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Sharp Wilker-type inequalities with applications

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Abstract

In this paper, we prove that the Wilker-type inequality

$$\frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp}+\frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p}>(<)1$$

holds for any fixed $k\geq 1$ and all $x\in(0,\pi/2)$ if and only if $p>0$ or $p\leq -\frac{\ln(k+2)-\ln 2}{k(\ln\pi-\ln 2)}$ ($-\frac{12}{5(k+2)}\leq p<0$), and the hyperbolic version of Wilker-type inequality

$$\frac{2}{k+2}\left(\frac{\sinh x}{x}\right)^{kp}+\frac{k}{k+2}\left(\frac{\tanh x}{x}\right)^{p}>(<)1$$

holds for any fixed $k\geq 1$ (<-2) and all $x\in(0,\infty)$ if and only if $p>0$ or $p\leq -\frac{12}{5(k+2)}$ ($p<0$ or $p\geq -\frac{12}{5(k+2)}$). As applications, several new analytic inequalities are presented.

MSC: 26D05; 33B10

Keywords: Wilker inequality; trigonometric function; hyperbolic function

1 Introduction

Wilker [1] proposed two open problems, the first of which states that the inequality

$$\left(\frac{\sin x}{x}\right)^{2}+\frac{\tan x}{x}>2 \tag{1.1}$$

holds for all $x\in(0,\pi/2)$. Inequality (1.1) was proved by Sumner *et al.* in [2].

Recently, the Wilker inequality (1.1) and its generalizations, improvements, refinements and applications have attracted the attention of many mathematicians (see [3–17] and related references therein).

In [9], Wu and Srivastava established the following Wilker-type inequality:

$$\left(\frac{x}{\sin x}\right)^{2}+\frac{x}{\tan x}>2 \quad \text{for } x\in(0,\pi/2) \tag{1.2}$$

and its weighted and exponential generalization.

Theorem Wu ([9, Theorem 1]) *Let* $\lambda>0$, $\mu>0$ *and* $p\leq 2q\mu/\lambda$. *If* $q>0$ *or* $q\leq \min(-1,-\lambda/\mu)$, *then the inequality*

$$\frac{\lambda}{\lambda+\mu}\left(\frac{\sin x}{x}\right)^p+\frac{\mu}{\lambda+\mu}\left(\frac{\tan x}{x}\right)^q>1 \tag{1.3}$$

holds for $x\in(0,\pi/2)$.

As an application of inequality (1.3), an open problem was proposed, answered and improved by Sándor and Bencze in [18]. Recently, inequality (1.3) and its related inequalities in [9] were extended to Bessel functions [3], and the hyperbolic version of Theorem Wu was presented in [12].

In 2009, Zhu [16] gave another exponential generalization of Wilker inequality (1.1) as follows.

Theorem Zh1 ([16, Theorems 1.1 and 1.2]) *Let* $0<x<\pi/2$. *Then the inequalities*

$$\left(\frac{\sin x}{x}\right)^{2p}+\left(\frac{\tan x}{x}\right)^p>\left(\frac{x}{\sin x}\right)^{2p}+\left(\frac{x}{\tan x}\right)^p>2 \tag{1.4}$$

hold if $p\geq 1$, *while the first one in* (1.4) *holds if and only if* $p>0$.

Theorem Zh2 ([16, Theorems 1.3 and 1.4]) *Let* $x>0$. *Then the inequalities*

$$\left(\frac{\sinh x}{x}\right)^{2p}+\left(\frac{\tanh x}{x}\right)^p>\left(\frac{x}{\sinh x}\right)^{2p}+\left(\frac{x}{\tanh x}\right)^p>2 \tag{1.5}$$

hold if $p\geq 1$, *while the first one in* (1.5) *holds if and only if* $p>0$.

In [16], Zhu also proposed an open problem: find the respectively largest range of p such that inequalities (1.4) and (1.5) hold. It was solved by Matejička in [19].

Another inequality associated with the Wilker inequality is the following:

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>3 \tag{1.6}$$

for $x\in(0,\pi/2)$, which is known as the Huygens inequality [20]. The following refinement of Huygens inequality is due to Neuman and Sándor [7]:

$$2\frac{\sin x}{x}+\frac{\tan x}{x}>2\frac{x}{\sin x}+\frac{x}{\tan x}>3 \tag{1.7}$$

for $x\in(0,\pi/2)$. Very recently, the generalizations of (1.7) were given by Neuman in [8]. In [21], Zhu proved that the inequalities

$$(1-\xi_1)\frac{\sin x}{x}+\xi_1\frac{\tan x}{x}>1>(1-\eta_1)\frac{\sin x}{x}+\eta_1\frac{\tan x}{x}, \tag{1.8}$$

$$(1-\xi_2)\frac{x}{\sin x}+\xi_2\frac{x}{\tan x}>1>(1-\eta_2)\frac{x}{\sin x}+\eta_2\frac{x}{\tan x} \tag{1.9}$$

hold for all $x\in(0,\pi/2)$ with the best constants $\xi_1=1/3$, $\eta_1=0$, $\xi_2=1/3$, $\eta_2=1-2/\pi$. Later, Zhu [15] generalized inequalities (1.8) and (1.9) to the exponential form as follows.

