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Some identities of q-Euler polynomials arising from q-umbral calculus

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Abstract

Recently, Araci-Acikgoz-Sen derived some interesting identities on weighted q-Euler polynomials and higher-order q-Euler polynomials from the applications of umbral calculus (see (Araci *et al.* in J. Number Theory 133(10):3348-3361, 2013)). In this paper, we develop the new method of q-umbral calculus due to Roman, and we study a new q-extension of Euler numbers and polynomials which are derived from q-umbral calculus. Finally, we give some interesting identities on our q-Euler polynomials related to the q-Bernoulli numbers and polynomials of Hegazi and Mansour.

1 Introduction

Throughout this paper we will assume q to be a fixed real number between 0 and 1. We define the q-shifted factorials by

$$(a:q)_0 = 1, \qquad (a:q)_n = \prod_{i=0}^{n-1}\left(1-aq^i\right), \qquad (a:q)_\infty = \prod_{i=0}^{\infty}\left(1-aq^i\right). \tag{1.1}$$

If x is a classical object, such as a complex number, its q-version is defined as $[x]_q = \frac{1-q^x}{1-q}$. We now introduce the q-extension of exponential function as follows:

$$e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \frac{1}{((1-q)z:q)_\infty} \quad \text{(see [1–4])}, \tag{1.2}$$

where $z \in \mathbb{C}$ with $|z| < 1$.

The Jackson definite q-integral of the function f is defined by

$$\int_0^x f(t)\,d_qt = (1-q)\sum_{a=0}^{\infty} f\left(q^a x\right)xq^a \quad \text{(see [1, 2, 5])}. \tag{1.3}$$

The q-difference operator D_q is defined by

$$D_qf(x) = \frac{d_qf(x)}{d_qx} = \begin{cases} \frac{f(x)-f(qx)}{(1-q)x} & \text{if } x \neq 0, \\ \frac{df(x)}{dx} & \text{if } x = 0, \end{cases} \tag{1.4}$$

where

$$\lim_{q\to 1} D_qf(x) = \frac{df(x)}{dx} \quad \text{(see [1, 2, 4, 6])}.$$

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By using an exponential function $e_q(x)$, Hegazi and Mansour defined q-Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{t}{e_q(t)-1}e_q(xt) \quad \text{(see [1, 2, 4, 7])}. \tag{1.5}$$

In the special case, $x = 0$, $B_{n,q}(0) = B_{n,q}$ are called the nth q-Bernoulli numbers.
From (1.5), we can easily derive the following equation:

$$B_{n,q}(x) = \sum_{l=0}^{n}\binom{n}{l}_q B_{n-l,q}x^l = \sum_{l=0}^{n}\binom{n}{l}_q B_{l,q}x^{n-l}, \tag{1.6}$$

where

$$\binom{n}{l}_q = \frac{[n]_q!}{[n-l]_q![l]_q!} = \frac{[n]_q[n-1]_q\cdots[n-l+1]_q}{[l]_q!} \quad \text{(see [2, 7])}.$$

In the next section, we will consider new q-extensions of Euler numbers and polynomials by using the method of Hegazi and Mansour. More than five decades ago, Carlitz [8] defined a q-extension of Euler polynomials. In a recent paper (see [3]), Kupershmidt constructed reflection symmetries of q-Bernoulli polynomials which differ from Carlitz's q-Bernoulli numbers and polynomials. By using the method of Kupershmidt, Hegazi and Mansour also introduced a new q-extension of Bernoulli numbers and polynomials (see [1, 3, 4]). From the q-exponential function, Kurt and Cenkci derived some interesting new formulae of q-extension of Genocchi polynomials. Recently, several authors have studied various q-extensions of Bernoulli and Euler polynomials (see [1–6, 8–11]). Let $\mathbb{C}$ be the complex number field, and let $\mathcal{F}$ be the set of all formal power series in variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty}\frac{a_k}{[k]_q!}t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{1.7}$$

