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A fuzzy characterization of QF rings

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Abstract

Let R be a ring. R is called a quasi-Frobenius (QF) ring if R is right artinian and R_R is an injective right R-module. In this article, we introduce (weak) fuzzy homomorphisms of modules to obtain a fuzzy characterization of QF rings. We also obtain some fuzzy characterizations of right artinian rings and right CF rings. These results throw new light on the research of QF rings and the related CF conjecture.

MSC: 03E72; 16L60

Keywords: fuzzy homomorphisms; weak fuzzy homomorphisms; artinian rings; QF rings

1 Introduction

Recall that a *fuzzy subset* of a nonempty set X is a map f from X into the closed interval $[0,1]$. The notion of fuzzy subset of a set was firstly introduced by Zadeh [1]. Then this important ideal has been applied to various algebraic structures such as groups and rings and so on (see [2–9] etc.). In this article, we introduce some special fuzzy subsets of modules to characterize quasi-Frobenius (QF) rings.

QF rings were introduced by Nakayama [10] as generalizations of group algebras of a finite group over a field. A ring R is called *quasi-Frobenius* (*QF*) if the right R-module R_R is both artinian and injective. QF rings became an important algebraic structure because of their beautiful characterizations and nice applications (see [11–16] etc.). For example, a ring R is QF if and only if every right R-module can be embedded into a free right R-module. Many results of QF rings have been applied into coding theory. During the progress of research on QF rings, many important conjectures arose. One of them is the CF conjecture (see [17, 18] etc.). It says that every right CF ring is right artinian. Recall that a ring R is called *right CF* if every cyclic right R-module can be embedded into a free right R-module.

Firstly, we introduce the fuzzy homomorphism and weak fuzzy homomorphism of R-modules in Section 2. Then in Section 3, we use weak fuzzy homomorphisms to give a characterization of injective right R-modules. We also obtain some new fuzzy characterizations of right artinian rings. In Section 4, we give a fuzzy characterization of right CF rings. We also give an approach to the CF conjecture through fuzzy viewpoints. Then based on the results we have obtained, we finally get a fuzzy characterization of QF rings.

2 Definitions and examples

Throughout the paper, R is an associative ring with identity and all modules are unitary. For a subset X of a ring R, the right annihilator of X in R is $\mathbf{r}(X) = \{r \in R : xr = 0 \text{ for all } x \in$

$X\}$. We write M_R to indicate that M is a right R-module. Let M_R and N_R be two right R-modules. $\mathrm{Hom}_R(M,N)$ denotes the set of all right R-module homomorphisms from M_R to N_R. $A \times B$ means the Cartesian cross product of two sets A and B. We use $\mathrm{Im}(f)$ to denote the image of a map f. For much more notations one is referred to [19].

We now consider the following conditions of a fuzzy subset f of $M_R \times N_R$.

(1) $\forall x \in M$, $\exists y \in N$ such that $f(x,y) > 0$;
(1′) $\exists x \in M$, $\exists y \in N$ such that $f(x,y) > 0$;
(2) $\forall x \in M$, $\forall y_1, y_2 \in N$, $f(x,y_1) > 0$ and $f(x,y_2) > 0$ implies $y_1 = y_2$;
(3) $\forall x_1, x_2 \in M$, $\forall y \in N$,

$$f(x_1 + x_2, y) \geq \sup\bigl\{\min\bigl\{f(x_1,y_1), f(x_2,y_2)\bigr\} : y_1 + y_2 = y, y_1, y_2 \in N\bigr\};$$

(4) $\forall x \in M$, $\forall y \in N$, $\forall r \in R$, $f(xr,y) \geq \sup\{f(x,y_1) : y = y_1 r, y_1 \in N\}$.

