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Some properties of higher-order Daehee polynomials of the second kind arising from umbral calculus

Dae San Kim[1] and Taekyun Kim[2*]

*Correspondence: tkkim@kw.ac.kr
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we study the higher-order Daehee polynomials of the second kind from the umbral calculus viewpoint and give various identities of the higher-order Daehee polynomials of the second kind arising from umbral calculus.

1 Introduction

Let $k \in \mathbb{Z}_{\geq 0}$. The Daehee polynomials of the second kind of order k are defined by the generating function to be

$$\left(\frac{(1+t)\log(1+t)}{t}\right)^k (1+t)^x = \sum_{n=0}^{\infty} \hat{D}_n^{(k)}(x)\frac{t^n}{n!} \tag{1}$$

(see [1]).

When $x = 0$, $\hat{D}_n^{(k)} = \hat{D}_n^{(k)}(0)$ are called the Daehee numbers of the second kind of order k.

The Stirling number of the first kind is defined by the falling factorial to be

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l. \tag{2}$$

Thus, by (2), we get

$$\big(\log(1+t)\big)^m = m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!} \tag{3}$$

(see [2–4]), where $m \in \mathbb{Z}_{\geq 0}$.

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order s ($\in \mathbb{N}$) are given by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x|\lambda)\frac{t^n}{n!} \tag{4}$$

(see [1–18]).

When $x = 0$, $H_n^{(s)}(\lambda) = H_n^{(s)}(\lambda|0)$ are called the Frobenius-Euler numbers of order s.

As is well known, the Bernoulli polynomials of order k ($\in \mathbb{N}$) are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!} \tag{5}$$

(see [1–18]).

When $x=0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the Bernoulli numbers of order k.

In this paper, we study the higher-order Daehee polynomials of the second kind with umbral calculus viewpoint and give various identities of the higher-order Daehee polynomials of the second kind arising from umbral calculus.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$, and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ indicates the action of the linear functional L on the polynomial $p(x)$. Then the vector space operations on $\mathbb{P}^*$ are given by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, and $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, the linear functional on $\mathbb{P}$ is defined by $\langle f(t)|x^n\rangle = a_n$. Then, in particular, we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k \geq 0) \tag{6}$$

(see [3, 18]), where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$. By (6), we get $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of the formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of the umbral algebra. The order $o(f(t))$ of the power series $f(t)$ ($\neq 0$) is the smallest integer for which the coefficient of t^k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series; if $o(f(t)) = 1$, then $f(t)$ is called a delta series.

Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$, for $n,k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$. For $f(t), g(t) \in \mathcal{F}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle. \tag{7}$$

From (6), we note that

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \qquad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!} \tag{8}$$

and, by (8), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x+y) \tag{9}$$

(see [3, 18]).

For $s_n(x) \sim (g(t), f(t))$, we have

$$\frac{ds_n(x)}{dx} = \sum_{l=0}^{n-1} \binom{n}{l} \langle \bar{f}(t) | x^{n-l} \rangle s_l(x), \tag{10}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = t$. We have

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad \text{for all } x \in \mathbb{C}, \tag{11}$$

$$f(t) s_n(x) = n s_{n-1}(x) \quad (n \geq 1), \qquad s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle x^j, \tag{12}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{13}$$

where $p_n(x) = g(t) s_n(x)$.

$$\langle f(t) | x p(x) \rangle = \langle \partial_t f(t) | p(x) \rangle, \tag{14}$$

with $\partial_t f(t) = \frac{df(t)}{dt}$, and

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x) \quad (n \geq 0) \tag{15}$$

(see [3, 18]).

Let us assume that $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$. Then we see that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0), \tag{16}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle \tag{17}$$

(see [3, 18]).