Let $\mathbb{P} = \mathbb{C}[t]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of linear functional L on the polynomial $p(x)$, and it is well known that the vector space operations on $\mathbb{P}^*$ are defined by

$$\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \qquad \langle cL|p(x)\rangle = c\langle L|p(x)\rangle,$$

where c is a complex constant (see [7, 9, 11]).
For $f(t) = \sum_{k=0}^{\infty}\frac{a_k}{[k]_q!}t^k \in \mathcal{F}$, we define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad \text{for all } n \geq 0. \tag{1.8}$$

From (1.7) and (1.8), we note that

$$\langle t^k|x^n\rangle = [n]_q!\delta_{n,k} \quad (n,k \geq 0), \tag{1.9}$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let us assume that $f_L(t) = \sum_{k=0}^{\infty}\langle L|x^n\rangle \frac{t^k}{k!}$. Then by (1.9) we easily see that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $f_L(t) = L$. Additionally, the map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as a formal power series and a linear functional. We call it the q-umbral algebra. The q-umbral calculus is the study of q-umbral algebra. By (1.2) and (1.3), we easily see that $\langle e_q(yt)|x^n\rangle = y^n$ and so $\langle e_q(yt)|p(x)\rangle = p(y)$ for $p(x) \in \mathbb{P}$. The order $o(f(t))$ of the power series $f(t) \neq 0$ is the smallest integer for which a_k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series. If $o(f(t)) = 1$, then$f(t)$ is called a delta series (see [7, 9, 11, 12]). For $f(t), g(t) \in \mathcal{F}$, we have $\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$. Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty}\left\langle f(t)|x^k\right\rangle \frac{t^k}{[k]_q!}, \qquad p(x) = \sum_{k=0}^{\infty}\left\langle t^k|p(x)\right\rangle \frac{x^k}{[k]_q!} \quad \text{(see [12])}. \tag{1.10}$$

From (1.10), we have

$$p^{(k)}(x) = D_q^k p(x) = \sum_{l=k}^{\infty}\frac{\langle t^l|p(x)\rangle}{[l]_q!}[l]_q \cdots [l-k+1]_q x^{l-k}. \tag{1.11}$$

By (1.11), we get

$$p^{(k)}(0) = \left\langle t^k|p(x)\right\rangle \quad \text{and} \quad \left\langle 1|p^{(k)}(x)\right\rangle = p^{(k)}(0). \tag{1.12}$$

Thus from (1.12), we note that

$$t^k p(x) = p^{(k)}(x) = D_q^k p(x). \tag{1.13}$$

Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ ($\deg S_n(x) = n$) of polynomials such that $\langle g(t)f(t)^k|S_n(x)\rangle = [n]_q!\delta_{n,k}$ $(n, k \geq 0)$. The sequence $S_n(x)$ is called the q-Sheffer sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$. Let $S_n(x) \sim (g(t), f(t))$. For $h(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$h(t) = \sum_{k=0}^{\infty}\frac{\langle h(t)|S_k(x)\rangle}{[k]_q!}g(t)f(t)^k, \qquad p(x) = \sum_{k=0}^{\infty}\frac{\langle g(t)f(t)^k|p(x)\rangle}{[k]_q!}S_k(x), \tag{1.14}$$

and

$$\frac{1}{g(\bar{f}(t)}e_q\bigl(y\bar{f}(t)\bigr) = \sum_{k=0}^{\infty}\frac{S_k(y)}{[k]_q!}t^k \quad \text{for all } y \in \mathbb{C}, \tag{1.15}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [7, 12]).

Recently, Araci-Acikgoz-Sen derived some new interesting properties on the new family of q-Euler numbers and polynomials from some applications of umbral algebra (see [9]). The properties of q-Euler and q-Bernoulli polynomials seem to be of interest and worthwhile in the areas of both number theory and mathematical physics. In this paper, we develop the new method of q-umbral calculus due to Roman and study a new q-extension of

Euler numbers and polynomials which are derived from q-umbral calculus. Finally, we give new explicit formulas on q-Euler polynomials related to Hegazi-Mansour's q-Bernoulli polynomials.

