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Hybrid methods for common solutions in Hilbert spaces with applications

Lijuan Sun*

*Correspondence: kfsunlj@yeah.net
Kaifeng Vocational College of Culture and Arts, Kaifeng, Henan, China

Abstract
In this paper, hybrid methods are investigated for treating common solutions of nonlinear problems. A strong convergence theorem is established in the framework of real Hilbert spaces.

Keywords: equilibrium problem; fixed point; projection; variational inequality; zero point

1 Introduction and preliminaries

Common solutions to variational inclusion, equilibrium and fixed point problems have been recently extensively investigated based on iterative methods; see [1–33] and the references therein. The motivation for this subject is mainly to its possible applications to mathematical modeling of concrete complex problems, which use more than one constraint. The aim of this paper is to investigate a common solution of variational inclusion, equilibrium and fixed point problems. The organization of this paper is as follows. In Section 1, we provide some necessary preliminaries. In Section 2, a hybrid method is introduced and analyzed. Strong convergence theorems are established in the framework of Hilbert spaces. In Section 3, applications of the main results are discussed.

In what follows, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty, closed, and convex subset of H and let P_C be the metric projection from H onto C. Let $S: C \to C$ be a mapping. $F(S)$ stands for the fixed point set of S; that is, $F(S) := \{x \in C : x = Sx\}$.

Recall that S is said to be *contractive* iff there exists a constant $\alpha \in [0,1)$ such that

$$\|Sx - Sy\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

If $\alpha = 1$, then S is said to be *nonexpansive*. Let $A: C \to H$ be a mapping. If C is nonempty closed and convex, then the fixed point set of S is nonempty.

Recall that A is said to be *monotone* iff

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

Recall that A is said to be *strongly monotone* iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be *α-strongly monotone*. Recall that A is said to be *inverse-strongly monotone* iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be *α-inverse-strongly monotone*.

Recall that a set-valued mapping $M : H \rightrightarrows H$ is said to be *monotone* iff, for all $x, y \in H$, $f \in Mx$, and $g \in My$ imply $\langle x - y, f - g\rangle > 0$. M is *maximal* iff the graph Graph(M) of R is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in \mathrm{Graph}(M)$ implies $f \in Rx$. For a maximal monotone operator M on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to D(M)$, where $D(M)$ denote the domain of M. It is well known that J_r is firmly nonexpansive, and $M^{-1}(0) = F(J_r)$, where $F(J_r) := \{x \in D(M) : x = J_r x\}$, and $M^{-1}(0) := \{x \in H : 0 \in Mx\}$.

Let $A : C \to H$ be a inverse-strongly monotone mapping, and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following generalized equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F, A)$.

To study the equilibrium problems (1.1), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F\big(tz + (1 - t)x, y\big) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semicontinuous.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 [34] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.2 [35] *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $T_r : H \to C$ *as follows*:

$$T_r x = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}, \quad x \in H,$$

then the following conclusions hold:

(1) T_r *is single-valued*;

(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and $\{\gamma_i\}$ be a nonnegative real sequence with $0 \leq \gamma_i < 1$, $\forall i \geq 1$. For $n \geq 1$ define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n S_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} S_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k S_k U_{n,k+1} + (1-\gamma_k)I,\\
&U_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 S_2 U_{n,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1-\gamma_1)I.
\end{aligned} \tag{1.2}$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$.

Lemma 1.3 [36] *Let* $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let* $\{\gamma_i\}$ *be a real sequence such that* $0 < \gamma_i \leq l < 1$, *where l is some real number,* $\forall i \geq 1$. *Then*

(1) W_n *is nonexpansive and* $F(W_n) = \bigcap_{i=1}^{\infty} F(S_i)$, *for each* $n \geq 1$;

(2) *for each* $x \in C$ *and for each positive integer* k, *the limit* $\lim_{n\to\infty} U_{n,k}$ *exists*;

(3) *the mapping* $W : C \to C$ *defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad x \in C, \tag{1.3}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$ *and it is called the* W*-mapping generated by* $S_1, S_2, \ldots$ *and* $\gamma_1, \gamma_2, \ldots$.

