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Sharp bounds for the arithmetic-geometric mean

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Abstract
In this article, we establish some new inequality chains for the ratio of certain bivariate means, and we present several sharp bounds for the arithmetic-geometric mean.
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1 Introduction

Let $\mathbb{R}_+$ be the set of positive real numbers. Then a two-variable continuous function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ is said to be a mean on $\mathbb{R}_+$ if the double inequality

$$\min(a,b) \leq M(a,b) \leq \max(a,b)$$

holds for all $a, b \in \mathbb{R}_+$.

The classical arithmetic-geometric mean $AGM(a,b)$ of two positive real numbers a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$AGM(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

where $a_0 = a$, $b_0 = b$, and for $n \in \mathbb{N}$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}. \tag{1.1}$$

The well-known Gauss identity shows that

$$AGM\big(1, \sqrt{1-r^2}\big) = \frac{\pi}{2\mathrm{K}(r)}$$

for $r \in (0,1)$, where $\mathrm{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2}\, dt$ [1] is the complete elliptic integral of the first kind.

Let $a, b > 0$ with $a \neq b$. Then the well-known Stolarsky mean [2] $S_{p,q}(a,b)$ can be expressed as

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$$S_{p,q}(a,b)=\begin{cases}(\frac{q}{p}\frac{a^p-b^p}{a^q-b^q})^{1/(p-q)} & \text{if } pq(p-q)\neq 0,\\ (\frac{b^p-a^p}{p(\ln b-\ln a)})^{1/p} & \text{if } p\neq 0, q=0,\\ (\frac{b^q-a^q}{q(\ln b-\ln a)})^{1/q} & \text{if } p=0, q\neq 0,\\ \exp(\frac{b^p\ln b-a^p\ln a}{b^p-a^p}-\frac{1}{p}) & \text{if } p=q\neq 0,\\ \sqrt{ab} & \text{if } p=q=0.\end{cases} \tag{1.2}$$

Many bivariate means are the special case of the Stolarsky mean, for example, $S_{2,1}(a,b)=(a+b)/2=A(a,b)$ is the arithmetic mean, $S_{0,0}(a,b)=\sqrt{ab}=G(a,b)$ is the geometric mean, $S_{3/2,1/2}(a,b)=(a+\sqrt{ab}+b)/3=He(a,b)$ is the Heronian mean, $S_{1,0}(a,b)=(b-a)/(\ln b-\ln a)=L(a,b)$ is the logarithmic mean, $S_{1,1}(a,b)=1/e(b^b/a^a)^{1/(b-a)}=I(a,b)$ is the identric (exponential) mean, $S_{2p,p}(a,b)=A^{1/p}(a^p,b^p)=A_p(a,b)$ the p-order arithmetic (power, Hölder) mean, $S_{3p/2,p/2}(a,b)=He^{1/p}(a^p,b^p)=He_p(a,b)$ is the p-order Heronian mean, $S_{p,0}(a,b)=L^{1/p}(a^p,b^p)=L_p(a,b)$ is the p-order logarithmic mean, $S_{p,p}(a,b)=I^{1/p}(a^p,b^p)=I_p(a,b)$ is the p-order identric (exponential) mean and $S_{p+1,p}(a,b)=[p(a^{p+1}-b^{p+1})]/[(p+1)(a^p-b^p)]=\mathcal{J}_p(a,b)$ is the one-parameter mean.

Another important family of means is the Gini means [3] defined by

$$G_{p,q}(a,b)=\begin{cases}(\frac{a^p+b^p}{a^q+b^q})^{1/(p-q)} & \text{if } p\neq q,\\ \exp(\frac{a^p\ln a+b^p\ln b}{a^p+b^p}) & \text{if } p=q,\end{cases} \tag{1.3}$$

it also contains many special means, for instance, $G_{1,0}(a,b)=A(a,b)$ is the arithmetic mean, $G_{1,1}(a,b)=a^{a/(a+b)}b^{b/(a+b)}=I(a^2,b^2)/I(a,b)=Z(a,b)$ is the power-exponential mean, $G_{p,0}(a,b)=A^{1/p}(a^p,b^p)=A_p(a,b)$ is the p-order arithmetic (power, Hölder) mean, $G_{p,p}(a,b)=Z^{1/p}(a^p,b^p)=Z_p(a,b)$ is the p-order power-exponential mean and $G_{p+1,p}(a,b)=(a^{p+1}+b^{p+1})/(a^p+b^p)=\mathcal{L}_p(a,b)$ is the Lehmer mean.

Recently, the inequalities for the bivariate means have been the subject of intensive research. In particular, the bounds for the arithmetic-geometric mean AGM have attracted the attention of many mathematicians. It is well known that the double inequality

$$L(a,b)<AGM(a,b)<L^{2/3}\bigl(a^{3/2},b^{3/2}\bigr) \tag{1.4}$$

holds for all $a,b>0$ with $a\neq b$. The first inequality of (1.4) was first proposed by Carlson and Vuorinen [4], it was proved in the literature [5–8] by different methods. Vamanamurthy and Vuorinen [9] (also see [5, 6]) proved that $AGM(a,b)<(\pi/2)L(a,b)$ for all $a,b>0$ with $a\neq b$. The second inequality of (1.4) is due to Borwein and Borwein [10], and Yang [8] presented a simple proof by use of the 'Comparison Lemma' [10, Lemma 2.1].