2 q-Euler numbers and polynomials

We consider the new q-extension of Euler polynomials which are generated by the generating function to be

$$\frac{2}{e_q(t)+1}e_q(xt)=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{2.1}$$

In the special case, $x=0$, $E_{n,q}(0)=E_{n,q}$ are called the nth q-Euler numbers. From (2.1), we note that

$$E_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}_q E_{l,q}x^{n-l}=\sum_{l=0}^{n}\binom{n}{l}_q E_{n-l,q}x^l. \tag{2.2}$$

By (2.1), we easily get

$$E_{0,q}=1, \qquad E_{n,q}(1)+E_{n,q}=2\delta_{0,n}. \tag{2.3}$$

For example, $E_{0,q}=1$, $E_{1,q}=-\frac{1}{2}$, $E_{2,q}=\frac{q-1}{4}$, $E_{3,q}=\frac{q+q^2-1}{4}+\frac{(1-q)[3]_q}{8},\ldots$. From (1.15) and (2.1), we have

$$E_{n,q}(x)\sim\left(\frac{e_q(t)+1}{2},t\right) \tag{2.4}$$

and

$$\frac{2}{e_q(t)+1}x^n=E_{n,q}(x) \quad (n\geq 0). \tag{2.5}$$

Thus, by (1.13) and (2.5), we get

$$tE_{n,q}(x)=\frac{2}{e_q(t)+1}tx^n=[n]_q\frac{2}{e_q(t)+1}x^{n-1}=[n]_qE_{n-1,q}(x) \quad (n\geq 0). \tag{2.6}$$

Indeed, by (1.9), we get

$$\begin{aligned}\left\langle\frac{e_q(t)+1}{2}t^k\middle|E_{n,q}(x)\right\rangle&=\frac{[k]_q!}{2}\binom{n}{k}_q\langle e_q(t)+1|E_{n-k,q}(x)\rangle\\&=\frac{[k]_q!}{2}\binom{n}{k}_q\bigl(E_{n-k,q}(1)+E_{n-k,q}\bigr).\end{aligned} \tag{2.7}$$

From (2.4), we have

$$\left\langle\left(\frac{e_q(t)+1}{2}\right)t^k\middle|E_{n,q}(x)\right\rangle=[n]_q!\delta_{n,k}. \tag{2.8}$$

Thus, by (2.7) and (2.8), we get

$$0 = E_{n-k,q}(1) + E_{n-k,q} = \sum_{l=0}^{n-k} \binom{n-k}{l}_q E_{l,q} + E_{n-k,q} \quad (n,k \in \mathbb{Z}_{\geq 0} \text{ with } n > k). \tag{2.9}$$

This is equivalent to

$$-2E_{n-k,q} = \sum_{l=0}^{n-k-1} \binom{n-k}{l}_q E_{l,q}, \quad \text{where } n,k \in \mathbb{Z}_{\geq 0} \text{ with } n > k. \tag{2.10}$$

Therefore, by (2.10), we obtain the following lemma.

Lemma 2.1 *For $n \geq 1$, we have*

$$-2E_{n,q} = \sum_{l=0}^{n-1} \binom{n}{l}_q E_{l,q}.$$

From (2.2) we have

$$\begin{aligned}
\int_x^{x+y} E_{n,q}(u)\, d_q u &= \sum_{l=0}^{n} \binom{n}{l}_q E_{n-l,q} \frac{1}{[l+1]_q} \{(x+y)^{l+1} - x^{l+1}\} \\
&= \frac{1}{[n+1]_q} \sum_{l=0}^{n} \binom{n+1}{l+1}_q E_{n-l,q} \{(x+y)^{l+1} - x^{l+1}\} \\
&= \frac{1}{[n+1]_q} \sum_{l=1}^{n+1} \binom{n+1}{l}_q E_{n+1-l,q} \{(x+y)^{l} - x^{l}\} \\
&= \frac{1}{[n+1]_q} \sum_{l=0}^{n+1} \binom{n+1}{l}_q E_{n+1-l,q} \{(x+y)^{l} - x^{l}\} \\
&= \frac{1}{[n+1]_q} \{E_{n+1,q}(x+y) - E_{n+1,q}(x)\}.
\end{aligned} \tag{2.11}$$