3 Higher-order Daehee polynomials of the second kind

By (1), we see that

$$\hat{D}_n^{(k)}(x) \sim \left(\left(\frac{e^t - 1}{te^t} \right)^k, e^t - 1 \right). \tag{18}$$

From (18), we have

$$\left(\frac{e^t-1}{te^t}\right)^k \hat{D}_n^{(k)}(x) \sim \left(1, e^t-1\right) \quad \text{and} \quad (x)_n \sim \left(1, e^t-1\right). \tag{19}$$

By (19), we get

$$\begin{aligned}
\hat{D}_n^{(k)}(x) &= \left(\frac{te^t}{e^t-1}\right)^k (x)_n \\
&= \sum_{m=0}^{n} S_1(n,m)\left(\frac{te^t}{e^t-1}\right)^k x^m \\
&= \sum_{m=0}^{n} S_1(n,m) e^{kt} B_n^{(k)}(x) \\
&= \sum_{m=0}^{n} S_1(n,m) B_m^{(k)}(x+k).
\end{aligned} \tag{20}$$

From (12) and (18), we have

$$\hat{D}_n^{(k)}(x) = \sum_{j=0}^{n} \frac{1}{j!}\left\langle \left(\frac{(1+t)\log(1+t)}{t}\right)^k \left(\log(1+t)\right)^j \middle| x^n \right\rangle x^j, \tag{21}$$

where

$$\begin{aligned}
&\left\langle \left(\frac{(1+t)\log(1+t)}{t}\right)^k \left(\log(1+t)\right)^j \middle| x^n \right\rangle \\
&= \left\langle \left(\frac{\log(1+t)}{t}\right)^{k+j} (1+t)^k \middle| t^j x^n \right\rangle \\
&= (n)_j \left\langle \left(\frac{\log(1+t)}{t}\right)^{k+j} \middle| \sum_{m=0}^{\min\{k,n-j\}} \binom{k}{m} t^m x^{n-j} \right\rangle \\
&= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \sum_{l=0}^{\infty} \frac{(k+j)!}{(l+k+j)!} S_1(l+k+j, k+j) \left\langle t^l \middle| x^{n-j-m} \right\rangle \\
&= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{(k+j)!}{(n+k-m)!} S_1(n+k-m, k+j)(n-j-m)! \\
&= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{S_1(n+k-m, k+j)}{\binom{n+k-m}{k+j}}.
\end{aligned} \tag{22}$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 1 *For $n \in \mathbb{Z}_{\geq 0}$ and $k \geq 1$, we have*

$$\hat{D}_n^{(k)}(x) = \sum_{j=0}^{n} \left\{ \binom{n}{j} \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{S_1(n+k-m, k+j)}{\binom{n+k-m}{k+j}} \right\} x^j.$$

By (1) and (6), we get

$$\begin{aligned}
\hat{D}_n^{(k)}(y) &= \left\langle \sum_{l=0}^{\infty} \hat{D}_l^{(k)}(y)\frac{t^l}{l!} \middle| x^n \right\rangle \\
&= \left\langle \left(\frac{\log(1+t)}{t}\right)^k (1+t)^y \middle| (1+t)^k x^n \right\rangle \\
&= \sum_{0\le r\le\min\{k,n\}} \binom{k}{r}(n)_r \left\langle \left(\frac{\log(1+t)}{t}\right)^k (1+t)^y \middle| x^{n-r} \right\rangle \\
&= \sum_{0\le r\le\min\{k,n\}} \binom{k}{r}(n)_r \sum_{0\le m\le n-r} \binom{y}{m}(n-r)_m \\
&\quad \times \sum_{0\le l\le n-r-m} \frac{k!S_1(l+k,k)}{(l+k)!} \langle t^l | x^{n-r-m} \rangle \\
&= \sum_{0\le r\le n}\sum_{0\le m\le n-r} \frac{(n)_r\binom{k}{r}\binom{n-r}{m}}{\binom{n-r-m+k}{k}} S_1(n-r-m+k,k)(y)_m. \qquad (23)
\end{aligned}$$

Therefore, by (23), we obtain the following theorem.