Definition 2.1 If f satisfies (1), (2), (3), and (4) of the above conditions, f is called a *fuzzy homomorphism* from M_R to N_R. If f satisfies (1′), (2), (3), and (4) of the above conditions, f is called a *weak fuzzy homomorphism* from M_R to N_R. We will use $\mathrm{FHom}_R(M,N)$ (resp., $\mathrm{WFHom}_R(M,N)$) to denote the set of all fuzzy homomorphisms (resp., weak fuzzy homomorphisms) from M_R to N_R. It is clear that $\mathrm{FHom}_R(M,N) \subseteq \mathrm{WFHom}_R(M,N)$.

Example 2.2 Let $u \in \mathrm{Hom}_R(M,N)$. f_u is a fuzzy subset of $M \times N$ constructed by

$$f_u(x,y) = \begin{cases} 1, & x \in M, \text{ and } y = u(x), \\ 0, & \text{others}. \end{cases}$$

Then $f_u \in \mathrm{FHom}_R(M,N)$.

Proof The conditions (1) and (2) of f_u are satisfied obviously. Let $x_1, x_2 \in M$ and $y \in N$. If $y = u(x_1 + x_2)$, it is clear that

$$1 = f_u(x_1 + x_2, y) \geq \sup\bigl\{\min\bigl\{f_u(x_1,y_1), f_u(x_2,y_2)\bigr\} : y_1 + y_2 = y, y_1, y_2 \in N\bigr\}.$$

If $y \neq u(x_1 + x_2)$ and $y_1 + y_2 = y$, then either $u(x_1) \neq y_1$ or $u(x_2) \neq y_2$. Thus, $\min\{f_u(x_1,y_1), f_u(x_2,y_2)\} = 0$. So

$$0 = f_u(x_1 + x_2, y) \geq 0 = \sup\bigl\{\min\bigl\{f_u(x_1,y_1), f_u(x_2,y_2)\bigr\} : y_1 + y_2 = y\bigr\}.$$

Hence f_u satisfies the condition (3). For the condition (4), let $x \in M$, $y \in N$, $r \in R$. If $y = u(xr)$, it is clear that $1 = f_u(xr,y) \geq \sup\{f_u(x,y_1) : y = y_1 r, y_1 \in N\}$. If $y \neq u(xr)$ and $y = y_1 r$, then $y_1 \neq u(x)$. So $f_u(x,y_1) = 0$. Hence $0 = f_u(xr,y) \geq 0 = \sup\{f_u(x,y_1) : y = y_1 r, y_1 \in N\}$. Thus, $f_u \in \mathrm{FHom}_R(M,N)$. □

Remark 2.3 Let $f \in \mathrm{WFHom}_R(M,N)$.

(i) According to the condition (3), $\forall x_1, x_2 \in M$, $\forall y_1, y_2 \in N$, $f(x_1 + x_2, y_1 + y_2) \geq \min\{f(x_1,y_1), f(x_2,y_2)\}$. According to the conditions (3) and (4), $\forall x \in M$, $\forall y \in N$, $f(0,0) \geq \min\{f(x,y), f(-x,-y)\} = f(x,y) = f(-x,-y)$.

(ii) Set $t \in (0,1]$ and $M_t = \{x \in M_R : \exists y \in N, f(x,y) \geq t\}$. If M_t is not empty, according to the conditions (3) and (4), M_t is a right R-submodule of M_R.

(iii) Let K_R be a submodule of M_R such that $f|_{K\times N} \in \mathrm{FHom}_R(K,N)$. Then by the conditions (1) and (2), for each $x \in K$, there exists a unique $y_x \in N$ such that $f(x,y_x) > 0$. Now define a map $u: K \to N$ with $u(x) = y_x$. Again by the conditions (2), (3), and (4), it is not difficult to see that $u \in \mathrm{Hom}_R(K,N)$.