Thus, by (2.11), we get

$$\begin{aligned}
\left\langle \frac{e_q(t)-1}{t} \middle| E_{n,q}(x) \right\rangle &= \frac{1}{[n+1]_q} \left\langle \frac{e_q(t)-1}{t} \middle| tE_{n+1,q}(x) \right\rangle \\
&= \frac{1}{[n+1]_q} \langle e_q(t) - 1 | E_{n+1,q}(x) \rangle \\
&= \frac{1}{[n+1]_q} \{E_{n+1,q}(1) - E_{n+1,q}\} \\
&= \int_0^1 E_{n,q}(u)\, d_q u.
\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$\left\langle \frac{e_q(t)-1}{t} \middle| E_{n,q}(x) \right\rangle = \int_0^1 E_{n,q}(u)\, d_q u.$$

Let

$$\mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] \,|\, \deg p(x) \leq n \}. \tag{2.13}$$

For $p(x) \in \mathbb{P}_n$, let us assume that

$$p(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x). \tag{2.14}$$

Then, by (2.4), we get

$$\left\langle \left(\frac{e_q(t)+1}{2} \right) t^k \middle| E_{n,q}(x) \right\rangle = [n]_q! \delta_{n,k}. \tag{2.15}$$

From (2.14) and (2.15), we can derive the following equation:

$$\begin{aligned} \left\langle \left(\frac{e_q(t)+1}{2} \right) t^k \middle| p(x) \right\rangle &= \sum_{l=0}^{n} b_{l,q} \left\langle \left(\frac{e_q(t)+1}{2} \right) t^k \middle| E_{l,q}(x) \right\rangle \\ &= \sum_{l=0}^{n} b_{l,q} [l]_q! \delta_{l,k} = [k]_q! b_{k,q}. \end{aligned} \tag{2.16}$$

Thus, by (2.16), we get

$$\begin{aligned} b_{k,q} &= \frac{1}{[k]_q!} \left\langle \left(\frac{e_q(t)+1}{2} \right) t^k \middle| p(x) \right\rangle = \frac{1}{2[k]_q!} \left\langle \left(e_q(t)+1 \right) t^k | p(x) \right\rangle \\ &= \frac{1}{2[k]_q!} \left\langle e_q(t)+1 | p^{(k)}(x) \right\rangle = \frac{1}{2[k]_q!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\}, \end{aligned} \tag{2.17}$$

where $p^{(k)}(x) = D_q^k p(x)$.

Therefore, by (2.14) and (2.17), we obtain the following theorem.

Theorem 2.3 *For $p(x) \in \mathbb{P}_n$, let $p(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x)$. Then we have*

$$\begin{aligned} b_{k,q} &= \frac{1}{2[k]_q!} \left\langle \left(e_q(t)+1 \right) t^k | p(x) \right\rangle \\ &= \frac{1}{2[k]_q!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\}, \end{aligned}$$

where $p^{(k)}(x) = D_q^k p(x)$.