Theorem 2 *For $n \ge 0$, we have*

$$\begin{aligned}
&\hat{D}_n^{(k)}(x) \\
&\quad = \sum_{0\le m\le n} \left\{ \sum_{0\le r\le n-m} \frac{(n)_r\binom{k}{r}\binom{n-r}{m}}{\binom{n-r-m+k}{k}} S_1(n-r-m+k,k) \right\} (x)_m \\
&\quad = \sum_{0\le m\le n} \left\{ \sum_{0\le r\le n-m} \frac{(n)_r\binom{k}{r}\binom{n-r}{n-m}}{\binom{m-r+k}{k}} S_1(m-r+k,k) \right\} (x)_{n-m}.
\end{aligned}$$

From (12) and (18), we have

$$\left(e^t-1\right)\hat{D}_n^{(k)}(x) = n\hat{D}_{n-1}^{(k)}(x) \qquad (24)$$

and

$$\left(e^t-1\right)\hat{D}_n^{(k)}(x) = \hat{D}_n^{(k)}(x+1) - \hat{D}_n^{(k)}(x).$$

Thus, by (24), we get

$$\hat{D}_n^{(k)}(x+1) - \hat{D}_n^{(k)}(x) = n\hat{D}_{n-1}^{(k)}(x) \quad (n \ge 1). \qquad (25)$$

From (15) and (18), we derive the following equation:

$$\begin{aligned}
\hat{D}_{n+1}^{(k)}(x) &= \left(x + k\frac{e^t-1-t}{t(e^t-1)}\right)e^{-t}\hat{D}_n^{(k)}(x) \\
&= x\hat{D}_n^{(k)}(x-1) + ke^{-t}\frac{e^t-1-t}{t(e^t-1)}\hat{D}_n^{(k)}(x), \qquad (26)
\end{aligned}$$

where

$$\begin{aligned}
&e^{-t}\frac{e^t-1-t}{t(e^t-1)}\hat{D}_n^{(k)}(x)\\
&\quad=e^{-t}\frac{e^t-1-t}{t(e^t-1)}\sum_{0\le j\le n}\left\{\binom{n}{j}\sum_{0\le m\le n-j}\frac{m!\binom{k}{m}\binom{n-j}{m}}{\binom{n+k-m}{k+j}}\right.\\
&\qquad\left.\times S_1(n+k-m,k+j)\right\}x^j\\
&\quad=\sum_{0\le j\le n}\binom{n}{j}\sum_{0\le m\le n-j}\frac{m!\binom{k}{m}\binom{n-j}{m}}{\binom{n+k-m}{k+j}}\\
&\qquad\times S_1(n+k-m,k+j)e^{-t}\frac{e^t-1-t}{t(e^t-1)}x^j\\
&\quad=\sum_{0\le j\le n}\binom{n}{j}\sum_{0\le m\le n-j}\frac{m!\binom{m+k}{m}\binom{n-j}{m}}{\binom{n+k-m}{k+j}}\\
&\qquad\times S_1(n+k-m,k+j)e^{-t}\left(\frac{e^t-1-t}{e^t-1}\right)\frac{x^{j+1}}{j+1}\\
&\quad=\sum_{0\le j\le n}\binom{n}{j}\sum_{0\le m\le n-j}\frac{m!\binom{m+k}{m}\binom{n-j}{m}}{\binom{n+k-m}{k+j}}\\
&\qquad\times\frac{S_1(n+k-m,k+j)}{j+1}e^{-t}\left(x^{j+1}-B_{j+1}(x)\right)\\
&\quad=\sum_{0\le j\le n}\binom{n}{j}\sum_{0\le m\le n-j}\frac{m!\binom{m+k}{m}\binom{n-j}{m}}{\binom{n+k-m}{k+j}}\\
&\qquad\times\frac{S_1(n+k-m,k+j)}{j+1}e^{-t}\left((x-1)^{j+1}-B_{j+1}(x-1)\right).
\end{aligned}\tag{27}$$

Therefore, from (26) and (27), we obtain the following theorem.