Definition 2.4 Let $f_1, f_2 \in \mathrm{WFHom}_R(M,N)$, we say $f_1 \leq f_2$ if $f_1(x,y) \leq f_2(x,y)$ for all $x \in M$, $y \in N$. $f_1 \wedge f_2$ is defined by $(f_1 \wedge f_2)(x,y) = \min\{f_1(x,y), f_2(x,y)\}$, $\forall x \in M$, $y \in N$. It is easy to prove that if $f_1, f_2 \in \mathrm{WFHom}_R(M,N)$, then $f_1 \wedge f_2 \in \mathrm{WFHom}_R(M,N)$. It is also clear that $f_1 \wedge f_2 \leq f_1$ and $f_1 \wedge f_2 \leq f_2$.

Definition 2.5 A weak fuzzy homomorphism $f \in \mathrm{WFHom}_R(M,N)$ is said to be *extendable* if there exists $g \in \mathrm{FHom}_R(M,N)$ such that $f \leq g$.

Example 2.6 Let M_R be a non-artinian right R-module. Then M has an infinite descending chain of submodules $M = M_1 \supsetneq M_2 \supsetneq M_3 \cdots$. Set $N = \bigcap_i M_i$. Now we define a fuzzy subset f of $M \times M$ by

$$f(x,y) = \begin{cases} 1, & x = y \in N, \\ 1 - \frac{1}{n+1}, & x = y \in M_n, x \notin M_{n+1}, \\ 0, & \text{others.} \end{cases}$$

Then $f \in \mathrm{FHom}_R(M,M)$.

Proof It is obvious that f satisfies the conditions (1) and (2).

For the condition (3), let $x_1, x_2, y \in R$ with $y = y_1 + y_2$, where $y_1, y_2 \in R$. We only need to consider the following three cases.

Case 1: $\min\{f(x_1,y_1), f(x_2,y_2)\} = 0$. It is clear that $f(x_1 + x_2, y) \geq \min\{f(x_1,y_1), f(x_2,y_2)\}$.

Case 2: $0 < \min\{f(x_1,y_1), f(x_2,y_2)\} < 1$. We can suppose that $f(x_1,y_1) = 1 - \frac{1}{j+1}$ or 1, $f(x_2,y_2) = 1 - \frac{1}{k+1}$ and $j \geq k$. Then $I \subseteq I_j \subseteq I_k$ and $x_1 = y_1 \in M_j$ or N, $x_2 = y_2 \in I_k$. So $x_1 + x_2 = y_1 + y_2 \in I_k$. Hence $f(x_1 + x_2, y) \geq 1 - \frac{1}{k+1} = \min\{f(x_1,y_1), f(x_2,y_2)\}$.

Case 3: $\min\{f(x_1,y_1), f(x_2,y_2)\} = 1$. Then $x_1 = y_1 \in N$ and $x_2 = y_2 \in N$. So $f(x_1 + x_2, y) = 1 \geq \min\{f(x_1,y_1), f(x_2,y_2)\}$.

From the above three cases, it is clear that f satisfies the condition (3).

Finally, let $x, r, y \in R$ with $y = y_1 r$, where $y_1 \in R$. If $f(x,y_1) = 0$, then $f(xr,y) \geq f(x,y_1)$. If $f(x,y_1) = 1 - \frac{1}{k+1}$ for a positive integer k, then $x = y_1 \in M_k$. So $xr = y_1 r \in M_k$. Hence $f(xr,y) \geq 1 - \frac{1}{k+1} = f(x,y_1)$. If $f(x,y_1) = 1$, then $x = y_1 \in N$. So $xr = y \in N$. Thus $f(xr,y) \geq f(x,y_1)$. Therefore, $f(xr,y) \geq \sup\{f(x,y_1) : y = y_1 r, y_1 \in R\}$ for all $x \in R$, $y \in R$ and $r \in R$. Then f satisfies the condition (4). □

Definition 2.7 A weak fuzzy homomorphism $f \in \mathrm{WFHom}_R(M,N)$ is said to be *bounded* if there exists $t \in (0,1]$ such that, $\forall m \in M$ and $\forall n \in N$, $f(m,n) \geq t$ or $f(m,n) = 0$.

