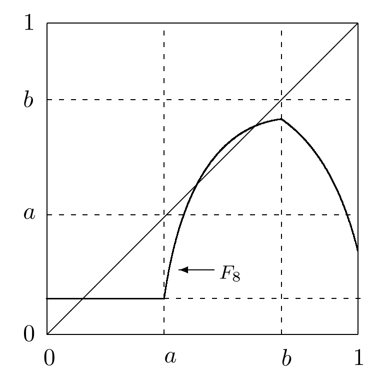
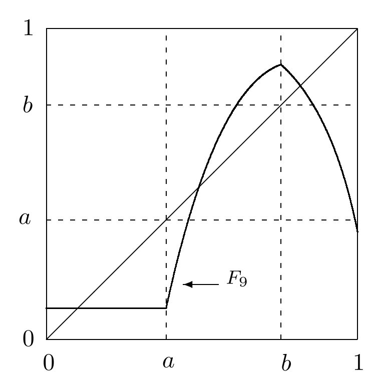


Figure 11 F_9.



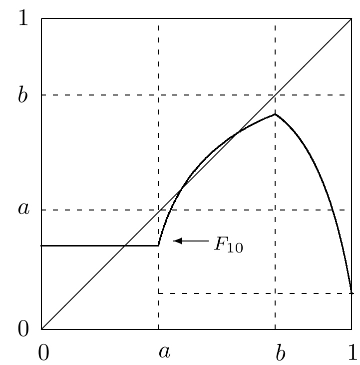
which implies that $F(b) = f(f^{n-1}(b)) \leq b$ and $F(1) = f^n(1) \geq F(a)$. Consequently, F satisfies the condition (ii) if F does not satisfy the condition (i), and necessity is proved. This completes the proof. □

Example 4.1 Let $F_7 : I \to I$ and $F_8 : I \to I$ be given by Figures 9 and 10, respectively.

We see, according to Theorem 4.3, that both F_7 and F_8 have iterative roots of order $n \geq 2$ on I.

Example 4.2 It follows from Theorem 4.3 that the functions $F_9 : I \to I$ and $F_{10} : I \to I$ given by Figures 11 and 12, respectively, have no iterative roots of order $n \geq 2$ on I.

Figure 12 F_{10}.



In this paper, we give all conditions of existence and nonexistence of iterative roots for the sickle-like functions. Meanwhile, we construct infinite many iterative roots if they exist. However, we are not sure if all continuous iterative roots are defined by the construction method given in this paper, in other words, it is unclear whether there are other forms of iterative roots if they exist. On the other hand, as the number of the subintervals on which the function is constant and isolated non-monotonic points increases, we do not know whether the existence of its iterative roots can be solved in a similar manner to the one described in this paper. Although we are not able to give answers for these problems, yet, they are our further directions of investigation.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author read and approved the final manuscript.

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References

1. Babbage, C: An essay towards the calculus of functions. Philos. Trans. R. Soc. Lond. Ser. A **105**, 389-423 (1815)
2. Babbage, C: An essay towards the calculus of functions, II. Philos. Trans. R. Soc. Lond. Ser. A **106**, 179-256 (1816)
3. Bödewadt, UT: Zur Iteration reeller Funktionen. Math. Z. **49**, 497-521 (1944)
4. Kuczma, M: On the functional equation $\varphi^n(x) = g(x)$. Ann. Pol. Math. **11**, 161-175 (1961)
5. Isaacs, R: Iterates of fractional order. Can. J. Math. **2**, 409-416 (1950)
6. Kobza, J: Iterative functional equation $x(x(t)) = f(t)$ with $f(t)$ piecewise linear. J. Comput. Appl. Math. **115**, 331-347 (2000)
7. Targonski, G: Progress of iteration theorem since 1981. Aequ. Math. **50**, 50-72 (1995)
8. Bogatyi, S: On the nonexistence of iterative roots. Topol. Appl. **76**, 97-123 (1997)
9. Kuczma, M: Functional Equations in a Single Variable. Polish Sci., Warsaw (1968)
10. Kuczma, M, Choczewski, B, Ger, R: Iterative Functional Equations. Cambridge University Press, Cambridge (1990)
11. Zhang, JZ, Yang, L: Discussion on iterative roots of piecewise monotone functions. Acta Math. Sin., **26**, 398-412 (1983) (in Chinese)
12. Zhang, WN: PM functions, their characteristic intervals and iterative roots. Ann. Pol. Math. **65**, 119-128 (1997)
13. Blokh, A, Coven, E, Misiurewicz, M, Nitecki, Z: Roots of continuous piecewise monotone maps of an interval. Acta Math. Univ. Comen. **60**, 3-10 (1991)
14. Li, L, Yang, DL, Zhang, WN: A note on iterative roots of PM functions. J. Math. Anal. Appl. **34**, 1482-1486 (2008)
15. Liu, L, Zhang, WN: Non-monotonic iterative roots extended from characteristic intervals. J. Math. Anal. Appl. **378**, 359-373 (2011)
16. Liu, L, Jarczyk, W, Li, L, Zhang, WN: Iterative roots of piecewise monotonic functions of nonmonotonicity height not less than 2. Nonlinear Anal. **75**, 286-303 (2012)
17. Shi, YG: Iterative roots with circuits for piecewise continuous and globally periodic maps. Topol. Appl. **159**, 2721-2727 (2012)

18. Shi, YG, Chen, L: Extension of iterative roots. Aequ. Math. (2013). doi:10.1007/s00010-013-0223-9
19. Schweizer, B, Sklar, A: All trapezoid functions are conjugate. C. R. Math. Acad. Sci. **5**, 275-280 (1983)
20. Schweizer, B, Sklar, A: Continuous functions that conjugate trapezoid functions. Aequ. Math. **28**, 300-304 (1985)
21. Zdun, MC: On the iteration of the hat-functions. Aequ. Math. **16**, 181-182 (1977)
22. Zhang, GY: Conjugation and iterative root of a class of self-mappings on the interval (II). Chin. Ann. Math., Ser. A **4**, 473-478 (1992) (in Chinese)
23. Sun, TX, Xi, HJ: Iterative solutions of level-top bimodal continuous self-maps on the interval. J. Syst. Sci. Math. Sci. **21**, 348-361 (2001) (in Chinese)

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