From (1.5), we note that

$$B_{n,q}(x) \sim \left(\frac{e_q(t)-1}{t}, t \right) \quad (n \geq 0). \tag{2.18}$$

Let us take $p(x) = B_{n,q}(x) \in \mathbb{P}_n$. Then $B_{n,q}(x)$ can be represented as a linear combination of $\{E_{0,q}(x), E_{1,q}(x), \ldots, E_{n,q}(x)\}$ as follows:

$$B_{n,q}(x) = p(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x) \quad (n \geq 0), \tag{2.19}$$

where

$$\begin{aligned} b_{k,q} &= \frac{1}{2[k]_q!} \langle (e_q(t)+1) t^k | B_{n,q}(x) \rangle \\ &= \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{2[k]_q!} \langle e_q(t)+1 | B_{n-k,q}(x) \rangle \\ &= \frac{1}{2} \binom{n}{k}_q \langle e_q(t)+1 | B_{n-k,q}(x) \rangle = \frac{1}{2} \binom{n}{k}_q \{ B_{n-k,q}(1) + B_{n-k,q} \}. \end{aligned} \tag{2.20}$$

From (1.5), we can derive the following recurrence relation for the q-Bernoulli numbers:

$$\begin{aligned} t &= \left(\sum_{l=0}^{\infty} B_{l,q} \frac{t^l}{[l]_q!} \right) (e_q(t) - 1) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l}_q B_{l,q} \right) \frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty} (B_{n,q}(1) - B_{n,q}) \frac{t^n}{[n]_q!}. \end{aligned} \tag{2.21}$$

Thus, by (2.21), we get

$$B_{0,q} = 1, \qquad B_{n,q}(1) - B_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{2.22}$$

For example, $B_{0,q} = 1$, $B_{1,q} = -\frac{1}{[2]_q}$, $B_{2,q} = \frac{q^2}{[3]_q[2]_q}, \ldots$.
By (2.19), (2.20) and (2.22), we get

$$\begin{aligned} B_{n,q}(x) &= b_{n,q} E_{n,q}(x) + b_{n-1,q} E_{n-1,q}(x) + \sum_{k=0}^{n-2} b_{k,q} E_{k,q}(x) \\ &= E_{n,q}(x) + \frac{[n]_q}{2} \left(1 - \frac{2}{[2]_q} \right) E_{n-1,q}(x) + \sum_{k=0}^{n-2} \binom{n}{k}_q B_{n-k,q} E_{k,q}(x) \\ &= E_{n,q}(x) - \frac{[n]_q(1-q)}{2[2]_q} E_{n-1,q}(x) + \sum_{k=0}^{n-2} \binom{n}{k}_q B_{n-k,q} E_{k,q}(x). \end{aligned} \tag{2.23}$$

Therefore, by (2.23), we obtain the following theorem.

Theorem 2.4 *For $n \geq 2$, we have*

$$B_{n,q}(x) = E_{n,q}(x) + \frac{[n]_q(q-1)}{2[2]_q} E_{n-1,q}(x) + \sum_{k=0}^{n-2} \binom{n}{k}_q B_{n-k,q} E_{k,q}(x).$$

For $r \in \mathbb{Z}_{\geq 0}$, the q-Euler polynomials, $E_{n,q}^{(r)}(x)$, of order r are defined by the generating function to be

$$\begin{aligned}\left(\frac{2}{e_q(t)+1}\right)^r e_q(xt) &= \underbrace{\left(\frac{2}{e_q(t)+1}\right) \times \cdots \times \left(\frac{2}{e_q(t)+1}\right)}_{r\text{-times}} e_q(xt) \\ &= \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{[n]_q!}. \qquad (2.24)\end{aligned}$$

In the special case, $x = 0$, $E_{n,q}^{(r)}(0) = E_{n,q}^{(r)}$ are called the nth q-Euler numbers of order r.