Example 2.8 Let M_R be a non-noetherian right R-module. Then M has an infinite ascending chain of submodules $M_1 \subsetneq M_2 \subsetneq M_3 \cdots$. Now we define a fuzzy subset f of $M \times M$

by

$$f(x,y)=\begin{cases}1, & x\in M_1, y=0,\\ \frac{1}{k}, & x\in M_k, x\notin M_{k-1}, k\geq 2, y=0,\\ 0, & \text{others}.\end{cases}$$

Then $f\in \mathrm{WFHom}_R(M,M)$. In particular, f is extendable and not bounded.

Proof By a similar discussion as that in Example 2.6, we have $f\in \mathrm{WFHom}_R(M,M)$. It is obvious that f is not bounded. Now set

$$g(x,y)=\begin{cases}1, & x\in M, y=0,\\ 0, & \text{others}.\end{cases}$$

It is clear that $g\in \mathrm{FHom}_R(M,M)$ and $f\leq g$. So f is extendable. □

3 Fuzzy characterizations of injective modules and artinian rings

According to Baer's Criterion, a right R-module M_R is said to be *injective* if every homomorphism from a right ideal I of R to M_R can be extended to a homomorphism from R_R to M_R.

Theorem 3.1 *Let R be a ring and M_R a right R-module. Then M is injective if and only if every $f\in \mathrm{WFHom}_R(R,M)$ is extendable.*

Proof ($\Leftarrow$) Let I be a right ideal of R. Suppose $u\in \mathrm{Hom}_R(I,M)$, we will show that u can be extended to a homomorphism $v\in \mathrm{Hom}_R(R,M)$. Firstly we construct a fuzzy subset of $R\times M$ by

$$f(x,y)=\begin{cases}1, & x\in I, \text{ and } y=u(x),\\ 0, & \text{others}.\end{cases}$$

By a similar proof of Example 2.2, $f\in \mathrm{WFHom}_R(R,M)$. Since f is extendable, there exists some $g\in \mathrm{FHom}_R(R,M)$ such that $f\leq g$. Now define $v: R\to M$ via $v(x)=y$, where $g(x,y)>0$. According to Remark 2.3(iii), $v\in \mathrm{Hom}_R(R,M)$. It is easy to see that $v|_I=u$. This shows that M is an injective right R-module.

($\Rightarrow$) Assume that M is an injective right R-module and $f\in \mathrm{WFHom}_R(R,M)$. Set $I=\sum_{0\neq t\in \mathrm{Im}f} R_t$. By Remark 2.3(ii), I is a right ideal of R. Now define $u: I\to M$ via $u(x)=y$, where $f(x,y)>0$. According to Remark 2.3(iii), $u\in \mathrm{Hom}_R(I,M)$. Since M is injective as a right R-module, u can be extended to a homomorphism v from R_R to M_R. By Example 2.2, we have $f_v\in \mathrm{FHom}_R(R,M)$. It is clear that $f\leq f_v$. So f is extendable. □

Next we will give some new fuzzy characterizations of right artinian rings. Recall that a fuzzy subset μ of a ring R is called a *fuzzy left* (*right*) *ideal* of R if μ satisfies: (i) $\mu(x-y)\geq \min\{\mu(x),\mu(y)\}$; (ii) $\mu(xy)\geq \mu(y)(\mu(xy)\geq \mu(x))$ for all $x,y\in R$. A fuzzy subset f is called *finite valued* if $\mathrm{Im}f$ is a finite set. If $\mathrm{Im}f$ is an infinite set, f is called *infinite valued*.

Theorem 3.2 *The following conditions are equivalent for a ring R.*

(a) *R is right artinian.*
(b) *Every fuzzy right ideal of R is finite valued.*
(c) *For every $f \in \mathrm{WFHom}_R(R_R, R_R)$, f is finite valued.*
(d) *For every $f \in \mathrm{FHom}_R(R_R, R_R)$, f is finite valued.*

Proof (a) $\Leftrightarrow$ (b) See [6, Theorem 3.2].