In [9] Vamanmurthy and Vuorinen presented the upper bounds for the arithmetic-geometric mean AGM in terms of the arithmetic mean A, geometric mean G and logarithmic mean L as follows:

$$\begin{aligned}&AGM(a,b)<L_2(a,b)=\bigl(A(a,b)L(a,b)\bigr)^{1/2},\\ &AGM(a,b)<I(a,b)<A(a,b),\\ &AGM(a,b)<A_{1/2}(a,b)\end{aligned}$$

for all $a,b>0$ with $a\neq b$.

In 1995, Sándor [5] proved that the double inequality

$$\frac{1}{\frac{12/(5\pi)}{L(a,b)}+\frac{1-12/(5\pi)}{A(a,b)}} < AGM(a,b) < \frac{1}{\frac{2/\pi}{L(a,b)}+\frac{1-2/\pi}{A(a,b)}}, \tag{1.5}$$

holds for all $a,b>0$ with $a\neq b$, and it was improved by Alzer and Qiu [11, Theorem 19] as

$$\frac{1}{\frac{\beta_2}{L(a,b)}+\frac{1-\beta_2}{A(a,b)}} < AGM(a,b) < \frac{1}{\frac{\alpha_2}{L(a,b)}+\frac{1-\alpha_2}{A(a,b)}}, \tag{1.6}$$

with the best possible parameters $\beta_2 = 3/4$ and $\alpha_2 = 2/\pi$.

Other inequalities involving AGM can be found in the literature [12–20].

The aim of this paper is to establish the new inequality chains for the ratio of certain bivariate means, and we present the sharp bounds for the arithmetic-geometric mean AGM.

2 Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

Lemma 1 ([21, Corollary 1.1]) *Let $a,b>0$ with $a\neq b$. Then both $S_{p,2m-p}(a,b)$ and $G_{p,2m-p}(a,b)$ are strictly increasing (decreasing) with respect to $p\in(-\infty,m)$ ($p\in(m,\infty)$) for fixed $m>0$.*

Lemma 2 ([22, Theorem 5], [23, Theorem 3.4]) *Let $a,b,c,d>0$ with $b/a>d/c\geq 1$. Then the ratio of Stolarsky means $R(p,2m-p;a,b;c,d)=S_{p,2m-p}(a,b)/S_{p,2m-p}(c,d)$ is strictly increasing (decreasing) with respect to $p\in(-\infty,m)$ ($p\in(m,\infty)$) for fixed $m>0$.*

Lemma 3 ([24, Theorem 4.1]) *Let $a,b,c,d>0$ with $b/a>d/c\geq 1$. Then the ratio of Stolarsky means $R(p,q;a,b;c,d)=S_{p,q}(a,b)/S_{p,q}(c,d)$ is strictly log-concave (log-convex) with respect to $p\in((|q|-q)/2,\infty)$ ($p\in(-\infty,-(|q|+q)/2)$) for fixed $q\in\mathbb{R}$.*

From Lemma 3, we have Corollary 1.

Corollary 1 *Let $\lambda>0$, $\alpha\in(0,1)$ and $a,b,c,d>0$ with $b/a>d/c\geq 1$. Then the function*

$$p\mapsto R^{\alpha}(p,0;a,b;c,d)R^{1-\alpha}\left(\frac{\lambda-\alpha p}{1-\alpha},0;a,b;c,d\right)=:r(p)$$

is strictly increasing in $(0,\lambda)$ and strictly decreasing in $(\lambda,\lambda/\alpha)$.

Proof Let $p_1=(\lambda-\alpha p)/(1-\alpha)$. Then

$$\begin{aligned}\frac{r'(p)}{r(p)} &= \alpha\big(\ln R(p,0;a,b;c,d)\big)'+(1-\alpha)\big(\ln R(p_1,0;a,b;c,d)\big)' \\ &\quad\times\frac{-\alpha}{1-\alpha} \\ &= \alpha\Big(\big(\ln R(p,0;a,b;c,d)\big)'-\big(\ln R(p_1,0;a,b;c,d)\big)'\Big)\end{aligned}$$

$$= \alpha(p - p_1)\big(\ln R(\xi, 0; a, b; c, d)\big)''$$
$$= \frac{\alpha}{1-\alpha}(p - \lambda)\big(\ln R(\xi, 0; a, b; c, d)\big)'',$$

where ξ is between p and p_1.

It follows from Lemma 3 that $R(p, 0; a, b; c, d)$ is strictly log-concave with respect to $p \in (0, \infty)$ and strictly log-convex with respect to $p \in (-\infty, 0)$. Therefore, $r'(p) > 0$ for $p \in (0, \lambda)$ and $r'(p) < 0$ for $p \in (\lambda, \lambda/\alpha)$. □

Lemma 4 ([25, Corollary 3.1]) *Let $a, b, c, d > 0$ with $b/a > d/c \geq 1$. Then the function*

$$Q(p) = \frac{\sqrt{S_{p,q}(a,b)S_{2k-p,q}(a,b)}}{\sqrt{S_{p,q}(c,d)S_{2k-p,q}(c,d)}}$$

is strictly decreasing (increasing) in (k, ∞) and strictly increasing (decreasing) in $(-\infty, k)$ for fixed $q \geq (\leq)0$, $k \geq (\leq)0$ with $q^2 + k^2 \neq 0$.

