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On variational inequality, fixed point and generalized mixed equilibrium problems

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Abstract
In this article, variational inequality, fixed point, and generalized mixed equilibrium problems are investigated based on an extragradient iterative algorithm. Weak convergence of the extragradient iterative algorithm is obtained in Hilbert spaces.

Keywords: fixed point; equilibrium problem; monotone mapping; nonexpansive mapping; projection

1 Introduction

In this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and C is a nonempty, closed, and convex subset of H. $\mathbb{R}$ is denoted by the set of real numbers. Let F be a bifunction of $C \times C$ into $\mathbb{R}$. Consider the problem: find a p such that

$$F(p,y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, the solution set of the problem is denoted by $\mathrm{EP}(F)$, *i.e.*,

$$\mathrm{EP}(F) = \{p \in C : F(p,y) \geq 0, \forall y \in C\}.$$

The above problem is first introduced by Ky Fan [1]. In the sense of Blum and Oettli [2], the Ky Fan problem is also called an equilibrium problem.

Recently, the 'so-called' generalized mixed equilibrium problem has been investigated by many authors: The generalized mixed equilibrium problem is to find $p \in C$ such that

$$F(p,y) + \langle Ap, y-p\rangle + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C, \tag{1.2}$$

where $\varphi : C \to \mathbb{R}$ is a real valued function and $A : C \to H$ is mapping. We use $\mathrm{GMEP}(F, A,\varphi)$ to denote the solution set of the equilibrium problem. That is,

$$\mathrm{GMEP}(F,A,\varphi) := \{p \in C : F(p,y) + \langle Ap, y-p\rangle + \varphi(y) - \varphi(z) \geq 0, \forall y \in C\}.$$

Next, we give some special cases.

If $A = 0$, then the problem (1.2) is equivalent to find $p \in C$ such that

$$F(p,y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which is called the mixed equilibrium problem.

If $F = 0$, then the problem (1.2) is equivalent to find $p \in C$ such that

$$\langle Ap, y - p\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.4}$$

which is called the mixed variational inequality of Browder type.

If $\varphi = 0$, then the problem (1.2) is equivalent to find $p \in C$ such that

$$F(p,y) + \langle Ap, y - p\rangle \geq 0, \quad \forall y \in C, \tag{1.5}$$

which is called the generalized equilibrium problem.

If $A = 0$ and $\varphi = 0$, then the problem (1.2) is equivalent to (1.1).

For solving the above equilibrium problems, let us assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$, $\forall x, y \in C$;

(A3)

$$\limsup_{t \downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y), \quad \forall x, y, z \in C;$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semicontinuous.

Equilibrium problems have intensively been studied. It has been shown that equilibrium problems cover fixed point problems, variational inequality problems, inclusion problems, saddle problems, complementarity problem, minimization problem, and Nash equilibrium problem; see [1–20] and the references therein.

Let $S : C \to C$ be a mapping. In this paper, we use $F(S)$ to stand for the set of fixed points. Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be κ-strictly pseudocontractive if there exists a constant $\kappa \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|x - y - Sx + Sy\|^2, \quad \forall x, y \in C.$$

It is clear that the class of κ-strictly pseudocontractive includes the class of nonexpansive mappings as a special case. The class of κ-strictly pseudocontractive mappings was introduced by Browder and Petryshyn [21]; for existence and approximation of fixed points of the class of mappings, see [22–29] and the references therein.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be κ-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \kappa \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is clear that the κ-inverse being strongly monotone is monotone and Lipschitz continuous.

A set-valued mapping $T : H \to 2^H$ is said to be monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle > 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. The class of monotone operators is one of the most important classes of operators. Within the past several decades, many authors have been devoting their efforts to the studies of the existence and convergence of zero points for maximal monotone operators.