(b) $\Rightarrow$ (c) Suppose $f \in \mathrm{WFHom}_R(R_R, R_R)$. We can define a fuzzy subset μ of R by

$$\mu(x) = \begin{cases} t, & f(x,y) = t > 0 \text{ for some } y \in R, \\ 0, & \text{others}. \end{cases}$$

Since $f \in \mathrm{WFHom}_R(R_R, R_R)$, according to the condition (4), $f(xr, yr) \geq f(x,y)$, $\forall x, y, r \in R$. We have $\mu(xr) \geq \mu(x)$. For any $x_1, x_2 \in R$, if $\mu(x_1) = 0$ or $\mu(x_2) = 0$, it is clear that $\mu(x_1 - x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$. If $\mu(x_1) > 0$ and $\mu(x_2) > 0$, then there exist $y_1, y_2 \in R$ such that $\mu(x_1) = f(x_1, y_1)$ and $\mu(x_2) = f(x_2, y_2)$. Again since $f \in \mathrm{WFHom}_R(R_R, R_R)$, according to the condition (3) and Remark 2.3(i), we have

$$f(x_1 - x_2, y_1 - y_2) \geq \min\bigl\{f(x_1, y_1), f(-x_2, -y_2)\bigr\} = \min\bigl\{f(x_1, y_1), f(x_2, y_2)\bigr\}.$$

Therefore $\mu(x_1 - x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$. So μ is a fuzzy right ideal of R. Since $\mathrm{Im} f \subseteq \mathrm{Im}\,\mu \cup \{0\}$ and μ is finite valued, f is finite valued.

(c) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (a) Assume that R is not right artinian. Let $R = I_1 \supsetneq I_2 \supsetneq I_3 \cdots$ be a descending chain of right ideals of R and $I = \bigcap_i I_i$. Define a fuzzy subset f of $R \times R$ by

$$f(x,y) = \begin{cases} 1, & x = y \in I, \\ 1 - \frac{1}{n+1}, & x = y \in I_n, x \notin I_{n+1}, \\ 0, & \text{others}. \end{cases}$$

By Example 2.6, $f \in \mathrm{FHom}_R(R_R, R_R)$. But f is infinite valued. This is a contradiction. So R is right artinian. □

4 Fuzzy characterizations of right CF rings and QF rings

In this section, we will firstly give a fuzzy characterization of right CF ring. It is well known that a ring R is right CF if and only if for every right ideal I of R, there exist $x_1, \ldots, x_n \in R$ such that $I = \mathbf{r}(x_1, \ldots, x_n)$ (see [14, Lemma 7.2]).

Theorem 4.1 *A ring R is a right CF ring if and only if for every bounded and extendable $f \in \mathrm{WFHom}(R_R, R_R)$, there exist $F_1, \ldots, F_n \in \mathrm{FHom}_R(R_R, R_R)$ such that $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$.*

Proof ($\Rightarrow$) Suppose R is a right CF ring and $f \in \mathrm{WFHom}(R, R)$ is bounded and extendable. Then there exist $F \in \mathrm{FHom}_R(R_R, R_R)$ and $s \in (0,1]$ such that $f \leq F$ and $f(x,y) \geq s$ for all $x, y \in R$ with $f(x,y) > 0$. Set $I = \sum_{0 \neq t \in \mathrm{Im} f} R_t$. By Remark 2.3(ii), I is a right ideal of R. Since R is a right CF ring, there exist $c_1, \ldots, c_n \in R$ such that $I = \mathbf{r}(c_1, \ldots, c_n)$. To be convenient, we assume that $c_1 = 0$. Since $F \in \mathrm{FHom}_R(R_R, R_R)$, for the identity $1 \in R$, there exists $c \in R$

such that $F(1,c) > 0$. We now can construct fuzzy subsets $F_1, F_2, \ldots, F_n$ of $R \times R$ by

$$F_i(x,y) = \begin{cases} f(x,y), & x \in I, y = cx, \\ s, & x \notin I, y = (c-c_i)x, \quad i = 1,2,\ldots,n, \\ 0, & \text{others.} \end{cases}$$

At first, we show that $F_i \in \mathrm{FHom}_R(R_R, R_R)$, $i = 1,2,\ldots,n$.