Let $(k, q) = (3/2, 0)$, $(1/2, 1)$, respectively. Then Lemma 4 leads to the following.

Corollary 2 *Let $a, b, c, d > 0$ with $b/a > d/c \geq 1$. Then*

(i) *the function*

$$p \mapsto \frac{\sqrt{S_{p,0}(a,b)S_{3-p,0}(a,b)}}{\sqrt{S_{p,0}(c,d)S_{3-p,0}(c,d)}}$$

is strictly decreasing in $(3/2, \infty)$ and strictly increasing in $(-\infty, 3/2)$;

(ii) *the function*

$$p \mapsto \frac{\sqrt{S_{p,1}(a,b)S_{1-p,1}(a,b)}}{\sqrt{S_{p,1}(c,d)S_{1-p,1}(c,d)}}$$

is strictly decreasing in $(1/2, \infty)$ and increasing in $(-\infty, 1/2)$.

Lemma 5 *Let $a, b > 0$ with $b > a$. Then $b/a > A(a,b)/G(a,b) > 1$.*

Proof Simple computations lead to

$$\frac{b}{a} - \frac{A(a,b)}{G(a,b)} = \frac{b}{a} - \frac{\sqrt{\frac{b}{a}} + \sqrt{\frac{a}{b}}}{2}$$
$$= \frac{1}{2}\left(\sqrt{\frac{b}{a}} - 1\right)\left(1 + 2\sqrt{\frac{b}{a}} + \sqrt{\frac{a}{b}}\right) > 0.$$ □

Lemma 6 ([26]) *Let $x \in (0, 1)$. Then*

$$AGM(1, x) \sim \frac{\pi/2}{\ln(1/x)}, \quad x \to 0^+, \tag{2.1}$$

$$\frac{1}{AGM(1, 1-x)} = 1 + \frac{1}{2}x + \frac{5}{16}x^2 + \frac{7}{32}x^3 + \frac{169}{1{,}024}x^4 + o\big(x^4\big). \tag{2.2}$$

Lemma 7 is a consequence of the 'Comparison Lemma' in [10, Lemma 2.1].

Lemma 7 *Let Φ be a bivariate mean such that $\Phi(G(x,y),A(x,y)) < (>) \Phi(x,y)$ for all $x,y > 0$ with $x \neq y$. Then*

$$AGM(a,b) < (>) \Phi(a,b)$$

for all $a,b > 0$ with $a \neq b$.

3 Inequality chains for the ratio of means

In this section, we give some inequality chains for the ratio of certain bivariate means, which will be used to prove our main results in next section.

Proposition 1 *Let $a,b,c,d > 0$ with $b/a > d/c \geq 1$. Then we have*

$$\frac{\sqrt{A(a,b)G(a,b)}}{\sqrt{A(c,d)G(c,d)}} < \frac{G_{3/4,-1/4}(a,b)}{G_{3/4,-1/4}(c,d)} < \frac{S_{7/4,-1/4}(a,b)}{S_{7/4,-1/4}(c,d)}. \tag{3.1}$$

Proof The second inequality of (3.1) can be rewritten as

$$\frac{S_{7/4,-1/4}(1,b/a)}{G_{3/4,-1/4}(1,b/a)} > \frac{S_{7/4,-1/4}(1,d/c)}{G_{3/4,-1/4}(1,d/c)}.$$

Therefore, it suffices to prove that the function

$$f_1(x) = \ln \frac{S_{7/4,-1/4}(1,x)}{G_{3/4,-1/4}(1,x)} = \frac{1}{2} \ln \frac{x^{7/4}-1}{7(1-x^{-1/4})} - \ln \frac{x^{3/4}+1}{x^{-1/4}+1}$$

is strictly increasing in $(1,\infty)$. Replacing x by x^4 and differentiating f_1 give

$$\begin{aligned} 4x^3 f_1'(x^4) &= \frac{7}{2} \frac{x^6}{(x^7-1)} - \frac{1}{2(x-1)} - \frac{1}{2x} - \frac{3x^2}{x^3+1} + \frac{1}{x+1} \\ &= \frac{(x+1)(x+x^2+1)(x-1)^5}{2x(x^2-x+1)(x^6+x^5+x^4+x^3+x^2+x+1)} > 0 \end{aligned}$$

for $x \in (1,\infty)$.