Let $F(x, y) = \langle Ax, y - x\rangle$, $\forall x, y \in C$. We see that the problem (1.1) is reduced to the following classical variational inequality. Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

It is well known that $x \in C$ is a solution to (1.6) if and only if x is a fixed point of the mapping $P_C(I - \rho A)$, where $\rho > 0$ is a constant, and I is the identity mapping. If C is bounded, closed, and convex, then the solution set of the variational inequality (1.6) is nonempty.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 [21] *Let $S : C \to C$ be a κ-strictly pseudocontractive mapping. Define $S_t : C \to C$ by $S_t x = tx + (1 - t)Sx$ for each $x \in C$. Then, as $t \in [\kappa, 1)$, S_t is nonexpansive such that $F(S_t) = F(S)$.*

Lemma 1.2 [2] *Let C be a nonempty, closed, and convex subset of H, and $F : C \times C \to \mathbb{R}$ a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) $F(T_r) = \mathrm{EP}(F)$;

(d) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 1.3 [30] *Let A be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, i.e.,*

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$$

and define a mapping T on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, u - v\rangle \geq 0$ for all $u \in C$.

Lemma 1.4 [31] *Let $\{a_n\}_{n=1}^{\infty}$ be real numbers in $[0,1]$ such that $\sum_{n=1}^{\infty} a_n = 1$. Then we have the following*:

$$\left\| \sum_{i=1}^{\infty} a_i x_i \right\|^2 \leq \sum_{i=1}^{\infty} a_i \|x_i\|^2$$

for any given bounded sequence $\{x_n\}_{n=1}^{\infty}$ in H.

Lemma 1.5 [32] *Let $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \qquad \limsup_{n\to\infty} \|y_n\| \leq d$$

and

$$\lim_{n\to\infty} \left\| t_n x_n + (1 - t_n) y_n \right\| = d$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.6 [21] *Let C be a nonempty, closed, and convex subset of H, and $S : C \to C$ a strictly pseudocontractive mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$, then $x = Sx$.*

Lemma 1.7 [33] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition*:

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

2 Main results

Theorem 2.1 *Let C be a nonempty, closed, and convex subset of H, $S : C \to C$ a κ-strictly pseudocontractive mapping with a nonempty fixed point set, and $A : C \to H$ an L-Lipschitz* *continuous and monotone mapping. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4), *$B_m : C \to H$ a continuous and monotone mapping, $\varphi_m : C \to \mathbb{R}$ a lower semicontinuous and convex function for each $m \geq 1$. Assume that $\mathcal{F} := \bigcap_{m=1}^{\infty} \mathrm{GMEP}(F_m, B_m, \varphi_m) \cap \mathrm{VI}(C, A) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_{n,m}\}$ be real number sequences in $(0,1)$. Let $\{\lambda_n\}$, $\{r_{n,m}\}$ be positive real number sequences. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n I + (1-\beta_n)S)\operatorname{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A y_n), \quad n \geq 1, \\ y_n = \operatorname{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A \sum_{m=1}^{\infty} \delta_{n,m} z_{n,m}), \end{cases}$$

where $z_{n,m}$ is such that

$$F_m(z_{n,m}, z) + \langle B_m z_{n,m}, z - z_{n,m}\rangle + \varphi_m(z) - \varphi_m(z_{n,m}) + \frac{1}{r_{n,m}}\langle z - z_{n,m}, z_{n,m} - x_n\rangle \geq 0, \quad \forall z \in C.$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, $\{r_{n,m}\}$ satisfy the following restrictions:

(a) $0 < a \leq \alpha_n \leq b < 1$;

(b) $\kappa \leq \beta_n \leq c < 1$;

(c) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, *and* $0 < d \leq \delta_{n,m} \leq 1$;

(d) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $e \leq \lambda_n \leq f$, *where* $e, f \in (0, 1/L)$.

Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in \mathcal{F}$.

Proof The proof is split into five steps.

Step 1. Show that the sequence $\{x_n\}$ is bounded.