Lemma 1.4 [27] *Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \| Wx - W_n x\| = 0.$$

Throughout this paper, we always assume that $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$.

Lemma 1.5 [37] *Let $B : C \to H$ be a mapping and let $M : H \rightrightarrows H$ be a maximal monotone operator. Then $F(J_r(I - sB)) = (B + M)^{-1}(0)$.*

Lemma 1.6 [38] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H and let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty} \bigl(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\bigr) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.7 [39] *Let $A : C \to H$ a Lipschitz monotone mapping and let $N_C x$ be the normal cone to C at $x \in C$; that is, $N_C x = \{y \in H : \langle x - u, y\rangle, \forall u \in C\}$. Define*

$$Dx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset & x \notin C. \end{cases}$$

Then D is maximal monotone and $0 \in Dx$ if and only if $x \in VI(C, A)$.

2 Main results

Theorem 2.1 *Let C be a nonempty closed convex subset of a Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $A_1 : C \to H$ be a δ_1-inverse-strongly monotone mapping, $A_2 : C \to H$ be a δ_2-inverse-strongly monotone mapping, $A_3 : C \to H$ be a δ_3-inverse-strongly monotone mapping, $M_1 : H \rightrightarrows H$ a maximal monotone operator such that $\mathrm{Dom}(M_1) \subset C$ and $M_2 : H \rightrightarrows H$ a maximal monotone operator such that $\mathrm{Dom}(M_2) \subset C$. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega := \bigcap_{i=1}^{\infty} F(S_i) \cap EP(F, A_3) \cap (A_1 + M_1)^{-1}(0) \cap (A_2 + M_2)^{-1}(0) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{s_n}(u_n - s_n A_2 u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n W_n J_{r_n}(z_n - r_n A_1 z_n), \quad \forall n \geq 1, \end{cases}$$

where u is a fixed element in C, u_n is such that

$$F(u_n, y) + \langle A_3 x_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

$\{W_n : C \to C\}$ is the sequence generated in (1.2), $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{r_n\}$, $\{s_n\}$, and $\{\lambda_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions:

(a) $0 < a \le \lambda_n \le b < 2\delta_3$, $0 < a' \le r_n \le b' < 2\delta_1$, $0 < \bar{a} \le s_n \le \bar{b} < 2\delta_2$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in \Omega$, *where* $\bar{x} = P_\Omega u$.

Proof First, we show that the mapping $I - r_nA_1$, $I - s_nA_2$, and $I - \lambda_nA_3$ are nonexpansive. Indeed, we find from the restriction (a) that

$$\begin{aligned}
&\big\|(I - r_nA_1)x - (I - r_nA_1)y\big\|^2 \\
&\quad= \|x - y\|^2 - 2r_n\langle x - y, A_1x - A_1y\rangle + r_n^2\|A_1x - A_1y\|^2 \\
&\quad\le \|x - y\|^2 - 2r_n\delta_1\|A_1x - A_1y\|^2 + r_n^2\|A_1x - A_1y\|^2 \\
&\quad= \|x - y\|^2 + r_n(r_n - 2\delta_1)\|A_1x - A_1y\|^2 \\
&\quad\le \|x - y\|^2, \quad \forall x, y \in C,
\end{aligned}$$

which implies that the mapping $I - r_nA_1$ is nonexpansive. In the same way, we find $I - s_nA_2$ and $I - \lambda_nA_3$ are also nonexpansive. Put $y_n = J_{r_n}(z_n - r_nA_1z_n)$. Fixing $x^* \in \Omega$, we find

$$\begin{aligned}
\big\|y_n - x^*\big\| &= \big\|J_{r_n}(z_n - r_nA_1z_n) - J_{r_n}\big(x^* - r_nA_1x^*\big)\big\| \\
&\le \big\|z_n - x^*\big\| \\
&= \big\|J_{s_n}(u_n - s_nA_2u_n) - J_{s_n}\big(x^* - s_nA_2x^*\big)\big\| \\
&\le \big\|T_{\lambda_n}(I - \lambda_nA_3)x_n - T_{\lambda_n}(I - \lambda_nA_3)x^*\big\| \\
&\le \big\|x_n - x^*\big\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\big\|x_{n+1} - x^*\big\| &= \big\|\alpha_nu + \beta_nx_n + \gamma_nW_ny_n - x^*\big\| \\
&\le \alpha_n\big\|u - x^*\big\| + \beta_n\big\|x_n - x^*\big\| + \gamma_n\big\|W_ny_n - x^*\big\| \\
&\le \alpha_n\big\|u - x^*\big\| + (1 - \alpha_n)\big\|x_n - x^*\big\|.
\end{aligned}$$