Let

$$g^r(t) = \left(\frac{e_q(t)+1}{2}\right)^r \quad (r \in \mathbb{Z}_{\geq 0}). \qquad (2.25)$$

Then $g^r(t)$ is an invertible series. From (2.24) and (2.25), we have

$$\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{[n]_q!} = \frac{1}{g^r(t)} e_q(xt) = \sum_{n=0}^{\infty} \frac{1}{g^r(t)} x^n \frac{t^n}{[n]_q!}. \qquad (2.26)$$

By (2.26), we get

$$E_{n,q}^{(r)}(x) = \frac{1}{g^r(t)} x^n, \qquad (2.27)$$

and

$$tE_{n,q}^{(r)}(x) = \frac{1}{g^r(t)} tx^n = [n]_q \frac{1}{g^r(t)} x^{n-1} = [n]_q E_{n-1,q}^{(r)}(x). \qquad (2.28)$$

Thus, by (2.26), (2.27) and (2.28), we see that

$$E_{n,q}^{(r)}(x) \sim \left(\left(\frac{e_q(t)+1}{2}\right)^r, t\right). \qquad (2.29)$$

By (1.9) and (2.24), we get

$$\left\langle \left(\frac{2}{e_q(t)+1}\right)^r e_q(yt) \middle| x^n \right\rangle = E_{n,q}^{(r)}(y) = \sum_{l=0}^{n} \binom{n}{l}_q E_{n-l,q}^{(r)} y^l. \qquad (2.30)$$

Thus, we have

$$\begin{aligned}\left\langle \left(\frac{2}{e_q(t)+1}\right)^r \middle| x^n \right\rangle &= \sum_{m=0}^{\infty} \left(\sum_{i_1+\cdots+i_r=m} \frac{E_{i_1,q} \cdots E_{i_r,q}}{[i_1]_q! \cdots [i_r]_q!} \right) \langle t^m | x^n \rangle \\ &= \sum_{i_1+\cdots+i_r=n} \frac{[n]_q!}{[i_1]_q! \cdots [i_r]_q!} E_{i_1,q} \cdots E_{i_r,q} \\ &= \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1, \ldots, i_r}_q E_{i_1,q} \cdots E_{i_r,q}, \qquad (2.31)\end{aligned}$$

where $\binom{n}{i_1,\ldots,i_r}_q = \frac{[n]_q!}{[i_1]_q! \cdots [i_r]_q!}$.

By (2.30), we easily get

$$\left\langle \left(\frac{2}{e_q(t)+1} \right)^r \middle| x^n \right\rangle = E_{n,q}^{(r)}. \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$E_{n,q}^{(r)} = \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1,\ldots,i_r}_q E_{i_1,q} \cdots E_{i_r,q},$$

where $\binom{n}{i_1,\ldots,i_r}_q = \frac{[n]_q!}{[i_1]_q!\cdots[i_r]_q!}$.

Let us take $p(x) = E_{n,q}^{(r)}(x) \in \mathbb{P}_n$. Then, by Theorem 2.3, we get

$$E_{n,q}^{(r)}(x) = p(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x), \tag{2.33}$$

where

$$\begin{aligned} b_{k,q} &= \frac{1}{2[k]_q!} \langle (e_q(t)+1) t^k | p(x) \rangle = \frac{1}{2[k]_q!} \langle (e_q(t)+1) | t^k p(x) \rangle \\ &= \frac{\binom{n}{k}_q}{2} \langle (e_q(t)+1) | E_{n-k,q}^{(r)}(x) \rangle = \frac{\binom{n}{k}_q}{2} \{ E_{n-k,q}^{(r)}(1) + E_{n-k,q}^{(r)} \}. \end{aligned} \tag{2.34}$$

From (2.24), we have

$$\begin{aligned} \sum_{k=0}^{\infty} \{ E_{n,q}^{(r)}(1) + E_{n,q}^{(r)} \} \frac{t^n}{[n]_q!} &= \left(\frac{2}{e_q(t)+1} \right)^r (e_q(t)+1) \\ &= 2 \left(\frac{2}{e_q(t)+1} \right)^{r-1} = 2 \sum_{n=0}^{\infty} E_{n,q}^{(r-1)} \frac{t^n}{[n]_q!}. \end{aligned} \tag{2.35}$$

By comparing the coefficients on the both sides of (2.35), we get

$$E_{n,q}^{(r)}(1) + E_{n,q}^{(r)} = 2E_{n,q}^{(r-1)} \quad (n \geq 0). \tag{2.36}$$

Therefore, by (2.33), (2.34) and (2.36), we obtain the following theorem.