Define $G_m(p,y) = F_m(p,y) + \langle B_m p, y - p\rangle + \varphi_m(y) - \varphi_m(p)$, $\forall p, y \in C$. Next, we prove that the bifunction G_m satisfies the conditions (A1)-(A4). Therefore, generalized mixed equilibrium problem is equivalent to the following equilibrium problem: find $p \in C$ such that $G_m(p,y) \geq 0$, $\forall y \in C$. It is clear that G_m satisfies (A1). Next, we prove G_m is monotone. Since B_m is a continuous and monotone operator, we find from the definition of G that

$$\begin{aligned} G_m(y,z) + G_m(z,y) &= F_m(y,z) + \langle B_m y, z - y\rangle + \varphi_m(z) - \varphi_m(y) + F_m(z,y) \\ &\quad + \langle B_m z, y - z\rangle + \varphi_m(y) - \varphi_m(z) \\ &= F_m(z,y) + F_m(y,z) + \langle B_m z, y - z\rangle + \langle B_m y, z - y\rangle \\ &\leq \langle B_m z - B_m y, y - z\rangle \\ &\leq 0. \end{aligned}$$

Next, we show G_m satisfies (A3), that is,

$$\limsup_{t\downarrow 0} G_m\bigl(tz + (1-t)x, y\bigr) \leq G_m(x,y), \quad \forall x, y, z \in C.$$

Since B_m is continuous and φ_m is lower semicontinuous, we have

$$\begin{aligned} \limsup_{t\downarrow 0} G_m\bigl(tz + (1-t)x, y\bigr) &= \limsup_{t\downarrow 0} F_m\bigl(tz + (1-t)x, y\bigr) \\ &\quad + \limsup_{t\downarrow 0}\bigl\langle B_m\bigl(tz + (1-t)x\bigr), y - \bigl(tz + (1-t)x\bigr)\bigr\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \limsup_{t\downarrow 0}\bigl(\varphi_m(y) - \varphi_m\bigl(tz + (1-t)x\bigr)\bigr) \\
&\le F_m(x,y) + \langle B_m x, y - x\rangle + \varphi_m(y) - \varphi_m(x) \\
&= G_m(x,y).
\end{aligned}
$$

Next, we show that, for each $x \in C$, $y \mapsto G_m(x,y)$ is a convex and lower semicontinuous. For each $x \in C$, for all $t \in (0,1)$ and for all $y, z \in C$, since F_m satisfies (A4) and φ_m is convex, we have

$$
\begin{aligned}
&G_m\bigl(x, ty + (1-t)z\bigr) \\
&\quad = F_m\bigl(x, ty + (1-t)z\bigr) + \bigl\langle B_m x, ty + (1-t)z - x\bigr\rangle + \varphi_m\bigl(ty + (1-t)z\bigr) - \varphi_m(x) \\
&\quad \le t\bigl(F_m(x,y) + \langle B_m x, y - x\rangle + \varphi_m(y) - \varphi_m(x)\bigr) \\
&\qquad + (1-t)\bigl(F_m(x,z) + \langle B_m x, z - x\rangle + \varphi_m(z) - \varphi_m(x)\bigr) \\
&\quad = tG_m(x,y) + (1-t)G_m(x,z).
\end{aligned}
$$

Thus, $y \mapsto G_m(x,y)$ is convex. Similarly, we find that $y \mapsto G_m(x,y)$ is also lower semicontinuous. Put $u_n = \mathrm{Proj}_C(\sum_{m=1}^{N} \delta_{n,m} z_{n,m} - \lambda_n A y_n)$ and $v_n = \sum_{m=1}^{N} \delta_{n,m} z_{n,m}$. Letting $p \in \mathcal{F}$, we see that

$$
\begin{aligned}
\|u_n - p\|^2 &\le \|v_n - \lambda_n A y_n - p\|^2 - \|v_n - \lambda_n A y_n - u_n\|^2 \\
&= \|v_n - p\|^2 - \|v_n - u_n\|^2 + 2\lambda_n\bigl(\langle Ay_n - Ap, p - y_n\rangle + \langle Ap, p - y_n\rangle \\
&\quad + \langle Ay_n, y_n - u_n\rangle\bigr) \\
&\le \|v_n - p\|^2 - \|v_n - y_n\|^2 - \|y_n - u_n\|^2 + 2\langle v_n - \lambda_n A y_n - y_n, u_n - y_n\rangle.
\end{aligned}
$$