This implies that $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$. Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that $x_n, y_n, z_n, u_n \in K$. Notice that

$$F(u_{n+1}, y) + \frac{1}{\lambda_{n+1}}\big\langle y - u_{n+1}, u_{n+1} - (I - r_{n+1}A_3)x_{n+1}\big\rangle \ge 0, \quad \forall y \in C, \tag{2.1}$$

and

$$F(u_n, y) + \frac{1}{\lambda_n}\big\langle y - u_n, u_n - (I - r_nA_3)x_n\big\rangle \ge 0, \quad \forall y \in C. \tag{2.2}$$

Let $y = u_n$ in (2.1) and $y = u_{n+1}$ in (2.2). By adding these two inequalities, we obtain

$$\left\langle u_{n+1} - u_n, \frac{u_n - (I - \lambda_nA_3)x_n}{\lambda_n} - \frac{u_{n+1} - (I - \lambda_{n+1}A_3)x_{n+1}}{\lambda_{n+1}}\right\rangle \ge 0.$$

It follows that

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\le \Big\langle u_{n+1}-u_n, (I-\lambda_{n+1}A_3)x_{n+1}-(I-\lambda_nA_3)x_n \\ &\quad + \left(1-\frac{\lambda_n}{\lambda_{n+1}}\right)\big(u_{n+1}-(I-\lambda_{n+1}A_3)x_{n+1}\big)\Big\rangle \\ &\le \|u_{n+1}-u_n\|\bigg(\big\|(I-\lambda_{n+1}A_3)x_{n+1}-(I-\lambda_nA_3)x_n\big\| \\ &\quad + \left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|\big\|u_{n+1}-(I-\lambda_{n+1}A_3)x_{n+1})\big\|\bigg).\end{aligned}$$

It follows that

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \big\|(I-\lambda_{n+1}A_3)x_{n+1}-(I-\lambda_nA_3)x_n\big\| \\ &\quad + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\big\|u_{n+1}-(I-\lambda_{n+1}A_3)x_{n+1}\big\| \\ &= \big\|(I-\lambda_{n+1}A_3)x_{n+1}-(I-\lambda_{n+1}A_3)x_n+(I-\lambda_{n+1}A_3)x_n-(I-\lambda_nA_3)x_n\big\| \\ &\quad + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\big\|u_{n+1}-(I-\lambda_{n+1}A_3)x_{n+1}\big\| \\ &\le \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|M_1,\end{aligned} \tag{2.3}$$

where M_1 is an appropriate constant such that

$$M_1=\sup_{n\ge 1}\left\{\|A_3x_n\|+\frac{\|u_{n+1}-(I-\lambda_{n+1}A_3)x_{n+1}\|}{a}\right\}.$$

Since J_{s_n} is firmly nonexpansive, we find that

$$\begin{aligned}&\|z_{n+1}-z_n\| \\ &\quad\le \big\|u_{n+1}-s_{n+1}A_2u_{n+1}-(u_n-s_nA_2u_n)\big\| \\ &\quad= \big\|(I-s_{n+1}A_2)u_{n+1}-(I-s_{n+1}A_2)u_n+(s_n-s_{n+1})A_2u_n\big\| \\ &\quad\le \|u_{n+1}-u_n\|+|s_n-s_{n+1}|\|A_2u_n\|.\end{aligned} \tag{2.4}$$

Combining (2.3) with (2.4) yields

$$\|z_{n+1}-z_n\|\le\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|M_1+|s_n-s_{n+1}|\|A_2u_n\|. \tag{2.5}$$