Theorem Zh3 ([15, Theorems 1.1 and 1.2]) *Let* $0 < x < \pi/2$. *Then we have*

(i) *If* $p \geq 1$, *then the double inequality*

$$(1-\lambda)\left(\frac{x}{\sin x}\right)^p + \lambda\left(\frac{x}{\tan x}\right)^p < 1 < (1-\eta)\left(\frac{x}{\sin x}\right)^p + \eta\left(\frac{x}{\tan x}\right)^p \tag{1.10}$$

holds if and only if $\eta \leq 1/3$ *and* $\lambda \geq 1-(2/\pi)^p$.

(ii) *If* $0 \leq p \leq 4/5$, *then double inequality* (1.10) *holds if and only if* $\lambda \geq 1/3$ *and* $\eta \leq 1-(2/\pi)^p$.

(iii) *If* $p < 0$, *then the second inequality in* (1.10) *holds if and only if* $\eta \geq 1/3$.

The hyperbolic version of inequalities (1.7) was given in [7] by Neuman and Sándor. Later, Zhu showed the following.

Theorem Zh4 ([17, Theorem 4.1]) *Let* $x > 0$. *Then one has*

(i) *If* $p \geq 4/5$, *then the double inequality*

$$(1-\lambda)\left(\frac{x}{\sinh x}\right)^p + \lambda\left(\frac{x}{\tanh x}\right)^p < 1 < (1-\eta)\left(\frac{x}{\sinh x}\right)^p + \eta\left(\frac{x}{\tanh x}\right)^p \tag{1.11}$$

holds if and only if $\eta \geq 1/3$ *and* $\lambda \leq 0$.

(ii) *If* $p < 0$, *then the inequality*

$$(1-\eta)\left(\frac{x}{\sinh x}\right)^p + \eta\left(\frac{x}{\tanh x}\right)^p > 1 \tag{1.12}$$

holds if and only if $\eta \leq 1/3$.

The main aim of this paper is to present the best possible parameter p such that the inequalities

$$\frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^p > 1 \quad \text{for } x \in (0, \pi/2), \tag{1.13}$$

$$\frac{2}{k+2}\left(\frac{\sinh x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tanh x}{x}\right)^p > 1 \quad \text{for } x \in (0, \infty) \tag{1.14}$$

or their reversed inequalities hold for certain fixed k with $k(k+2) \neq 0$. As applications, we also present several new analytic inequalities.

2 Lemmas

In order to establish our main results, we need several lemmas, which we present in this section.

Lemma 1 *Let* A, B *and* C *be defined on* $(0, \pi/2)$ *by*

$$A = A(x) = \cos x(\sin x - x\cos x)^2(x - \cos x\sin x), \tag{2.1}$$

$$B = B(x) = (x - \cos x\sin x)^2(\sin x - x\cos x), \tag{2.2}$$

$$C = C(x) = \sin^2 x\left(-2x^2\cos x + x\sin x + \cos x\sin^2 x\right). \tag{2.3}$$

Then, for fixed $k \geq 1$, the function $x \mapsto C(x)/(kA(x)+B(x))$ is increasing on $(0,\pi/2)$. Moreover, we have

$$\frac{5}{12(k+2)} < \frac{C(x)}{kA(x)+B(x)} < 1. \tag{2.4}$$

Proof We clearly see that $A, B > 0$ for $x \in (0,\pi/2)$ because of $\sin x - x\cos x > 0$ and $x - \cos x \sin x = (2x - \sin 2x)/2 > 0$, and $C > 0$ because of

$$\left(-2x^2\cos x + x\sin x + \cos x\sin^2 x\right) = x^2\cos x\left(\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right) > 0$$

by Wilker inequality (1.1).