Notice that A is L-Lipschitz continuous and $y_n = \mathrm{Proj}_C(v_n - \lambda_n A v_n)$. It follows that

$$
\langle v_n - \lambda_n A y_n - y_n, u_n - y_n\rangle \le \lambda_n L\|v_n - y_n\|\|u_n - y_n\|.
$$

It follows that

$$
\|u_n - p\|^2 \le \|v_n - p\|^2 + \bigl(\lambda_n^2 L^2 - 1\bigr)\|v_n - y_n\|^2. \tag{2.1}
$$

On the other hand, we have

$$
\begin{aligned}
\|v_n - p\|^2 &\le \left\|\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - p\right\|^2 \\
&\le \sum_{m=1}^{\infty} \delta_{n,m}\|T_{r_{n,m}} x_n - p\|^2 \\
&\le \|x_n - p\|^2,
\end{aligned} \tag{2.2}
$$

where $T_{r_{n,m}} = \{z \in C : G_m(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}$. Substituting (2.2) into (2.1), we obtain

$$
\|u_n - p\|^2 \le \|x_n - p\|^2 + \bigl(\lambda_n^2 L^2 - 1\bigr)\|v_n - y_n\|^2.
$$

Putting $S_n = \beta_n I + (1-\beta_n)S$, we find from Lemma 1.1 that S_n is nonexpansive and $F(S_n) = F(S)$. It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|S_nu_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|u_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\bigl(\|x_n-p\|^2 + \bigl(\lambda_n^2L^2-1\bigr)\|v_n-y_n\|^2\bigr)\\ &\le \|x_n-p\|^2 + (1-\alpha_n)\bigl(\lambda_n^2L^2-1\bigr)\|v_n-y_n\|^2\\ &\le \|x_n-p\|^2. \end{aligned}\tag{2.3}$$

It follows from Lemma 1.7 that the $\lim_{n\to\infty}\|x_n-p\|$ exists. This shows that $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ.

Step 2. Show that $\xi \in \mathrm{VI}(C,A)$

From (2.3), we find that $\beta_n(1-\lambda_n^2L^2)\|v_n-y_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2$. In view of the restrictions (b) and (d), we see that $\lim_{n\to\infty}\|v_n-y_n\| = 0$. Since $\|y_n-u_n\| \le \lambda L\|v_n-y_n\|$, we have that $\lim_{n\to\infty}\|y_n-u_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|v_n-u_n\| = 0.\tag{2.4}$$

Notice that

$$\begin{aligned}\|z_{n,m}-p\|^2 &= \|T_{r_{n,m}}x_n - T_{r_{n,m}}p\|^2\\ &\le \langle T_{r_{n,m}}x_n - T_{r_{n,m}}p, x_n-p\rangle\\ &= \frac{1}{2}\bigl(\|z_{n,m}-p\|^2 + \|x_n-p\|^2 - \|z_{n,m}-x_n\|^2\bigr).\end{aligned}$$

This implies that $\|z_{n,m}-p\|^2 \le \|x_n-p\|^2 - \|z_{n,m}-x_n\|^2$. Since $v_n = \sum_{m=1}^{\infty}\delta_{n,m}z_{n,m}$, where $\sum_{m=1}^{\infty}\delta_{n,m} = 1$, we find that

$$\begin{aligned}\|v_n-p\|^2 &\le \sum_{m=1}^{\infty}\delta_{n,m}\|z_{n,m}-p\|^2\\ &\le \|x_n-p\|^2 - \sum_{m=1}^{\infty}\delta_{n,m}\|z_{n,m}-x_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|S_nu_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|u_n-p\|^2\\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|v_n-p\|^2\\ &\le \|x_n-p\|^2 - (1-\alpha_n)\sum_{m=1}^{\infty}\delta_{n,m}\|z_{n,m}-x_n\|^2.\end{aligned}$$