Similarly, to prove the first inequality of (3.1), it suffices to prove that the function

$$f_2(x) = \ln \frac{G_{3/4,-1/4}(1,x)}{\sqrt{A(1,x)G(1,x)}} = \ln \frac{x^{3/4}+1}{(x^{-1/4}+1)\sqrt{\frac{x+1}{2}\sqrt{x}}} = \ln \frac{\sqrt{x} - \sqrt[4]{x} + 1}{\sqrt{\frac{x+1}{2}}}$$

is strictly increasing on $(1,\infty)$. Replacing x by x^4 and differentiating f_2 yield

$$4x^3 f_2'(x^4) = \frac{(x+1)(x-1)^3}{(x^2-x+1)(x^4+1)} > 0$$

for $x \in (1,\infty)$, which completes the proof. □

Proposition 2 *Let $a, b > 0$ with $a \neq b$. Then we have*

$$\frac{\sqrt{A(a,b)G(a,b)}}{\sqrt{A(G,A)G(G,A)}} < \frac{G_{3/4,-1/4}(a,b)}{G_{3/4,-1/4}(G,A)} < \frac{S_{7/4,-1/4}(a,b)}{S_{7/4,-1/4}(G,A)} < 1, \tag{3.2}$$

where $G = G(a,b)$ and $A = A(a,b)$.

Proof By symmetry, without loss of generality, we assume that $a < b$. Then from Lemma 5 and Proposition 1 we clearly see that the first and second inequalities of (3.2) hold. Next we prove the last inequality of (3.2). Let $t = \ln\sqrt{b/a} > 0$, then the last inequality of (3.2) can be rewritten as

$$\left(\frac{\sinh\frac{7t}{4}}{7\sinh\frac{t}{4}}\right)^{1/2} < \left(\frac{1}{7}\frac{(\cosh t)^{7/4}-1}{1-(\cosh t)^{-1/4}}\right)^{1/2} = \left(\frac{1}{7}\frac{(\cosh t)^2-(\cosh t)^{1/4}}{(\cosh t)^{1/4}-1}\right)^{1/2}.$$

It suffices to prove that the function

$$g(t) = \cosh t - \left(\frac{\sinh\frac{7t}{4} + (\sinh\frac{t}{4})(\cosh t)^2}{\sinh\frac{7t}{4} + \sinh\frac{t}{4}}\right)^4 < 0$$

for $t > 0$.

Simple computations lead to

$$\begin{aligned}
&\sinh 7t + (\sinh t)(\cosh 4t)^2 = 8\left(\sinh t\cosh^2 t\right)\left(2\cosh^2 t - 1\right)\left(4\cosh^4 t - 2\cosh^2 t - 1\right),\\
&\sinh 7t + \sinh t = 8\left(\sinh t\cosh^2 t\right)\left(4\cosh^2 t - 3\right)\left(2\cosh^2 t - 1\right),\\
&\cosh 4t = 8\cosh^4 t - 8\cosh^2 t + 1,\\
&g(4t) = \cosh 4t - \left(\frac{\sinh 7t + (\sinh t)(\cosh 4t)^2}{\sinh 7t + \sinh t}\right)^4\\
&\qquad = \left(8\cosh^4 t - 8\cosh^2 t + 1\right) - \left(\frac{4\cosh^4 t - 2\cosh^2 t - 1}{4\cosh^2 t - 3}\right)^4 := g_1\left(\cosh^2 t\right),
\end{aligned}$$

where

$$g_1(x) = \left(8x^2 - 8x + 1\right) - \left(\frac{4x^2 - 2x - 1}{4x - 3}\right)^4, \quad x = \cosh^2 t > 1.$$

$g_1(x)$ can be rewritten as

$$\begin{aligned}
g_1(x) &= -16(x-1)^4\frac{16x^4 + 32x^3 - 88x^2 + 48x - 5}{(4x-3)^4}\\
&= -16(x-1)^4\frac{16x^2(x-1)^2 + 64x(x-1)^2 + 16x(x-1) + 5(x^2-1) + 3x^2}{(4x-3)^4} < 0
\end{aligned}$$

for $x > 1$. Therefore, $g(t) < 0$ for $t > 0$.

Thus we complete the proof. □

Proposition 3 *Let $a, b, c, d > 0$ with $b/a > d/c \geq 1$ and $p \in (3/2, 2)$. Then*

$$\begin{aligned}\frac{A^{1/4}(a,b)L^{3/4}(a,b)}{A^{1/4}(c,d)L^{3/4}(c,d)} &< \frac{\sqrt{L_p(a,b)L_{3-p}(a,b)}}{\sqrt{L_p(c,d)L_{3-p}(c,d)}} < \frac{L_{3/2}(a,b)}{L_{3/2}(c,d)} \\ &< \frac{\sqrt{L(a,b)A_{2/3}(a,b)}}{\sqrt{L(c,d)A_{2/3}(c,d)}} < \frac{\sqrt{L(a,b)I(a,b)}}{\sqrt{L(c,d)I(c,d)}} < \frac{S_{5/4,1/4}(a,b)}{S_{5/4,1/4}(c,d)} \\ &< \frac{He_{3/4}(a,b)}{He_{3/4}(c,d)} < \frac{A_{1/2}(a,b)}{A_{1/2}(c,d)} < \frac{I_{3/4}(a,b)}{I_{3/4}(c,d)}.\end{aligned}$$

Proof (i) From part one of Corollary 2 we see that

$$\frac{\sqrt{S_{2,0}(a,b)S_{1,0}(a,b)}}{\sqrt{S_{2,0}(c,d)S_{1,0}(c,d)}} < \frac{\sqrt{S_{p,0}(a,b)S_{3-p,0}(a,b)}}{\sqrt{S_{p,0}(c,d)S_{3-p,0}(c,d)}} < \frac{\sqrt{S_{3/2,0}(a,b)S_{3/2,0}(a,b)}}{\sqrt{S_{3/2,0}(c,d)S_{3/2,0}(c,d)}}$$

for $p \in (3/2, 2)$.