Let $D = (kA+B)/C$, then simple computations lead to

$$\begin{aligned} D(x) &= \frac{x\sin^2 x(-2x^2\cos x + x\sin x + \cos x\sin^2 x)}{(\sin x - x\cos x)(x - \cos x\sin x)((1-k\cos^2 x)x + (k-1)\cos x\sin x)} \\ &= \frac{-2x^2\cos x + x\sin x + \cos x\sin^2 x}{(\sin x - x\cos x)(x-\cos x\sin x)} \times \frac{x\sin^2 x}{k(\sin x - x\cos x)\cos x + (x - \cos x\sin x)} \\ &:= D_1(x) \times D_2(x). \end{aligned}$$

It follows from [16, Lemma 2.9] that the function D_1 is positive and increasing on $(0,\pi/2)$. Hence it remains to prove that the function D_2 is also positive and increasing. Clearly, $D_2(x) > 0$, we only need to show that $D_2'(x) > 0$ for $x \in (0,\pi/2)$. Indeed,

$$\begin{aligned} D_2'(x) &= (k-1)\sin x\frac{(-2x^2\cos x + \cos x\sin^2 x + x\sin x)}{(k(\sin x - x\cos x)\cos x + (x - \cos x\sin x))^2} \\ &= \frac{(k-1)x^2\sin x\cos x}{(k(\sin x - x\cos x)\cos x + (x-\cos x\sin x))^2}\left(\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right), \end{aligned}$$

which is clearly positive due to Wilker inequality (1.1). Therefore, $C/(kA+B)$ is increasing on $(0,\pi/2)$, and

$$\frac{5}{12(k+2)} = \lim_{x\to 0}\frac{C(x)}{kA(x)+B(x)} < D(x) < \lim_{x\to\pi/2^-}\frac{C(x)}{kA(x)+B(x)} = 1.$$

This completes the proof. □

Lemma 2 *Let E, F and G be defined on $(0,\infty)$ by*

$$E = E(x) = \cosh x(\sinh x - x\cosh x)^2(x - \cosh x\sinh x), \tag{2.5}$$

$$F = F(x) = (\sinh x - x\cosh x)(x - \cosh x\sinh x)^2, \tag{2.6}$$

$$G = G(x) = x\sinh^2 x\left(2x^2\cosh x - x\sinh x - \cosh x\sinh^2 x\right). \tag{2.7}$$

Then, for fixed $k \geq 1$ $(k < -2)$, the function $x \mapsto G(x)/(kE(x)+F(x))$ is decreasing (increasing) on $(0,\infty)$. Moreover, we have

$$\min\left(0, \frac{12}{5(k+2)}\right) < \frac{G(x)}{kE(x)+F(x)} < \max\left(0, \frac{12}{5(k+2)}\right). \tag{2.8}$$

Proof It is easy to verify that $E, F < 0$ for $x \in (0, \infty)$ due to

$$(x - \cosh x \sinh x) = (2x - \sinh 2x)/2 < 0,$$
$$(\sinh x - x \cosh x) = x\left(\frac{\sinh x}{x} - \cos x\right) < 0.$$

While $G < 0$ because of

$$\left(2x^2 \cosh x - x \sinh x - \cosh x \sinh^2 x\right) = -x^2 \cosh x\left(\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2\right) < 0$$

by Wilker inequality (1.5).

Denote $G/(kE + F)$ by H and simple computations give

$$\begin{aligned} H(x) &= \frac{x \sinh^2 x(2x^2 \cosh x - x \sinh x - \cosh x \sinh^2 x)}{\cosh x(\sinh x - x \cosh x)^2(x - \sinh x \cosh x)k + (\sinh x - x \cosh x)(x - \sinh x \cosh x)^2} \\ &= \frac{-2x^2 \cosh x + x \sinh x + \cosh x \sinh^2 x}{(x \cosh x - \sinh x)(\sinh x \cosh x - x)} \times \frac{x \sinh^2 x}{(k(x \cosh x - \sinh x) \cosh x + \sinh x \cosh x - x)} \\ &:= H_1(x) \times H_2(x). \end{aligned}$$

Clearly, $H_1(x) > 0$, and it was proved in [19, Proof of Lemma 2.2] that H_1 is decreasing on $(0, \infty)$. In order to prove the monotonicity of H, we only need to deal with the sign and monotonicity of H_2.