Theorem 3 *For $n\ge 0$, $k\ge 1$, we have*

$$\begin{aligned}
&\hat{D}_{n+1}^{(k)}(x)\\
&\quad=x\hat{D}_n^{(k)}(x-1)+k\sum_{0\le j\le n}\binom{n}{j}\sum_{0\le m\le n-j}\frac{m!\binom{m+k}{m}\binom{n-j}{m}}{\binom{n+k-m}{k+j}}\\
&\qquad\times\frac{S_1(n+k-m,k+j)}{j+1}\left\{(x-1)^{j+1}-B_{j+1}(x-1)\right\}.
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
&e^{-t}\frac{e^t-1-t}{t(e^t-1)}\hat{D}_n^{(k)}(x)\\
&\quad=\sum_{j=0}^{n}\frac{\binom{n}{j}}{\binom{n+k}{j+k}}S_1(n+k,j+k)e^{-t}\frac{e^t-1-t}{t(e^t-1)}(x+k)^j
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n+k}{j+k}} S_1(n+k,j+k) e^{(k-1)t} \frac{e^t-1-t}{t(e^t-1)} x^j \\
&= \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n+k}{j+k}} \frac{S_1(n+k,j+k)}{j+1} e^{(k-1)t} \bigl(x^{j+1} - B_{j+1}(x)\bigr) \\
&= \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n+k}{j+k}} \frac{S_1(n+k,j+k)}{j+1} \bigl((x+k-1)^{j+1} - B_{j+1}(x+k-1)\bigr).
\end{aligned} \tag{28}
$$

Thus, by (28), we get

$$
\hat{D}_{n+1}^{(k)}(x) = x\hat{D}_n^{(k)}(x-1) + k \sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{n+k}{j+k}} \frac{S_1(n+k,j+k)}{j+1} \bigl((x+k-1)^{j+1} - B_{j+1}(x+k-1)\bigr).
$$

From (10) and (18), we note that

$$
\frac{d}{dx}\hat{D}_n^{(k)}(x) = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \hat{D}_l^{(k)}(x). \tag{29}
$$

By (6) and (18), we see that

$$
\begin{aligned}
\hat{D}_n^{(k)}(y) &= \left\langle \sum_{l=0}^{\infty} \hat{D}_l^{(k)}(y) \frac{t^l}{l!} \,\middle|\, x^n \right\rangle \quad (n \geq 1) \\
&= \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (1+t)^y \,\middle|\, x^n \right\rangle \\
&= \left\langle \partial_t \left(\left(\frac{(1+t)\log(1+t)}{t} \right)^k (1+t)^y \right) \,\middle|\, x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \left(\frac{(1+t)\log(1+t)}{t} \right)^k \right) (1+t)^y \,\middle|\, x^{n-1} \right\rangle \\
&\quad + y \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (1+t)^{y-1} \,\middle|\, x^{n-1} \right\rangle \\
&= y\hat{D}_{n-1}^{(k)}(y-1) \\
&\quad + k \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} (1+t)^y \,\middle|\, \left(\log(1+t) + 1 - \frac{(1+t)\log(1+t)}{t} \right) \frac{x^n}{n} \right\rangle \\
&= y\hat{D}_{n-1}^{(k)}(y-1) + \frac{k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} (1+t)^y \,\middle|\, \log(1+t) x^n \right\rangle \\
&\quad + \frac{k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} (1+t)^y \,\middle|\, x^n \right\rangle \\
&\quad - \frac{k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k} (1+t)^y \,\middle|\, x^n \right\rangle \\
&= y\hat{D}_{n-1}^{(k)}(y-1) + \frac{k}{n}\hat{D}_n^{(k-1)}(y) - \frac{k}{n}\hat{D}_n^{(k)}(y) \\
&\quad + \frac{k}{n} \sum_{1 \leq l \leq n} \frac{(-1)^{l-1}(n)_l}{l} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} (1+t)^y \,\middle|\, x^{n-l} \right\rangle.
\end{aligned} \tag{30}
$$

Thus, by (30), we get

$$\hat{D}_n^{(k)}(x) = \frac{n}{n+k}x\hat{D}_{n-1}^{(k)}(x-1) + \frac{k}{n+k}\hat{D}_n^{(k-1)}(x) + \frac{k}{n+k}\sum_{1\le l\le n}(-1)^{l-1}\binom{n}{l}(l-1)!\hat{D}_{n-l}^{(k-1)}(x). \tag{31}$$

Therefore, by (31), we obtain the following theorem.