This implies that $(1-\alpha_n)\delta_{n,m}\|z_{n,m}-x_n\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2$. In view of the restrictions (a) and (c), we find that

$$\lim_{n\to\infty}\|z_{n,m}-x_n\| = 0. \tag{2.5}$$

Let T be the maximal monotone mapping defined by

$$Tx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x,y) \in G(T)$, we have $y - Ax \in N_C x$. So, we have $\langle x-m, y-Ax\rangle \geq 0$, for all $m \in C$. On the other hand, we have $u_n = \mathrm{Proj}_C(v_n - \lambda_n A y_n)$. We obtain

$$\left\langle x-u_n, \frac{u_n-v_n}{\lambda_n} + Ay_n\right\rangle \geq 0.$$

In view of the monotonicity of A, we see that

$$\begin{aligned} \langle x-u_{n_i}, y\rangle &\geq \langle x-u_{n_i}, Ax\rangle \\ &\geq \langle x-u_{n_i}, Ax\rangle - \left\langle x-u_{n_i}, \frac{u_{n_i}-v_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle \\ &= \langle x-u_{n_i}, Ax-Au_{n_i}\rangle + \langle x-u_{n_i}, Au_{n_i}-Ay_{n_i}\rangle \\ &\quad - \left\langle x-u_{n_i}, \frac{u_{n_i}-v_{n_i}}{\lambda_{n_i}}\right\rangle \\ &\geq \langle x-u_{n_i}, Au_{n_i}-Ay_{n_i}\rangle - \left\langle x-u_{n_i}, \frac{u_{n_i}-v_{n_i}}{\lambda_{n_i}}\right\rangle \end{aligned}$$

in view of $\|v_n-x_n\| \leq \sum_{m=1}^{\infty}\delta_{n,m}\|z_{n,m}-x_n\|$. It follows from (2.5) that $\lim_{n\to\infty}\|v_n-x_n\| = 0$. Notice that $\|u_n-x_n\| \leq \|u_n-v_n\| + \|v_n-x_n\|$. It follows that

$$\lim_{n\to\infty}\|u_n-x_n\| = 0. \tag{2.6}$$

This in turn implies that $u_{n_i} \rightharpoonup \xi$. It follows that $\langle x-\xi, y\rangle \geq 0$. Notice that T is maximal monotone and hence $0 \in T\xi$. This shows from Lemma 1.3 that $\xi \in \mathrm{VI}(C,A)$.

Step 3. Show that $\xi \in \mathrm{GMEP}(F_m, B_m, \varphi_m)$.

It follows from (2.5) that $\{z_{n_i,m}\}$ converges weakly to ξ for each $m \geq 1$. Since $z_{n,m} = T_{r_{n,m}}x_n$, we have

$$G_m(z_{n,m}, z) + \frac{1}{r_{n,m}}\langle z-z_{n,m}, z_{n,m}-x_n\rangle \geq 0, \quad \forall z \in C.$$

From the assumption (A2), we see that

$$\left\langle z-z_{n_i,m}, \frac{z_{n_i,m}-x_{n_i}}{r_{n_i,m}}\right\rangle \geq G_m(z, z_{n_i,m}), \quad \forall z \in C.$$

In view of the assumption (A4), we find from (2.5) that $G_m(z,\xi) \leq 0$, $\forall z \in C$. For t_m with $0 < t_m \leq 1$ and $z \in C$, let $z_{t_m} = t_m z + (1-t_m)\xi$, for each $1 \leq m \leq N$. Since $z \in C$ and $\xi \in C$, we have $z_{t_m} \in C$. It follows that $G_m(z_{t_m}, \xi) \leq 0$. Notice that

$$0 = G_m(z_{t_m}, z_{t_m}) \leq t_m G_m(z_{t_m}, z) + (1 - t_m) G_m(z_{t_m}, \xi) \leq t_m G_m(z_{t_m}, z),$$

which yields $G_m(z_{t_m}, z) \geq 0$, $\forall z \in C$. Letting $t_m \downarrow 0$, one sees that $G_m(\xi, z) \geq 0$, $\forall z \in C$. This implies that $\xi \in \mathrm{GMEP}(F_m, B_m, \varphi_m)$ for each $m \geq 1$. This proves that $\xi \in \bigcap_{m=1}^{\infty} \mathrm{GMEP}(F_m, B_m, \varphi_m)$.