Theorem 2.6 *For $n \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{>0}$, we have*

$$E_{n,q}^{(r)}(x) = \sum_{k=0}^{\infty} \binom{n}{k}_q E_{n-k,q}^{(r-1)} E_{k,q}(x).$$

Let us assume that

$$p(x) = \sum_{k=0}^{n} b_{k,q}^r E_{k,q}^{(r)}(x) \in \mathbb{P}_n. \tag{2.37}$$

By (2.29) and (2.37), we get

$$\left\langle\left(\frac{e_q(t)+1}{2}\right)^r t^k \middle| p(x)\right\rangle = \sum_{l=0}^{n} b_{l,q}^r \left\langle\left(\frac{e_q(t)+1}{2}\right)^r t^k \middle| E_{l,q}^{(r)}(x)\right\rangle$$
$$= \sum_{l=0}^{n} b_{l,q}^r [l]_q! \delta_{l,k} = [k]_q! b_{k,q}^r. \tag{2.38}$$

From (2.38), we have

$$\begin{aligned} b_{k,q}^r &= \frac{1}{[k]_q!}\left\langle\left(\frac{e_q(t)+1}{2}\right)^r t^k \middle| p(x)\right\rangle = \frac{1}{2^r[k]_q!}\left\langle\left(e_q(t)+1\right)^r \middle| t^k p(x)\right\rangle \\ &= \frac{1}{2^r[k]_q!}\sum_{l=0}^{r}\binom{r}{l}\sum_{m\geq 0}\left(\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}_q\right)\frac{1}{[m]_q!}\left\langle 1 \middle| t^{m+k}p(x)\right\rangle \\ &= \frac{1}{2^r[k]_q!}\sum_{l=0}^{r}\binom{r}{l}\sum_{m\geq 0}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}_q\frac{1}{[m]_q!}p^{(m+k)}(0). \end{aligned} \tag{2.39}$$

Therefore by (2.37) and (2.39), we obtain the following theorem.

Theorem 2.7 *For $n \geq 0$, let $p(x) = \sum_{k=0}^{n} b_{k,q}^r E_{k,q}^{(r)}(x) \in \mathbb{P}_n$.*
Then we have

$$\begin{aligned} b_{k,q}^r &= \frac{1}{2^r[k]_q!} = \left\langle\left(e_q(t)+1\right)^r t^k \middle| p(x)\right\rangle \\ &= \frac{1}{2^r[k]_q!}\sum_{m\geq 0}\sum_{l=0}^{r}\binom{r}{l}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}_q\frac{1}{[m]_q!}p^{(m+k)}(0), \end{aligned}$$

where $p^{(k)}(x) = D_q^k p(x)$.

Let us take $p(x) = E_{n,q}(x) \in \mathbb{P}_n$. Then, by Theorem 2.7, we get

$$E_{n,q}(x) = p(x) = \sum_{k=0}^{n} b_{k,q}^r E_{k,q}^{(r)}(x), \tag{2.40}$$

where

$$\begin{aligned} b_{k,q} &= \frac{1}{2^r[k]_q!}\sum_{m=0}^{n-k}\sum_{l=0}^{r}\binom{r}{l}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}_q \\ &\quad\times\frac{1}{[m]_q!}[n]_q\cdots[n-m-k+1]_q E_{n-m-k,q} \\ &= \frac{1}{2^r}\sum_{m=0}^{n-k}\sum_{l=0}^{r}\binom{r}{l}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}_q \\ &\quad\times\frac{[m+k]_q!}{[m]_q![k]_q!}\frac{[n]_q\cdots[n-m-k+1]_q}{[m+k]_q!}E_{n-m-k,q} \\ &= \frac{1}{2^r}\sum_{m=0}^{n-k}\sum_{l=0}^{r}\sum_{i_1+\cdots+i_l=m}\binom{r}{l}\binom{m}{i_1,\ldots,i_l}_q\binom{m+k}{m}_q\binom{n}{m+k}_q E_{n-m-k,q}. \end{aligned} \tag{2.41}$$

Therefore, by (2.40) and (2.41), we obtain the following theorem.