Theorem 4 *For $n \ge 0$, $k \ge 1$, we have*

$$\hat{D}_n^{(k)}(x) = \frac{n}{n+k}x\hat{D}_{n-1}^{(k)}(x-1) + \frac{k}{n+k}\hat{D}_n^{(k-1)}(x) + \frac{k}{n+k}\sum_{1\le l\le n}(-1)^{l-1}\binom{n}{l}(l-1)!\hat{D}_{n-l}^{(k-1)}(x).$$

Now, we compute $\langle(\frac{(1+t)\log(1+t)}{t})^k(\log(1+t))^m|x^n\rangle$ in two different ways:

$$\begin{aligned}
&\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^k\bigl(\log(1+t)\bigr)^m\middle|x^n\right\rangle \\
&= \left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^k\middle|\bigl(\log(1+t)\bigr)^m x^n\right\rangle \\
&= \sum_{0\le l\le n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^k\middle|x^{n-l-m}\right\rangle \\
&= \sum_{0\le l\le n-m}m!\binom{n}{l+m}S_1(l+m,m)\hat{D}_{n-l-m}^{(k)} \\
&= \sum_{0\le l\le n-m}m!\binom{n}{l}S_1(n-l,m)\hat{D}_l^{(k)}.
\end{aligned} \tag{32}$$

On the other hand,

$$\begin{aligned}
&\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^k\bigl(\log(1+t)\bigr)^m\middle|x^n\right\rangle \\
&= \left\langle\partial_t\left(\left(\frac{(1+t)\log(1+t)}{t}\right)^k\bigl(\log(1+t)\bigr)^m\right)\middle|x^{n-1}\right\rangle \\
&= k\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k-1}\left(\frac{\log(1+t)+1-\frac{(1+t)\log(1+t)}{t}}{t}\right)\bigl(\log(1+t)\bigr)^m\middle|x^{n-1}\right\rangle \\
&\quad + m\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^k(1+t)^{-1}\bigl(\log(1+t)\bigr)^{m-1}\middle|x^{n-1}\right\rangle \\
&= \frac{k}{n}\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k-1}\bigl(\log(1+t)\bigr)^{m+1}\middle|x^n\right\rangle \\
&\quad + \frac{k}{n}\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k-1}\bigl(\log(1+t)\bigr)^m\middle|x^n\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&-\frac{k}{n}\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k}\big(\log(1+t)\big)^{m}\Big|x^{n}\right\rangle\\
&+m\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k}(1+t)^{-1}\big(\log(1+t)\big)^{m-1}\Big|x^{n-1}\right\rangle.
\end{aligned}\tag{33}
$$

Thus, by (33), we get

$$
\begin{aligned}
&\frac{n+k}{n}\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k}\big(\log(1+t)\big)^{m}\Big|x^{n}\right\rangle\\
&\quad=\frac{k}{n}\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k-1}\big(\log(1+t)\big)^{m+1}\Big|x^{n}\right\rangle\\
&\qquad+\frac{k}{n}\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k-1}\big(\log(1+t)\big)^{m}\Big|x^{n}\right\rangle\\
&\qquad+m\left\langle\left(\frac{(1+t)\log(1+t)}{t}\right)^{k}(1+t)^{-1}\big(\log(1+t)\big)^{m-1}\Big|x^{n-1}\right\rangle.
\end{aligned}\tag{34}
$$

From (34), we derive the following equation:

$$
\begin{aligned}
&\frac{n+k}{k}\sum_{0\le l\le n-m}m!\binom{n}{l}S_1(n-l,m)\hat{D}_l^{(k)}\\
&\quad=\frac{k}{n}\sum_{0\le l\le n-m-1}(m+1)!\binom{n}{l}S_1(n-l,m+1)\hat{D}_l^{(k-1)}\\
&\qquad+\frac{k}{n}\sum_{0\le l\le n-m}m!\binom{n}{l}S_1(n-l,m)\hat{D}_l^{(k-1)}\\
&\qquad+m\sum_{0\le l\le n-m}(m-1)!\binom{n-1}{l}S_1(n-l-1,m-1)\hat{D}_l^{(k)}(-1).
\end{aligned}\tag{35}
$$

Therefore, by (35), we obtain the following theorem.