(a) If $x \in I$, according to Remark 2.3(i), there exists $y \in R$, such that $f(x,y) > 0$. As $F \in \mathrm{FHom}_R(R_R, R_R)$, by the condition (4), $F(x,cx) \geq F(1,c) > 0$. Then by the condition (2), we have $F(x,y) = 0$ for all $y \neq cx$. Since $f \leq F$, $f(x,y) = 0$ for all $y \neq cx$. This implies that $f(x,cx) = f(x,y) > 0$. Therefore, $F_i(x,cx) = f(x,cx) > 0$. If $x \notin I$, let $y = (c-c_i)x$, then $F_i(x,y) = s > 0$. So F_i satisfies the condition (1).

(b) If $x \in I$ and $y_1, y_2 \in R$ with $F_i(x,y_1) > 0$ and $F_i(x,y_2) > 0$, then $y_1 = y_2 = cx$. If $x \notin I$ and $y_1, y_2 \in R$ with $F_i(x,y_1) > 0$ and $F_i(x,y_2) > 0$, then $y_1 = y_2 = (c-c_i)x$. So F_i satisfies the condition (2).

(c) Let $x_1, x_2, y \in R$ with $y = y_1 + y_2$, where $y_1, y_2 \in R$.

If $\min\{F_i(x_1,y_1), F_i(x_2,y_2)\} = 0$, then $F_i(x_1+x_2,y) \geq \min\{F_i(x_1,y_1), F_i(x_2,y_2)\}$. If $\min\{F_i(x_1, y_1), F_i(x_2,y_2)\} > 0$, we only need to discuss the following three cases.

Case 1: $x_1, x_2 \in I$. Then $y_1 = cx_1$ and $y_2 = cx_2$.

So $y = c(x_1+x_2)$, and

$$\begin{aligned} F_i(x_1+x_2,y) &= f(x_1+x_2,y) \\ &\geq \min\bigl\{f(x_1,y_1), f(x_2,y_2)\bigr\} \\ &= \min\bigl\{F_i(x_1,y_1), F_i(x_2,y_2)\bigr\}. \end{aligned}$$

Case 2: $x_1 \in I$ and $x_2 \notin I$. Then $y_1 = cx_1$, $y_2 = (c-c_i)x_2$ and $x_1 + x_2 \notin I$.

So $\min\{F_i(x_1,y_1), F_i(x_2,y_2)\} = s$. Since $I = \mathbf{r}(c_1,\ldots,c_n)$, $c_ix_1 = 0$. Then $y_1 + y_2 = cx_1 + (c - c_i)x_2 = (c-c_i)(x_1+x_2)$. Thus, $F_i(x_1+x_2,y) = s = \min\{F_i(x_1,y_1), F_i(x_2,y_2)\}$.

Case 3: $x_1 \notin I$ and $x_2 \notin I$. Then $y_1 = (c-c_i)x_1$ and $y_2 = (c-c_i)x_2$.

So $\min\{F_i(x_1,y_1), F_i(x_2,y_2)\} = s$ and $y_1 + y_2 = (c-c_i)(x_1+x_2)$. If $(x_1+x_2) \in I$, then $y_1 + y_2 = (c-c_i)(x_1+x_2) = c(x_1+x_2)$ because $c_ix_1 = c_ix_2 = 0$. Thus, $F_i(x_1+x_2,y) = f(x_1+x_2,y) \geq s = \min\{F_i(x_1,y_1), F_i(x_2,y_2)\}$. If $(x_1+x_2) \notin I$, then $y_1 + y_2 = (c-c_i)(x_1+x_2)$. So $F_i(x_1+x_2,y) = s \geq \min\{F_i(x_1,y_1), F_i(x_2,y_2)\}$.