Since J_{r_n} is also firmly nonexpansive, we find that

$$\|y_{n+1}-y_n\|\le\|z_{n+1}-z_n\|+|r_n-r_{n+1}|\|A_1z_n\|. \tag{2.6}$$

Substituting (2.5) into (2.6), we see that

$$\|y_{n+1}-y_n\|\le\|x_{n+1}-x_n\|+\big(|r_{n+1}-r_n|+|\lambda_n-\lambda_{n+1}|+|s_n-s_{n+1}|\big)M_2, \tag{2.7}$$

where M_2 is an appropriate constant such that

$$M_2 = \max\Big\{\sup_{n\geq 1}\big\{\|A_1 z_n\|\big\}, \sup_{n\geq 1}\big\{\|A_2 u_n\|\big\}, M_1\Big\}.$$

Since W_n is nonexpansive, we find that

$$\begin{aligned}
&\|W_{n+1}y_{n+1} - W_n y_n\| \\
&\quad = \|W_{n+1}y_{n+1} - Wy_{n+1} + Wy_{n+1} - Wy_n + Wy_n - W_n y_n\| \\
&\quad \leq \|W_{n+1}y_{n+1} - Wy_{n+1}\| + \|Wy_{n+1} - Wy_n\| + \|Wy_n - W_n y_n\| \\
&\quad \leq \sup_{x\in K}\big\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\big\} + \|y_{n+1} - y_n\|,
\end{aligned} \tag{2.8}$$

where K is the bounded subset of C defined as above. Substituting (2.7) into (2.8), we find that

$$\begin{aligned}
\|W_{n+1}y_{n+1} - W_n y_n\| \leq{}& \sup_{x\in K}\big\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\big\} + \|x_{n+1} - x_n\| \\
&+ \big(|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}| + |s_n - s_{n+1}|\big)M_2.
\end{aligned} \tag{2.9}$$

Letting

$$x_{n+1} = (1-\beta_n)v_n + \beta_n x_n,$$

we see that

$$\begin{aligned}
v_{n+1} - v_n &= \frac{\alpha_{n+1}u + \gamma_{n+1}W_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n W_n y_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}u + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\beta_{n+1}}W_{n+1}y_{n+1} \\
&\quad - \left(\frac{\alpha_n}{1-\beta_n}u + \frac{1-\alpha_n-\beta_n}{1-\beta_n}W_n y_n\right) \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u - W_{n+1}y_{n+1}) - \frac{\alpha_n}{1-\beta_n}(u - W_n y_n) \\
&\quad + W_{n+1}y_{n+1} - W_n y_n.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - W_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u - W_n y_n\| \\
&+ \|W_{n+1}y_{n+1} - W_n y_n\|.
\end{aligned} \tag{2.10}$$

Substituting (2.9) into (2.10), we find that

$$\begin{aligned}
\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - W_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u - W_n y_n\| \\
&+ \sup_{x\in K}\big\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\big\} \\
&+ \big(|r_{n+1} - r_n| + |\lambda_n - \lambda_{n+1}| + |s_n - s_{n+1}|\big)M_2.
\end{aligned}$$

It follows from Lemma 1.4 that

$$\limsup_{n\to\infty}\big(\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|\big)\le 0.$$

In view of Lemma 1.6, we find that $\lim_{n\to\infty}\|v_n-x_n\|=0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.11}$$

For any $x^*\in\Omega$, we see that

$$\|x_{n+1}-x^*\|^2\le\alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|y_n-x^*\|^2. \tag{2.12}$$

Since

$$\begin{aligned}\|y_n-x^*\|^2&\le\|(I-r_nA_1)z_n-(I-r_nA_1)x^*\|^2\\&=\|z_n-x^*\|^2-2r_n\langle z_n-x^*,A_1z_n-A_1x^*\rangle+r_n^2\|A_1z_n-A_1x^*\|^2\\&\le\|z_n-x^*\|^2-2r_n\delta_1\|A_1z_n-A_1x^*\|^2+r_n^2\|A_1z_n-A_1x^*\|^2\\&=\|x_n-x^*\|^2+r_n(r_n-2\delta_1)\|A_1z_n-A_1x^*\|^2,\end{aligned}$$

we find that

$$\begin{aligned}&\gamma_nr_n(2\delta_1-r_n)\|A_1z_n-A_1x^*\|^2\\&\quad\le\alpha_n\|u-x^*\|^2+\big(\|x_n-x^*\|+\|x_{n+1}-x^*\|\big)\|x_n-x_{n+1}\|.\end{aligned}$$