Theorem 5 *For $n-1\ge m\ge 1$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\hat{D}_l^{(k)}\\
&\quad=\frac{k(m+1)}{n+k}\sum_{0\le l\le n-m-1}\binom{n}{l}S_1(n-l,m+1)\hat{D}_l^{(k-1)}\\
&\qquad+\frac{k}{n+k}\sum_{0\le l\le n-m}\binom{n}{l}S_1(n-l,m)\hat{D}_l^{(k-1)}\\
&\qquad+\frac{n}{n+k}\sum_{0\le l\le n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\hat{D}_l^{(k)}(-1).
\end{aligned}
$$

For $\hat{D}_n^{(k)}(x)\sim((\frac{e^t-1}{te^t})^k,e^t-1)$, and $(x)_n\sim(1,e^t-1)$, let us assume that

$$
\hat{D}_n^{(k)}(x)=\sum_{m=0}^{n}C_{n,m}(x)_m.\tag{36}
$$

Then, by (16) and (17), we get

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k \middle| t^m x^n \right\rangle \\
&= \binom{n}{m} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k \middle| x^{n-m} \right\rangle \\
&= \binom{n}{m} \hat{D}_{n-m}^{(k)}.
\end{aligned} \tag{37}
$$

Therefore, by (36) and (37), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$
\begin{aligned}
\hat{D}_n^{(k)}(x) &= \sum_{0\leq m\leq n} \binom{n}{m} \hat{D}_{n-m}^{(k)}(x)_m \\
&= \sum_{0\leq m\leq n} m! \binom{n}{m} \hat{D}_{n-m}^{(k)} \binom{x}{m}.
\end{aligned}
$$

Now, we consider the following two Sheffer sequences:

$$
\hat{D}_n^{(k)}(x) \sim \left(\left(\frac{e^t - 1}{te^t} \right)^k, e^t - 1 \right) \tag{38}
$$

and

$$
H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t - \lambda}{1-\lambda} \right)^s, t \right), \quad s \in \mathbb{N}, \lambda \in \mathbb{C} \text{ with } \lambda \neq 1.
$$

Let

$$
\hat{D}_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\lambda). \tag{39}
$$

Here

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!(1-\lambda)^s} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k \big(\log(1+t)\big)^m (1-\lambda+t)^s \middle| x^n \right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \sum_{j=0}^{n} \binom{s}{j} (1-\lambda)^{s-j} (n)_j \\
&\quad \times \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k \big(\log(1+t)\big)^m \middle| x^{n-j} \right\rangle \\
&= \sum_{j=0}^{n-m} \binom{s}{j} (1-\lambda)^{-j} (n)_j \sum_{l=0}^{n-m-j} \binom{n-j}{l+m} S_1(l+m, m) \hat{D}_{n-j-l-m}^{(k)} \\
&= \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-j}{l} (n)_j (1-\lambda)^{-j} S_1(n-j-l, m) \hat{D}_l^{(k)}.
\end{aligned} \tag{40}
$$

Therefore, by (39) and (40), we obtain the following theorem.

Theorem 7 *For $n \geq 0$, $k \geq 1$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, we have*

$$\hat{D}_n^{(k)}(x) = \sum_{m=0}^{n} \Biggl\{ \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-j}{l} (n)_j \times (1-\lambda)^{-j} S_1(n-j-l,m) \hat{D}_l^{(k)} \Biggr\} H_m^{(s)}(x|\lambda).$$

We consider the following two Sheffer sequences:

$$\hat{D}_n^{(k)}(x) \sim \left(\left(\frac{e^t-1}{te^t} \right)^k, e^t - 1 \right), \qquad B_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t} \right)^s, t \right).$$