Therefore, the first and second inequalities of Proposition 3 follow from the above inequalities and $\sqrt{S_{1,0}(a,b)S_{2,0}(a,b)} = A^{1/4}(a,b)L^{3/4}(a,b)$ together with $S_{3/2,0}(a,b) = L_{3/2}(a,b)$.

(ii) For the third inequality of Proposition 3. From

$$A_{2/3}(a,b) = S_{4/3,2/3}(a,b) = \frac{S^2_{4/3,0}(a,b)}{S_{2/3,0}(a,b)}$$

we clearly see that it suffices to prove

$$\frac{S_{3/2,0}(a,b)\sqrt{S_{2/3,0}(a,b)}}{S_{3/2,0}(c,d)\sqrt{S_{2/3,0}(c,d)}} < \frac{S_{4/3,0}(a,b)\sqrt{S_{1,0}(a,b)}}{S_{4/3,0}(c,d)\sqrt{S_{1,0}(c,d)}}.$$

Let $(\alpha, \lambda) = (1/3, 11/9)$. Then Corollary 1 leads to the conclusion that the function

$$p \mapsto R^{1/3}(p,0;a,b;c,d)R^{2/3}\left(\frac{11-3p}{6}, 0; a,b;c,d\right) =: r(p)$$

is increasing in $(0, 11/9)$. Therefore, $r(2/3) < r(1)$, that is,

$$\frac{S_{3/2,0}(a,b)\sqrt{S_{2/3,0}(a,b)}}{S_{3/2,0}(c,d)\sqrt{S_{2/3,0}(c,d)}} < \frac{S_{4/3,0}(a,b)\sqrt{S_{1,0}(a,b)}}{S_{4/3,0}(c,d)\sqrt{S_{1,0}(c,d)}}.$$

(iii) The fourth inequality of Proposition 3 can be written as $A_{2/3}(a,b)/A_{2/3}(c,d) < I(a,b)/I(c,d)$, that is, $R(4/3, 2/3; a,b;c,d) < R(1,1;a,b;c,d)$. Let $m = 1$, then from Lemma 2 we know that $R(p, 2-p; a,b;c,d)$ is strictly decreasing with respect to $p \in (1, \infty)$.

(iv) For the sixth, seventh, and eighth inequalities, let $m = 3/4$, then Lemma 2 leads to the conclusion that $R(p, 3/2 - p; a,b;c,d)$ is strictly decreasing with respect to $p \in (3/4, \infty)$. Consequently,

$$\begin{aligned}&R\left(\frac{5}{4}, \frac{3}{2} - \frac{5}{4}; a,b;c,d\right) < R\left(\frac{9}{8}, \frac{3}{2} - \frac{9}{8}; a,b;c,d\right) \\ &< R\left(1, \frac{3}{2} - 1; a,b;c,d\right) < R\left(\frac{3}{4}, \frac{3}{2} - \frac{3}{4}; a,b;c,d\right),\end{aligned}$$

which gives the desired results.

(v) Finally, we prove the fifth inequality. It can be written as

$$\frac{\sqrt{L(1,b/a)I(1,b/a)}}{S_{5/4,1/4}(1,b/a)} < \frac{\sqrt{L(1,d/c)I(1,d/c)}}{S_{5/4,1/4}(1,d/c)}.$$

Thus we need only to prove that the function

$$h(x) = \ln\frac{\sqrt{L(1,x)I(1,x)}}{S_{5/4,1/4}(1,x)}$$

is strictly decreasing in $(1,\infty)$. Let $t = \ln\sqrt{x} \in (0,\infty)$. Then

$$h(x) = \frac{1}{2}\ln\frac{\sinh t}{t} + \frac{1}{2}\left(\frac{t\cosh t}{\sinh t} - 1\right) - \ln\frac{\sinh\frac{5t}{4}}{5\sinh\frac{t}{4}} := h_1(t).$$

Differentiating $h_1(t)$ yields

$$h_1'(t) = -\frac{h_2(t)}{4t\sinh\frac{1}{4}t\sinh\frac{5}{4}t\sinh^2 t},$$

where

$$\begin{aligned} h_2(t) &= 2t^2\sinh\frac{t}{4}\sinh\frac{5t}{4} + 2\sinh\frac{t}{4}\sinh\frac{5t}{4}\sinh^2 t - t\cosh\frac{t}{4}\sinh\frac{5t}{4}\sinh^2 t \\ &\quad + 5t\cosh\frac{5t}{4}\sinh\frac{t}{4}\sinh^2 t - 4t\sinh\frac{t}{4}\sinh\frac{5t}{4}\cosh t\sinh t. \end{aligned}$$

We clearly see that it is enough to prove $h_2(t) > 0$ for $t > 0$.