Theorem 2.8 *For $n, r \geq 0$, we have*

$$E_{n,q}(x) = \frac{1}{2^r}\sum_{k=0}^{n}\left\{\sum_{m=0}^{n-k}\sum_{l=0}^{r}\sum_{i_1+\cdots+i_l=m}\binom{r}{l}\binom{m}{i_1,\ldots,i_l}_q\binom{m+k}{m}_q\binom{n}{m+k}_q \times E_{n-m-k,q}\right\}E_{k,q}^{(r)}(x).$$

For $r \in \mathbb{Z}_{\geq 0}$, let us consider q-Bernoulli polynomials of order r which are defined by the generating function to be

$$\begin{aligned}\left(\frac{t}{e_q(t)-1}\right)^r e_q(xt) &= \underbrace{\left(\frac{t}{e_q(t)-1}\right)\times\cdots\times\left(\frac{t}{e_q(t)-1}\right)}_{r\text{-times}} e_q(xt)\\ &= \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x)\frac{t^n}{[n]_q!}.\end{aligned} \tag{2.42}$$

In the special case, $x = 0$, $B_{n,q}^{(r)}(0) = B_{n,q}^{(r)}$ are called the nth q-Bernoulli numbers of order r. By (2.42), we easily get

$$B_{n,q}^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}_q B_{l,q}^{(r)}x^{n-l} \in \mathbb{P}_n. \tag{2.43}$$

Let us take $p(x) = B_{n,q}^{(r)}(x) \in \mathbb{P}_n$. Then, by Theorem 2.7, we get

$$B_{n,q}^{(r)}(x) = p(x) = \sum_{k=0}^{n} b_{k,q}^{r} E_{k,q}^{(r)}(x), \tag{2.44}$$

where

$$\begin{aligned}b_{k,q}^{r} &= \frac{1}{2^r[k]_q!}\left\langle \left(e_q(t)+1\right)^r t^k \middle| B_{n,q}^{(r)}(x)\right\rangle\\ &= \frac{1}{2^r[k]_q!}\sum_{m=0}^{n-k}\sum_{l=0}^{r}\binom{r}{l}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\ldots,i_l}_q\frac{[n]_q\cdots[n-m-k+1]_q}{[m]_q!}B_{n-m-k,q}^{(r)}\\ &= \frac{1}{2^r}\sum_{m=0}^{n-k}\sum_{l=0}^{r}\sum_{i_1+\cdots+i_l=m}\binom{r}{l}\binom{m}{i_1,\ldots,i_l}_q\binom{m+k}{m}_q\binom{n}{m+k}_q B_{n-m-k,q}^{(r)}.\end{aligned} \tag{2.45}$$

Therefore, by (2.44) and (2.45), we obtain the following theorem.

Theorem 2.9 *For $n, r \geq 0$, we have*

$$B_{n,q}^{(r)}(x) = \frac{1}{2^r}\sum_{k=0}^{n}\left\{\sum_{m=0}^{n-k}\sum_{l=0}^{r}\sum_{i_1+\cdots+i_l=m}\binom{r}{l}\binom{m}{i_1,\ldots,i_l}_q\binom{m+k}{m}_q\binom{n}{m+k}_q \times B_{n-m-k,q}^{(r)}\right\}E_{k,q}^{(r)}(x).$$

Remark Recently, Aral, Gupta and Agarwal introduced many interesting properties and applications of q-calculus which are related to this paper (see [13]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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