Let

$$\hat{D}_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x). \tag{41}$$

Here

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{(\frac{t}{\log(1+t)})^s}{(\frac{t}{(1+t)\log(1+t)})^k} \bigl(\log(1+t)\bigr)^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle (1+t)^s \frac{(\frac{t}{(1+t)\log(1+t)})^s}{(\frac{t}{(1+t)\log(1+t)})^k} \bigl(\log(1+t)\bigr)^m \middle| x^n \right\rangle. \end{aligned} \tag{42}$$

Case 1. For $s > k$, we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \left(\frac{t}{(1+t)\log(1+t)} \right)^{s-k} \bigl(\log(1+t)\bigr)^m \middle| (1+t)^s x^n \right\rangle \\ &= \frac{1}{m!} \sum_{0 \leq j \leq n} \binom{s}{j} (n)_j \left\langle \left(\frac{t}{(1+t)\log(1+t)} \right)^{s-k} \middle| \bigl(\log(1+t)\bigr)^m x^{n-j} \right\rangle \\ &= \sum_{0 \leq j \leq n-m} \binom{s}{j} (n)_j \sum_{m \leq l \leq n-j} S_1(l,m) \\ &\quad \times \binom{n-j}{l} \left\langle \left(\frac{t}{(1+t)\log(1+t)} \right)^{s-k} \middle| x^{n-j-l} \right\rangle \\ &= \sum_{0 \leq j \leq n-m} \sum_{m \leq l \leq n-j} \binom{s}{j} \binom{n-j}{l} (n)_j S_1(l,m) \hat{C}_{n-j-l}^{(s-k)}, \end{aligned} \tag{43}$$

where $\hat{C}_i^{(s-k)}$ is the ith Cauchy number of the second kind of order $s-k$ (see [14]).

Case 2. For $s = k$, we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \bigl\langle \bigl(\log(1+t)\bigr)^m \big| (1+t)^s x^n \bigr\rangle \\ &= \frac{1}{m!} \left\langle \bigl(\log(1+t)\bigr)^m \middle| \sum_{j=0}^{s} \binom{s}{j} t^j x^n \right\rangle \end{aligned}$$

$$= \sum_{0\le j\le n-m} \binom{s}{j}(n)_j \sum_{m\le l<\infty} \frac{S_1(l,m)}{l!}\langle t^l | x^{n-j}\rangle$$

$$= \sum_{0\le j\le n-m} \binom{s}{j}(n)_j S_1(n-j,m). \tag{44}$$

Case 3. For $s < k$, we have

$$C_{n,m} = \frac{1}{m!}\left\langle \left(\frac{(1+t)\log(1+t)}{t}\right)^{k-s} \bigl(\log(1+t)\bigr)^m \Big| (1+t)^s x^n \right\rangle$$

$$= \sum_{0\le j\le n-m} \sum_{m\le l\le n-j} \binom{s}{j}\binom{n-j}{l}(n)_j S_1(l,m) \hat{D}_{n-j-l}^{(k-s)}. \tag{45}$$

Therefore, by (41), (42), (43), (44), and (45), we obtain the following theorem.

Theorem 8 *Let $n \ge 0$, we have*:

(I) *For $s > k$, we have*

$$\hat{D}_n^{(k)}(x) = \sum_{0\le m\le n} \left\{ \sum_{0\le j\le n-m} \sum_{m\le l\le n-j} \binom{s}{j}\binom{n-j}{l} \times (n)_j S_1(l,m) \hat{C}_{n-j-l}^{(s-k)} \right\} B_m^{(s)}(x).$$

(II) *For $s = k$, we have*

$$\hat{D}_n^{(k)}(x) = \sum_{0\le m\le n} \left\{ \sum_{0\le j\le n-m} \binom{s}{j}(n)_j S_1(n-j,m) \right\} B_m^{(s)}(x).$$

(III) *For $s < k$, we have*

$$\hat{D}_n^{(k)}(x) = \sum_{0\le m\le n} \left\{ \sum_{0\le j\le n-m} \sum_{m\le l\le n-j} \binom{s}{j}\binom{n-j}{l} \times (n)_j S_1(l,m) \hat{D}_{n-j-l}^{(k-s)} \right\} B_m^{(s)}(x).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea.

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