Making use of 'product to sum' and power series formulas we get

$$\begin{aligned} h_2(t) &= -t\sinh\frac{t}{2} + \frac{1}{4}\cosh\frac{t}{2} - \frac{1}{2}\cosh\frac{3t}{2} + t^2\cosh\frac{3t}{2} - t\sinh\frac{3t}{2} + \frac{1}{4}\cosh\frac{7t}{2} \\ &\quad + \frac{1}{4}\cosh t + \frac{11}{4}t\sinh t - t^2\cosh t - \frac{1}{4}\cosh 3t - \frac{1}{4}t\sinh 3t \\ &= \sum_{n=1}^{\infty}\frac{s(n)}{4^{n+1}(2n)!}t^{2n}, \end{aligned}$$

where

$$s(n) = 7^{2n} - \left(\frac{2}{3}n + 1\right)6^{2n} + \left(64n^2 - 80n - 18\right)3^{2n-2} - \left(16n^2 - 30n - 1\right)2^{2n} - 16n + 1.$$

It is easy to verify that $s(1) = s(2) = s(3) = 0$, $s(4) = 71{,}680$. Next we show that $s(n) > 0$ for $n \geq 5$. To this end, we rewrite $s(n)$ as

$$s(n) = 6^{2n}s_1(n) + \frac{1}{9}\left(16n^2 - 30n - 1\right)2^{2n}s_2(n) + s_3(n),$$

where

$$s_1(n) = \left(\frac{7}{6}\right)^{2n} - \left(\frac{2}{3}n + 1\right),$$

$$s_2(n) = 3^{2n} - 9 \times 2^{2n} = \left(\frac{3}{2}\right)^{2n} - 9,$$

$$s_3(n) = \left(48n^2 - 50n - 17\right)3^{2n-2} - 16n + 1.$$

Due to $(16n^2 - 30n - 1) = 16n(n-2) + (2n-1) > 0$ for $n \geq 2$, it suffices to prove $s_i(n) > 0$ for $n \geq 5$, $i = 1, 2, 3$. Indeed,

$$s_1'(x) = 2\left(\frac{7}{6}\right)^{2x} \ln\frac{7}{6} - \frac{2}{3} \geq 2\left(\frac{7}{6}\right)^{10} \ln\frac{7}{6} - \frac{2}{3} = 0.77\ldots > 0,$$

therefore, $s_1(n) \geq s_1(5) = 20{,}455{,}153/60{,}466{,}176 > 0$; $s_2(n) > s_2(3) = 153/64 > 0$; $s_3(n) > (48n^2 - 50n - 17) - 16n + 1 = 48(n-2)^2 + 126(n-2) + 44 > 0$.

This completes the proof. □

Proposition 4 *Let $a, b > 0$ with $a \neq b$. Then for $p \in (3/2, 2)$ we have*

$$\begin{aligned} 1 &< \frac{A^{1/4}(a,b)L^{3/4}(a,b)}{A^{1/4}(G,A)L^{3/4}(G,A)} < \frac{\sqrt{L_p(a,b)L_{3-p}(a,b)}}{\sqrt{L_p(G,A)L_{3-p}(G,A)}} \\ &< \frac{L_{3/2}(a,b)}{L_{3/2}(G,A)} < \frac{\sqrt{L(a,b)A_{2/3}(a,b)}}{\sqrt{L(G,A)A_{2/3}(G,A)}} < \frac{\sqrt{L(a,b)I(a,b)}}{\sqrt{L(G,A)I(G,A)}} \\ &< \frac{S_{5/4,1/4}(a,b)}{S_{5/4,1/4}(G,A)} < \frac{He_{3/4}(a,b)}{He_{3/4}(G,A)} < \frac{A_{1/2}(a,b)}{A_{1/2}(G,A)} < \frac{I_{3/4}(a,b)}{I_{3/4}(G,A)}, \end{aligned} \tag{3.3}$$

where $G = G(a,b)$ and $A = A(a,b)$.

Proof Without loss of generality, we assume that $a < b$. Then the second inequality to the last inequality in (3.3) follows easily from Proposition 3 and Lemma 5.

Next we prove the first inequality of (3.3). Let $t = \ln\sqrt{b/a} > 0$. Then it equivalent to the inequality

$$u(t) = \frac{1}{4}\ln\cosh t + \frac{3}{4}\ln\frac{\sinh t}{t} - \frac{1}{4}\ln\frac{\cosh t + 1}{2} - \frac{3}{4}\ln\frac{\cosh t - 1}{\ln\cosh t} > 0.$$

Differentiating $u(t)$ gives

$$\begin{aligned} u'(t) &= \frac{3t\sinh^2 t - (t + 3\sinh t\cosh t + 2t\cosh t)\ln(\cosh t)}{2t(\sinh 2t)\ln(\cosh t)} \\ &= \frac{t + 3\sinh t\cosh t + 2t\cosh t}{2t(\sinh 2t)\ln(\cosh t)} \times u_1(t), \end{aligned}$$

where

$$u_1(t) = \frac{3t\sinh^2 t}{t + 3\sinh t\cosh t + 2t\cosh t} - \ln(\cosh t).$$

Differentiating $u_1(t)$ leads to

$$u_1'(t) = \frac{t\sinh t}{(t + 2t\cosh t + 3\cosh t\sinh t)^2\cosh t} \times u_2(t),$$

where

$$u_2(t) = \frac{13}{2} t \cosh t - 3 \sinh t + t \cosh 2t + \frac{3}{2} \sinh 2t + \frac{3}{2} t \cosh 3t - 3 \sinh 3t.$$