Using the restrictions (a) and (b), we obtain

$$\lim_{n\to\infty}\|A_1z_n-A_1x^*\|=0. \tag{2.13}$$

It follows from (2.12) that

$$\|x_{n+1}-x^*\|^2\le\alpha_n\|u-x^*\|^2+\beta_n\|x_n-x^*\|^2+\gamma_n\|z_n-x^*\|^2.$$

Since

$$\begin{aligned}\|z_n-x^*\|^2&=\|J_{r_n}(u_n-s_nA_2u_n)-x^*\|^2\\&\le\|(I-s_nA_2)u_n-(I-s_nA_2)x^*\|^2\\&=\|u_n-x^*\|^2-2s_n\langle u_n-x^*,A_2u_n-A_2x^*\rangle+s_n^2\|A_2u_n-A_2x^*\|^2\\&\le\|u_n-x^*\|^2-2s_n\delta_2\|A_2u_n-A_2x^*\|^2+s_n^2\|A_2u_n-A_2x^*\|^2\\&=\|u_n-x^*\|^2+s_n(s_n-2\delta_2)\|A_2u_n-A_2x^*\|^2,\end{aligned}$$

we have

$$\|x_{n+1}-x^*\|^2\le\alpha_n\|f(x_n)-x^*\|^2+\|x_n-x^*\|^2+\gamma_ns_n(s_n-2\delta_2)\|A_2u_n-A_2x^*\|^2,$$

which implies that

$$\begin{aligned}&\gamma_n s_n(2\delta_2 - s_n)\left\|A_2 u_n - A_2 x^*\right\|^2\\&\quad\le \alpha_n\left\|f(x_n) - x^*\right\|^2 + \left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right)\|x_n - x_{n+1}\|.\end{aligned}$$

Using the restrictions (a) and (b), we obtain

$$\lim_{n\to\infty}\left\|A_2 u_n - A_2 x^*\right\| = 0. \tag{2.14}$$

Note that

$$\begin{aligned}\left\|x_{n+1} - x^*\right\|^2 &\le \alpha_n\left\|u - x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 + \gamma_n\left\|x_n - x^* - r_n\left(A_3 x_n - A_3 x^*\right)\right\|^2\\&\le \alpha_n\left\|u - x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 + \gamma_n\big(\left\|x_n - x^*\right\|^2\\&\quad + \lambda_n^2\left\|A_3 x_n - A_3 x^*\right\|^2 - 2\lambda_n\left\langle A_3 x_n - A_3 x^*, x_n - x^*\right\rangle\big)\\&\le \alpha_n\left\|u - x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 + \gamma_n\left\|x_n - x^*\right\|^2\\&\quad - \lambda_n\gamma_n(2\delta_3 - \lambda_n)\left\|A_3 x_n - A_3 x^*\right\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}&\lambda_n\gamma_n(2\delta_3 - \lambda_n)\left\|A_3 x_n - A_3 x^*\right\|^2\\&\quad\le \alpha_n\left\|u - x^*\right\|^2 + \left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right)\|x_n - x_{n+1}\|.\end{aligned}$$

Using the restrictions (a) and (b), we see that

$$\lim_{n\to\infty}\left\|A_3 x_n - A_3 x^*\right\| = 0. \tag{2.15}$$

Since T_{λ_n} is firmly nonexpansive, we find that

$$\begin{aligned}\left\|u_n - x^*\right\|^2 &\le \left\langle (I - \lambda_n A_3)x_n - (I - \lambda_n A_3)x^*, u_n - x^*\right\rangle\\&\le \frac{1}{2}\big(\left\|x_n - x^*\right\|^2 + \left\|u_n - x^*\right\|^2 - \|x_n - u_n\|^2 - \lambda_n^2\left\|A_3 x_n - A_3 x^*\right\|^2\\&\quad + 2\lambda_n\left\langle A_3 x_n - A_3 x^*, x_n - u_n\right\rangle\big).\end{aligned}$$