Therefore, F_i satisfies the condition (3).

(d) For all $x, y_1r \in R$ with $y = y_1r$, where $y_1 \in R$. If $F_i(x,y_1) = 0$, $F_i(xr,y) \geq F_i(x,y_1)$. If $F_i(x,y_1) > 0$, we consider the following two cases.

Case 1: $x \in I$. Then $y_1 = cx$ and $F_i(x,y_1) = f(x,y_1)$. Thus, $xr \in I$ and $y = y_1r = cxr$. So $F_i(xr,y) = f(xr,y) \geq f(x,y_1) = F_i(x,y_1)$.

Case 2: $x \notin I$. Then $y_1 = (c-c_i)x$ and $F_i(x,y_1) = s$. Since $xr \in I$, $c_ixr = 0$. Then $y = y_1r = (c-c_i)xr = cxr$. Thus, $F_i(xr,y) = f(xr,y) \geq s = F_i(x,y_1)$. If $xr \notin I$ then $y = y_1r = (c-c_i)xr$. So $F_i(xr,y) = s \geq F_i(x,y_1)$.

Hence F_i satisfies the condition (4).

From the above, $F_i \in \mathrm{FHom}_R(R_R, R_R)$ for $i = 1,2,\ldots,n$.

Next we show that $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$.

Case 1: $x \in I$. If $y = cx$, then $F_i(x,y) = f(x,y)$, $i = 1,2,\dots,n$. So $f(x,y) = \min\{F_1(x,y), F_2(x,y), \dots, F_n(x,y)\}$. By (a) in the above proof, $f(x,cx) > 0$. If $y \neq cx$, then $f(x,y) = 0$. So $F_i(x,y) = 0$, $i = 1,\dots,n$. Hence $f(x,y) = \min\{F_1(x,y), F_2(x,y), \dots, F_n(x,y)\}$ for every $x \in I$.

Case 2: $x \notin I$. By the definition of I, $\forall y \in R$, $f(x,y) = 0$. If $y \neq (c - c_j)x$ for some $j \in \{1,2,\dots,n\}$, then $F_j(x,y) = 0$. Thus, $f(x,y) = \min\{F_1(x,y), F_2(x,y), \dots, F_n(x,y)\}$. If $y = (c - c_i)x$ for any $i \in \{1,2,\dots,n\}$, since $c_1 = 0$, $cx = (c - c_1)x = (c - c_2)x = \cdots = (c - c_n)x$. So $c_i x = 0$, $i = 1,2,\dots,n$. Hence $x \in \mathbf{r}(c_1,\dots,c_n) = I$. This is a contradiction.

So $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$.

($\Leftarrow$) Let I be a right ideal of R. We can construct a fuzzy subset f of $R \times R$ by

$$f(x,y) = \begin{cases} 1, & x \in I, \text{ and } y = 0, \\ 0, & \text{others.} \end{cases}$$