Making use of the power series we get

$$\begin{aligned} u_2(t) &= \frac{13}{2} \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-2)!} - 3 \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} + \sum_{n=1}^{\infty} \frac{2^{2n-2} t^{2n-1}}{(2n-2)!} \\ &\quad + \frac{3}{2} \sum_{n=1}^{\infty} \frac{2^{2n-1} t^{2n-1}}{(2n-1)!} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{3^{2n-2} t^{2n-1}}{(2n-2)!} - 3 \sum_{n=1}^{\infty} \frac{3^{2n-1} t^{2n-1}}{(2n-1)!} \\ &= \sum_{n=1}^{\infty} \frac{v(n)}{(2n-1)!} t^{2n-1}, \end{aligned}$$

where

$$v(n) = \left(n - \frac{7}{2}\right) 3^{2n-1} + (n+1) 2^{2n-1} + \left(13n - \frac{19}{2}\right).$$

Clearly, $v(1) = v(2) = 0$, $v(3) = 36$ and $v(n) > 0$ for $n \geq 4$. Therefore, $u_2(t) > 0$, $u_1(t)$ is strictly increasing in $(0, \infty)$, $u_1(t) > u_1(0^+) = 0$, $u'(t) > 0$, and $u(t) > u(0^+) = 0$ for $t > 0$.

Thus the proof is finished. □

4 Sharp bounds for *AGM*

In this section, we present several sharp bounds for the arithmetic-geometric mean AGM.

Theorem 1 can be derived from Propositions 1-4 and Lemma 7.

Theorem 1 *Let $a, b > 0$ with $a \neq b$. Then the inequalities*

$$\begin{aligned} \sqrt{A(a,b)G(a,b)} &< G_{3/4,-1/4}(a,b) < S_{7/4,-1/4}(a,b) < AGM(a,b) \\ &< A^{1/4}(a,b) L^{3/4}(a,b) < \sqrt{L_p(a,b) L_{3-p}(a,b)} \\ &< L_{3/2}(a,b) < \sqrt{L(a,b) A_{2/3}(a,b)} < \sqrt{L(a,b) I(a,b)} \\ &< S_{5/4,1/4}(a,b) < He_{3/4}(a,b) < A_{1/2}(a,b) < I_{3/4}(a,b) \end{aligned}$$

hold for $p \in (3/2, 2)$.

Remark 1 We clearly see that the upper bound $A^{1/4} L^{3/4}$ for AGM is better than $L_{3/2}$. Moreover, we have

$$AGM(a,b) < A^{1/4}(G,A) L^{3/4}(G,A) = A_{1/2}^{1/4}(a,b) L^{3/4}(G,A).$$

Theorem 2 *The inequality*

$$AGM(a,b) < A^p(a,b) L^{1-p}(a,b) \tag{4.1}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \geq 1/4$.

Proof Let $x > 0$ and $x \to 0^+$. Then (2.2) and the power series

$$\frac{1}{A^p(1,1-x)L^{1-p}(1,1-x)} = 1 + \frac{1}{2}x + \frac{4-p}{12}x^2 + o(x^2)$$

lead to

$$\lim_{x\to 0^+} \frac{\frac{1}{AGM(1,1-x)} - \frac{1}{A^p(1,1-x)L^{1-p}(1,1-x)}}{x^2} = \frac{1}{12}\left(p - \frac{1}{4}\right).$$

Therefore, $p \geq 1/4$ is the necessary condition for the inequality $AGM(a,b) < A^p(a,b) \times L^{1-p}(a,b)$ to hold for all $a, b > 0$ with $a \neq b$. The sufficiency follows easily from the function $p \mapsto A^p(a,b)L^{1-p}(a,b)$ is strictly increasing and Theorem 1. □

Let $a, b > 0$ with $a \neq b$, $r \in \mathbb{R}$ and $\alpha \in (0,1)$. Then we define $M_r(A(a,b), L(a,b); \frac{1}{4})$ by

$$M_r\left(A, L; \frac{1}{4}\right) = \left(\frac{1}{4}A^r + \frac{3}{4}L^r\right)^{1/r} \quad \text{if } r \neq 0 \text{ and } M_0\left(A, L; \frac{1}{4}\right) = A^{1/4}L^{3/4}. \tag{4.2}$$

Remark 2 From Theorem 1 and (1.6) we clearly see that the double inequality

$$M_{-1}\big(A(a,b), L(a,b); 1/4\big) < AGM(a,b) < M_0\big(A(a,b), L(a,b); 1/4\big)$$

holds for all $a, b > 0$ with $a \neq b$.