This in turn implies that

$$\begin{aligned}\gamma_n\|x_n - u_n\|^2 &\le \alpha_n\left\|f(x_n) - x^*\right\|^2 + \left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right)\|x_n - x_{n+1}\|\\&\quad + 2\lambda_n\left\|A_3 x_n - A_3 x^*\right\|\|x_n - u_n\|.\end{aligned}$$

Using the restrictions (a) and (b), we see that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{2.16}$$

Since J_{s_n} is also firmly nonexpansive mapping, we see that

$$\begin{aligned}\|z_n - x^*\|^2 &\le \langle (I - s_nA_2)u_n - (I - s_nA_2)x^*, z_n - x^*\rangle \\ &\le \frac{1}{2}\big(\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n - s_n(A_2u_n - A_2x^*)\|^2\big) \\ &\le \frac{1}{2}\big(\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n\|^2 \\ &\quad + 2s_n\|u_n - z_n\|\|A_2u_n - A_2x^*\| - s_n^2\|A_2u_n - A_2x^*\|^2\big),\end{aligned}$$

which implies that

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2s_n\|u_n - z_n\|\|A_2u_n - A_2x^*\|.$$

It follows that

$$\begin{aligned}\gamma_n\|u_n - z_n\|^2 &\le \alpha_n\|f(x_n) - x^*\|^2 + \big(\|x_n - x^*\| + \|x_{n+1} - x^*\|\big)\|x_n - x_{n+1}\| \\ &\quad + 2s_n\|u_n - z_n\|\|A_2u_n - A_2x^*\|.\end{aligned}$$

Using the restrictions (a) and (b), we obtain

$$\lim_{n\to\infty}\|u_n - z_n\| = 0 \tag{2.17}$$

and

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{2.18}$$

Note that

$$(1 - \beta_n)\|W_ny_n - x_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|u - W_ny_n\|.$$

Using the restrictions (b) and (c), we obtain

$$\lim_{n\to\infty}\|W_ny_n - x_n\| = 0. \tag{2.19}$$

On the other hand, one has

$$\|W_ny_n - y_n\| \le \|y_n - z_n\| + \|z_n - u_n\| + \|u_n - x_n\| + \|x_n - W_ny_n\|.$$

Using (2.6), (2.7), (2.8), and (2.9), we find that

$$\lim_{n\to\infty}\|W_ny_n - y_n\| = 0. \tag{2.20}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle u - \bar{x}, x_n - \bar{x}\rangle \le 0.$$

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-\bar{x}, x_n-\bar{x}\rangle = \lim_{i\to\infty}\langle u-\bar{x}, x_{n_i}-\bar{x}\rangle. \tag{2.21}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup q$. Therefore, we see that $y_{n_i} \rightharpoonup q$. We also have $z_{n_i} \rightharpoonup q$.

Next, we show that $q \in \bigcap_{i=1}^{\infty} F(S_i)$. Suppose the contrary, $q \notin CFPS$, *i.e.*, $Wq \neq q$. Since $y_{n_i} \rightharpoonup q$, we see from Opial's condition that

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i}-q\| &< \liminf_{i\to\infty}\|y_{n_i}-Wq\| \\ &\le \liminf_{i\to\infty}\big\{\|y_{n_i}-Wy_{n_i}\| + \|Wy_{n_i}-Wq\|\big\} \\ &\le \liminf_{i\to\infty}\big\{\|y_{n_i}-Wy_{n_i}\| + \|y_{n_i}-q\|\big\}. \end{aligned} \tag{2.22}$$

On the other hand, we have

$$\begin{aligned}\|Wy_n-y_n\| &\le \|Wy_n-W_ny_n\| + \|W_ny_n-y_n\| \\ &\le \sup_{x\in K}\|Wx-W_nx\| + \|W_ny_n-y_n\|.\end{aligned}$$

In view of Lemma 1.4, we obtain that $\lim_{n\to\infty}\|Wy_n-y_n\| = 0$. This implies from (2.22) that $\liminf_{i\to\infty}\|y_{n_i}-q\| < \liminf_{i\to\infty}\|y_{n_i}-q\|$. This is a contradiction. Thus, we have $q \in \bigcap_{i=1}^{\infty} F(S_i)$.