Step 4. Show that $\xi \in F(S)$.

Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we put $\lim_{n\to\infty} \|x_n - p\| = d > 0$. It follows that

$$\lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \bigl\|\alpha_n(x_n - p) + (1 - \alpha_n)(S_n u_n - p)\bigr\| = d.$$

Notice that $\limsup_{n\to\infty} \|S_n u_n - p\| \leq d$. From Lemma 1.5, we see that

$$\lim_{n\to\infty} \|x_n - S_n u_n\| = 0. \tag{2.7}$$

Since

$$\begin{aligned} \|S_n x_n - x_n\| &\leq \|S_n x_n - S_n u_n\| + \|S_n u_n - x_n\| \\ &\leq \|x_n - u_n\| + \|S_n u_n - x_n\|, \end{aligned}$$

we find from (2.6) and (2.7) that

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{2.8}$$

In view of $\|S x_n - x_n\| \leq \|S x_n - S_n x_n\| + \|S_n x_n - x_n\|$, we find from (2.8) that $\lim_{n\to\infty} \|x_n - S x_n\| = 0$. This implies from Lemma 1.6 that $\xi \in F(S)$. This completes the proof that $\xi \in \mathcal{F}$.

Step 5. Show that the whole sequence $\{x_n\}$ weakly converges to ξ.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ converging weakly to ξ', where $\xi' \neq \xi$. In the same way, we can show that $\xi' \in \mathcal{F}$. Since the space H enjoys Opial's condition, we, therefore, obtain

$$\begin{aligned} d &= \liminf_{i\to\infty} \|x_{n_i} - \xi\| < \liminf_{i\to\infty} \bigl\|x_{n_i} - \xi'\bigr\| \\ &= \liminf_{j\to\infty} \bigl\|x_j - \xi'\bigr\| < \liminf_{j\to\infty} \|x_j - \xi\| = d. \end{aligned}$$

This is a contradiction. Hence $\xi = \xi'$. This completes the proof. □

3 Applications

In this section, we consider solutions of the mixed equilibrium problem (1.3), which includes the Ky Fan inequality as a special case.

The so-called mixed equilibrium problem is to find $p \in C$ such that

$$F(p, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C.$$

The mixed equilibrium problem includes the Ky Fan inequality, fixed point problems, saddle problems, and complementary problems as special cases.

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of H, $S : C \to C$ a κ-strictly pseudocontractive mapping with a nonempty fixed point set, and $A : C \to H$ a L-Lipschitz continuous and monotone mapping. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4), *and $\varphi_m : C \to \mathbb{R}$ a lower semicontinuous and convex function for each $m \geq 1$. Assume that $\mathcal{F} := \bigcap_{m=1}^{\infty} \mathrm{MEP}(F_m, \varphi_m) \cap \mathrm{VI}(C, A) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_{n,m}\}$ be real number sequences in $(0,1)$. Let $\{\lambda_n\}$, $\{r_{n,m}\}$ be positive real number sequences. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n I + (1-\beta_n)S)\operatorname{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A y_n), \quad n \geq 1, \\ y_n = \operatorname{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A \sum_{m=1}^{\infty} \delta_{n,m} z_{n,m}), \end{cases}$$

where $z_{n,m}$ is such that

$$F_m(z_{n,m}, z) + \varphi_m(z) - \varphi_m(z_{n,m}) + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \geq 0, \quad \forall z \in C.$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, $\{r_{n,m}\}$ satisfy the following restrictions:
(a) $0 < a \leq \alpha_n \leq b < 1$;
(b) $\kappa \leq \beta_n \leq c < 1$;
(c) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, *and* $0 < d \leq \delta_{n,m} \leq 1$;
(d) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $e \leq \lambda_n \leq f$, *where* $e, f \in (0, 1/L)$.
Then the sequence $\{x_n\}$ weakly converges to some point $\bar{x} \in \mathcal{F}$.