It is easy to see that $f \in \mathrm{WFHom}_R(R_R, R_R)$ and it is bounded and extendable. So there exist $F_1,\dots,F_n \in \mathrm{FHom}_R(R_R, R_R)$ such that $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$. Then there exists $c_i \in R$ such that $F_i(1,c_i) > 0$, $i = 1,\dots,n$. By the condition (4), for each $x \in R$, $F_i(x, c_i x) \geq F_i(1,c_i) > 0$, $i = 1,2,\dots,n$. But for every $x \in I$, $F_i(x,0) \geq f(x,0) = 1$. By the condition (2), $c_i x = 0$, $i = 1,2,\dots,n$. Hence $x \in \mathbf{r}(c_1,\dots,c_n)$. This shows that $I \subseteq \mathbf{r}(c_1,\dots,c_n)$. On the contrary, if $x \in \mathbf{r}(c_1,\dots,c_n)$, then $F_i(x,0) = F_i(x,c_i x) > 0$. So $f(x,0) = \min\{F_1(x,0),\dots,F_n(x,0)\} > 0$. Hence $x \in I$. This shows that $\mathbf{r}(c_1,\dots,c_n) \subseteq I$. Therefore, $I = \mathbf{r}_R(c_1,\dots,c_n)$. So R is a right CF ring. □

The following proposition can be looked on as an approach to the CF conjecture.

Proposition 4.2 *Let R be a ring. If for every extendable $f \in \mathrm{WFHom}_R(R_R, R_R)$, there exist $F_1,\dots,F_n \in \mathrm{FHom}_R(R_R, R_R)$ such that $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$. Then R is right artinian.*

Proof By Theorem 4.1, R is a right CF ring. It is well known that a right CF and right noetherian ring is right artinian. So we only need to prove that R is right noetherian.

Assume R is not right noetherian, then there is a strictly ascending chain $I_1 \subsetneq I_2 \subsetneq I_3 \cdots$ of right ideals of R. Let f be a fuzzy subset of $R \times R$ constructed by

$$f(x,y) = \begin{cases} 1, & x \in I_1 \text{ and } y = 0, \\ \frac{1}{k}, & k \geq 2, x \in I_k, x \notin I_{k-1} \text{ and } y = 0, \\ 0, & \text{others.} \end{cases}$$

By Example 2.8, $f \in \mathrm{WFHom}_R(R_R, R_R)$. f is extendable and not bounded. So there exist some $F_1,\dots,F_n \in \mathrm{FHom}_R(R_R, R_R)$ such that $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$. According to the condition (1), there exist $c_i \in R$ such that $F_i(1,c_i) > 0$, $i = 1,2,\dots$. Then by the condition (4), $F_i(x,c_i x) \geq F_i(1,c_i) > 0$. So $F_i(x,y) = 0$ or $F_i(x,y) \geq F_i(1,c_i)$ for all $i = 1,2,\dots,n$. Thus, $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$ is bounded. This is a contradiction. So R is right noetherian. Thus, R is right artinian. □

Remark 4.3 According to Theorem 3.2, Theorem 4.1 and Proposition 4.2, the CF conjecture is equivalent to saying that every extendable $f \in \mathrm{WFHom}_R(R_R, R_R)$ of a right CF ring R is bounded.

At last, we obtain a fuzzy characterization of QF rings.

Theorem 4.4 *R is a QF ring if and only if, for every $f \in \mathrm{WFHom}_R(R_R, R_R)$, there exist $F_1, \dots, F_n \in \mathrm{FHom}_R(R_R, R_R)$ such that $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$.*

Proof Suppose R is a QF ring. Then R is right artinian and the right R-module R_R is injective. By Theorem 3.2, every $f \in \mathrm{WFHom}_R(R_R, R_R)$ is finite valued. So f is bounded. Since R_R is injective, by Theorem 3.1, f is extendable. As QF rings are right CF rings, by Theorem 4.1, there exist $F_1, \dots, F_n \in \mathrm{FHom}_R(R_R, R_R)$ such that $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$.

Conversely, if for every $f \in \mathrm{WFHom}_R(R_R, R_R)$, there exist $F_1, \dots, F_n \in \mathrm{FHom}_R(R_R, R_R)$, such that $f = F_1 \wedge F_2 \wedge \cdots \wedge F_n$. By Proposition 4.2, R is right artinian. According to Definition 2.4, $f \leq F_1$. Then by Theorem 3.1, R_R is injective. So R is a QF ring. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The three authors contributed equally in writing this article. They read and approved the final manuscript.

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