Now, we are in a position to prove that $q \in (A_1+M_1)^{-1}(0)$. Notice that $\frac{z_n-y_n}{r_n} - A_1z_n \in M_1y_n$. Let $\mu \in M_1\nu$. Since M_1 is monotone, we find that

$$\left\langle \frac{z_n-y_n}{r_n} - A_1z_n - \mu, y_n-\nu\right\rangle \ge 0.$$

This implies that $\langle -A_1q-\mu, q-\nu\rangle \ge 0$. This implies that $-A_1q \in M_1q$, that is, $q \in (A_1+M_1)^{-1}(0)$.

Now, we prove that $q \in (A_2+M_2)^{-1}(0)$. Notice that $\frac{u_n-z_n}{s_n} - A_2u_n \in M_2z_n$. Let $\mu' \in M_2\nu'$. Since M_2 is monotone, we find that

$$\left\langle \frac{u_n-z_n}{s_n} - A_2u_n - \mu', z_n-\nu'\right\rangle \ge 0.$$

This implies that $\langle -A_2q-\mu', q-\nu'\rangle \ge 0$. This implies that $-A_2q \in M_2q$, that is, $q \in (A_2+M_2)^{-1}(0)$.

Next, we show that $q \in EP(F, A_3)$. Since $u_n = T_{\lambda_n}(I-\lambda_nA_3)x_n$, for any $y \in C$, we have

$$F(u_n, y) + \langle A_3x_n, y-u_n\rangle + \frac{1}{\lambda_n}\langle y-u_n, u_n-x_n\rangle \ge 0.$$

Replacing n by n_i, we find from (A2) that

$$\langle A_3x_{n_i}, y-u_{n_i}\rangle + \left\langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{\lambda_{n_i}}\right\rangle \ge F(y, u_{n_i}), \quad \forall y \in C.$$

Putting $y_t = ty + (1-t)q$ for any $t \in (0,1]$ and $y \in C$, we see that $y_t \in C$. It follows that

$$\begin{aligned}
&\langle y_t - u_{n_i}, A_3 y_t\rangle \\
&\quad\geq \langle y_t - u_{n_i}, A_3 y_t\rangle - \langle A_3 x_{n_i}, y_t - u_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\
&\quad= \langle y_t - u_{n_i}, A_3 y_t - A_3 u_{n_i}\rangle + \langle y_t - u_{n_i}, A_3 u_{n_i} - A_3 x_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\qquad+ F(y_t, u_{n_i}).
\end{aligned}$$

In view of the monotonicity of A_3, and the restriction (a), we obtain from (A4) that

$$\langle y_t - q, A_3 y_t\rangle \geq F(y_t, q).$$

From (A1) and (A4), we see that

$$\begin{aligned}
0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, q) \\
&\leq tF(y_t, y) + (1-t)\langle y_t - q, A_3 y_t\rangle \\
&= tF(y_t, y) + (1-t)t\langle y - q, A_3 y_t\rangle.
\end{aligned}$$

It follows that

$$0 \leq F(y_t, y) + (1-t)\langle y - w, A_3 y_t\rangle, \quad \forall y \in C.$$

It follows from (A3) that $q \in EP(F, A_3)$. Hence,

$$\limsup_{n\to\infty}\langle u - \bar{x}, x_n - \bar{x}\rangle \leq 0.$$

Finally, we show that $x_n \to \bar{x}$. Note that

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 \\
&\quad\leq \alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + \beta_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \gamma_n\|y_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\quad\leq \alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + \frac{1-\alpha_n}{2}\big(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2\big).
\end{aligned}$$

This implies that

$$\|x_{n+1} - \bar{x}\|^2 \leq 2\alpha_n\langle u - \bar{x}, x_{n+1} - \bar{x}\rangle + (1-\alpha_n)\|x_n - \bar{x}\|^2.$$

Using Lemma 1.1, we find that $\lim_{n\to\infty}\|x_n - \bar{x}\| = 0$. This completes the proof. □

3 Applications

In this section, we consider some applications of the main results.