Proof If $B_m = 0$, we draw the desired conclusion immediately from Theorem 2.1. □

Further, if S is nonexpansive, we find from Theorem 3.1 the following result.

Corollary 3.2 *Let C be a nonempty, closed, and convex subset of H, $S : C \to C$ a nonexpansive mapping with a nonempty fixed point set, and $A : C \to H$ an L-Lipschitz continuous and monotone mapping. Let F_m be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4), *and $\varphi_m : C \to \mathbb{R}$ a lower semicontinuous and convex function for each $m \geq 1$. Assume that $\mathcal{F} := \bigcap_{m=1}^{\infty} \mathrm{MEP}(F_m, \varphi_m) \cap \mathrm{VI}(C, A) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_{n,m}\}$ be real number sequences in $(0,1)$. Let $\{\lambda_n\}$, $\{r_{n,m}\}$ be positive real number sequences. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S \operatorname{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A y_n), \quad n \geq 1, \\ y_n = \operatorname{Proj}_C(\sum_{m=1}^{\infty} \delta_{n,m} z_{n,m} - \lambda_n A \sum_{m=1}^{\infty} \delta_{n,m} z_{n,m}), \end{cases}$$

where $z_{n,m}$ is such that

$$F_m(z_{n,m}, z) + \varphi_m(z) - \varphi_m(z_{n,m}) + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n \rangle \geq 0, \quad \forall z \in C.$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,m}\}$, $\{\lambda_n\}$, $\{r_{n,m}\}$ satisfy the following restrictions:
(a) $0 < a \leq \alpha_n \leq b < 1$;

(b) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, *and* $0 < d \le \delta_{n,m} \le 1$;
(c) $\liminf_{n\to\infty} r_{n,m} > 0$ *and* $e \le \lambda_n \le f$, *where* $e, f \in (0, 1/L)$.
Then the sequence $\{x_n\}$ *weakly converges to some point* $\bar{x} \in \mathcal{F}$.

If $A = 0$, we find from Theorem 2.1 the following result.

Theorem 3.3 *Let* C *be a nonempty, closed, and convex subset of* H, $S : C \to C$ *a* κ*-strictly pseudocontractive mapping with a nonempty fixed point set. Let* F_m *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4), $B_m : C \to H$ *a continuous and monotone mapping,* $\varphi_m : C \to \mathbb{R}$ *a lower semicontinuous and convex function for each* $m \ge 1$. *Assume that* $\mathcal{F} := \bigcap_{m=1}^{\infty} \mathrm{GMEP}(F_m, B_m, \varphi_m) \cap F(S)$ *is not empty. Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\delta_{n,m}\}$ *be real number sequences in* $(0, 1)$. *Let* $\{r_{n,m}\}$ *be a positive real number sequence. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$x_1 \in H, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)\big(\beta_n I + (1-\beta_n)S\big) \sum_{m=1}^{\infty} \delta_{n,m} z_{n,m}, \quad n \ge 1,$$

where $z_{n,m}$ *is such that*

$$F_m(z_{n,m}, z) + \langle B_m z_{n,m}, z - z_{n,m}\rangle + \varphi_m(z) - \varphi_m(z_{n,m}) + \frac{1}{r_{n,m}} \langle z - z_{n,m}, z_{n,m} - x_n\rangle \ge 0, \quad \forall z \in C.$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,m}\}$, *and* $\{r_{n,m}\}$ *satisfy the following restrictions*:
(a) $0 < a \le \alpha_n \le b < 1$;
(b) $\kappa \le \beta_n \le c < 1$;
(c) $\sum_{m=1}^{\infty} \delta_{n,m} = 1$, *and* $0 < d \le \delta_{n,m} \le 1$;
(d) $\liminf_{n\to\infty} r_{n,m} > 0$.
Then the sequence $\{x_n\}$ *weakly converges to some point* $\bar{x} \in \mathcal{F}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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