Moreover, making use of (2.1) and (2.2) we get

$$\lim_{x\to 0^+} \frac{AGM(1,x) - (\frac{1}{4}(\frac{x+1}{2})^r + \frac{3}{4}(\frac{x-1}{\ln x})^r)^{1/r}}{1/\ln(1/x)} = \begin{cases} \frac{\pi}{2} - \infty, & r \geq 0, \\ \frac{\pi}{2} - (\frac{3}{4})^{1/r}, & r < 0, \end{cases}$$

$$\lim_{x\to 0^+} \frac{\frac{1}{AGM(1,1-x)} - (\frac{1}{4}(\frac{2-x}{2})^r + \frac{3}{4}(\frac{-x}{\ln(1-x)})^r)^{1/r}}{x^4} = \frac{r + 1/10}{1{,}536}.$$

Therefore, $p \leq -(\ln 4 - \ln 3)/(\ln \pi - \ln 2) = -0.63\ldots$ and $q \geq -1/10$ are the necessary conditions such that the double inequality

$$M_p\left(A(a,b), L(a,b); \frac{1}{4}\right) < AGM(a,b) < M_q\left(A(a,b), L(a,b); \frac{1}{4}\right) \tag{4.3}$$

holds for all $a, b > 0$ with $a \neq b$.

Conjecture 1 *The double inequality*

$$M_p\left(A(a,b), L(a,b); \frac{1}{4}\right) < AGM(a,b) < M_q\left(A(a,b), L(a,b); \frac{1}{4}\right)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq -(\ln 4 - \ln 3)/(\ln \pi - \ln 2)$ and $q \geq -1/10$.

Theorem 3 *Let $p, q > 3/4$. Then the double inequality*

$$S_{p,3/2-p}(a,b) < AGM(a,b) < S_{q,3/2-q}(a,b) \tag{4.4}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \geq 7/4$ and $3/4 < q \leq 3/2$.

Proof The sufficiency follows from the function $p \mapsto S_{p,3/2-p}(a,b)$ is strictly decreasing in $(3/4, \infty)$ by Lemma 2 and Theorem 1.

Next we prove the necessity. It follows from (2.1) and (2.2) together with the power series

$$\frac{1}{S_{p,3/2-p}(1,1-x)} = 1 + \frac{1}{2}x + \frac{5}{16}x^2 + \frac{7}{32}x^3 + \frac{2p^2 - 3p + 316}{1,920}x^4 + o(x^4),$$

that

$$\lim_{x\to 0^+} \frac{\frac{1}{AGM(1,1-x)} - \frac{1}{S_{p,3/2-p}(1,1-x)}}{x^4} = \frac{169}{1,024} - \frac{2p^2 - 3p + 316}{1,920},$$
$$\lim_{x\to 0^+} \frac{AGM(1,x) - S_{q,3/2-q}(1,x)}{1/(\ln(1/x))} = \begin{cases} \frac{\pi}{2} - \infty, & 3/4 < q \le 3/2, \\ \frac{\pi}{2}, & q > 3/2. \end{cases}$$

Therefore, $p \ge 7/4$ and $3/4 < q \le 3/2$ are the necessary conditions the double inequality (4.4) to hold for all $a, b > 0$ with $a \ne b$. □

Theorem 4 *Let $p \ge 1/2$. Then the inequality*

$$AGM(a,b) < \sqrt{S_{p,1}(a,b)S_{1-p,1}(a,b)} \tag{4.5}$$

holds for all $a, b > 0$ with $a \ne b$ if and only if $1/2 \le p \le 1$.

Proof The sufficiency follows from the function $p \mapsto \sqrt{S_{p,1}(a,b)S_{1-p,1}(a,b)}$ is strictly decreasing in $(1/2, \infty)$ by Corollary 2(ii) and the inequality

$$AGM(a,b) < \sqrt{L(a,b)I(a,b)} = \sqrt{S_{1,1}(a,b)S_{0,1}(a,b)}$$

in Theorem 1, and the necessity can be derived from the inequality

$$\lim_{x\to 0^+} \frac{AGM(1,x) - \sqrt{S_{p,1}(1,1-x)S_{1-p,1}(1,1-x)}}{1/(\ln(1/x))} = \begin{cases} \frac{\pi}{2} - \infty, & 1/2 \le p \le 1, \\ \frac{\pi}{2}, & p > 1 \end{cases} \le 0.$$ □

Making use of the similar methods, we can prove Theorems 5-7, we omit the proofs here.

Theorem 5 *Let and $p, q > 1/4$. Then the double inequality*

$$G_{p,1/2-p}(a,b) < AGM(a,b) < G_{q,1/2-q}(a,b)$$

holds for all $a, b > 0$ with $a \ne b$ if and only if $p \ge 3/4$ and $1/4 < q \le 1/2$.

Theorem 6 *The double inequality*

$$S_{p+1,p}(a,b) < AGM(a,b) < S_{q+1,q}(a,b)$$

holds for all $a, b > 0$ with $a \ne b$ if and only if $p \le 0$ and $q \ge 1/4$.

Theorem 7 *The double inequality*

$$He_p(a,b) < AGM(a,b) < He_q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq 0$ and $q \geq 3/4$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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