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C.$$

In this paper, we use $VI(C,A)$ to denote the solution set of the inequality. It is well known that $x \in C$ is a solution of the inequality iff x is a fixed point of the mapping $P_C(I - rA)$, where $r > 0$ is a constant, I stands for the identity mapping. If A is α-inverse-strongly monotone and $r \in (0, 2\alpha]$, then the mapping $I - rA$ is nonexpansive. It follows that $VI(C,A)$ is closed and convex.

Let $g : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂g of g is defined as follows:

$$\partial fg(x) = \{y \in H : g(z) \geq g(x) + \langle z - x, y\rangle, z \in H\}, \quad \forall x \in H.$$

From Rockafellar [39], we know that ∂g is maximal monotone. It is not hard to verify that $0 \in \partial g(x)$ if and only if $g(x) = \min_{y \in H} g(y)$.

Let I_C be the indicator function of C, *i.e.*,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Since I_C is a proper lower semicontinuous convex function on H, we see that the subdifferential ∂I_C of I_C is a maximal monotone operator. It is clearly that $J_r x = P_C x$, $\forall x \in H$, $(A_1 + \partial I_C)^{-1}(0) = VI(C, A_1)$ and $(A_2 + \partial I_C)^{-1}(0) = VI(C, A_2)$.

Theorem 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $A_1 : C \to H$ be a δ_1-inverse-strongly monotone mapping, $A_2 : C \to H$ be a δ_2-inverse-strongly monotone mapping, $A_3 : C \to H$ be a δ_3-inverse-strongly monotone mapping, and $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega := \bigcap_{i=1}^{\infty} F(S_i) \cap EP(F, A_3) \cap VI(C, A_1) \cap VI(C, A_2) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(u_n - s_n A_2 u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C(z_n - r_n A_1 z_n), \quad \forall n \geq 1, \end{cases}$$

where u_n is such that

$$F(u_n, y) + \langle A_3 x_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

$\{W_n : C \to C\}$ is the sequence generated in (1.2), *$\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in* $(0,1)$ *such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{r_n\}$, $\{s_n\}$, and $\{\lambda_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions*:

(a) $0 < a \leq \lambda_n \leq b < 2\delta_3$, $0 < a' \leq r_n \leq b' < 2\delta_1$, $0 < \bar{a} \leq s_n \leq \bar{b} < 2\delta_2$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega u$.

Recall that a mapping $T : C \to C$ is said to be a k-strict pseudo-contraction if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

Putting $A = I - T$, where $T : C \to C$ is a k-strict pseudo-contraction, we find that A is $\frac{1-k}{2}$-inverse-strongly monotone.

Next, we consider fixed points of strict pseudo-contractions.

Theorem 3.2 *Let C be a nonempty closed convex subset of a Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4). *Let $T_1 : C \to H$ be a k_1-strict pseudo-contraction, $T_2 : C \to H$ be a k_2-strict pseudo-contraction, $A_3 : C \to H$ be a δ-inverse-strongly monotone mapping, and $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega := \bigcap_{i=1}^{\infty} F(S_i) \cap EP(F, A_3) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = (1 - s_n)u_n + s_n T_2 u_n, \\ y_n = (1 - r_n)u_n + r_n T_1 u_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 1, \end{cases}$$

where u_n is such that

$$F(u_n, y) + \langle A_3 x_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$

$\{W_n : C \to C\}$ is the sequence generated in (1.2), *$\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in* $(0,1)$ *such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$ and $\{r_n\}$, $\{s_n\}$, and $\{\lambda_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions*:

(a) $0 < a \leq \lambda_n \leq b < 2\delta$, $0 < a' \leq r_n \leq b' < 1 - k_1$, $0 < \bar{a} \leq s_n \leq \bar{b} < 1 - k_2$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = \lim_{n\to\infty} |s_n - s_{n+1}| = \lim_{n\to\infty} |r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega u$.

Proof Taking $A_i = I - T_i$, wee see that $A_i : C \to H$ is a δ_i-strict pseudo-contraction with $\delta_i = \frac{1-k_i}{2}$ and $F(T_i) = VI(C, A_i)$ for $i = 1, 2$. In view of Theorem 3.1, we find the desired conclusion immediately. □

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The author declares that he has no competing interests.

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