(i) Clearly, $H_2(x) > 0$ for $k \geq 1$. And we claim that H_2 is also decreasing on $(0, \infty)$. Indeed,

$$\begin{aligned} H_2'(x) &= -(k-1) \sinh x \frac{(-2x^2 \cosh x + \cosh x \sinh^2 x + x \sinh x)}{(x \cosh x - \sinh x)^2(\cosh x \sinh x - x)^2} \\ &= -\frac{(k-1)x^2 \sinh x \cosh x}{(x \cosh x - \sinh x)^2(\cosh x \sinh x - x)^2}\left(\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2\right) < 0. \end{aligned}$$

Consequently, $H = H_1 \times H_2$ is positive and decreasing on $(0, \infty)$, and so

$$0 = \lim_{x\to\infty} \frac{G(x)}{kE(x) + F(x)} < \frac{G(x)}{kE(x) + F(x)} < \lim_{x\to 0} \frac{G(x)}{kE(x) + F(x)} = \frac{12}{5(k+2)}.$$

(ii) For $k < -2$, by the previous proof we clearly see that $-H_2'$ is decreasing on $(0, \infty)$, and so

$$0 < -\frac{1}{k} = \lim_{x\to\infty}\left(-H_2(x)\right) < -H_2(x) < \lim_{x\to 0}\left(-H_2(x)\right) = -\frac{3}{k+2},$$

which implies that $-H_2$ is positive and decreasing on $(0, \infty)$, and so is the function $-H = H_1 \times (-H_2)$. That is, H is negative and increasing on $(0, \infty)$, and inequality (2.8) holds true.

This completes the proof. □

Remark 1 It should be noted that $kE(x) + F(x) < 0$ for $k \geq 1$ and $kE(x) + F(x) > 0$ for $k < -2$. In fact, it suffices to notice (2.8) and $G(x) < 0$ for $x \in (0, \infty)$.

Lemma 3 *For $k \geq 1$, we have*

$$1 > \frac{\ln(k+2) - \ln 2}{k(\ln\pi - \ln 2)} > \frac{12}{5(k+2)}.$$

Proof It suffices to show that

$$\delta_1(k) = \frac{\ln(k+2) - \ln 2}{\ln\pi - \ln 2} - k < 0,$$
$$\delta_2(k) = \frac{\ln(k+2) - \ln 2}{\ln\pi - \ln 2} - \frac{12k}{5(k+2)} > 0$$

for $k \geq 1$.

Differentiation gives

$$\delta_1'(k) = \frac{1}{(\ln\pi - \ln 2)(k+2)} - 1 < 0,$$
$$\delta_2'(k) = \frac{1}{5}\frac{5k + 24\ln 2 - 24\ln\pi + 10}{(k+2)^2(\ln\pi - \ln 2)} > 0$$

for $k \geq 1$. Therefore, Lemma 3 follows from $\delta_1(k) \leq \delta_1(1) = (\ln 3 - \ln 2)/(\ln 3 - \ln\pi) < 0$ and $\delta_2(k) \geq \delta_2(1) = (\ln 3 - \ln 2)/(\ln\pi - \ln 2) - 4/5 > 0$. □

3 Main results

Theorem 1 *For fixed $k \geq 1$, inequality* (1.13) *holds for $x \in (0, \pi/2)$ if and only if $p > 0$ or $p \leq -\frac{\ln(k+2)-\ln 2}{k(\ln\pi-\ln 2)}$.*

Proof Inequality (1.13) is equivalent to

$$f(x) = \frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} - 1 > 0 \tag{3.1}$$

for $x \in (0, \pi/2)$. Differentiation yields

$$\begin{aligned} f'(x) &= -\frac{2kp}{k+2}\frac{\sin x - x\cos x}{x^2}\left(\frac{\sin x}{x}\right)^{kp-1} + \frac{kp}{k+2}\frac{x - \sin x\cos x}{x^2\cos^2 x}\left(\frac{\tan x}{x}\right)^{p-1} \\ &= \frac{kp}{k+2}\frac{x - \sin x\cos x}{x^2\cos^2 x}\left(\frac{\tan x}{x}\right)^{p-1} g(x), \end{aligned} \tag{3.2}$$

where

$$g(x) = 1 - 4\frac{\sin x - x\cos x}{2x - \sin 2x}\left(\frac{\sin x}{x}\right)^{(k-1)p}(\cos x)^{p+1}. \tag{3.3}$$

A simple computation leads to $g(0^+) = 0$.

Differentiation again and simplifying give

$$g'(x) = 8\frac{(\frac{\sin x}{x})^{(k-1)p}(\cos x)^p}{x\sin x(2x - \sin 2x)^2} h(x), \tag{3.4}$$

where

$$\begin{aligned} h(x) &= \cos x(\sin x - x\cos x)^2(x - \cos x\sin x)kp \\ &\quad + (x - \cos x\sin x)^2(\sin x - x\cos x)p \\ &\quad + x\sin^2 x\bigl(-2x^2\cos x + x\sin x + \cos x\sin^2 x\bigr) \\ &= kpA(x) + pB(x) + C(x) \\ &= (kA + B)\left(p + \frac{C}{kA + B}\right), \end{aligned} \tag{3.5}$$

where $A(x)$, $B(x)$ and $C(x)$ are defined as in (2.1), (2.2) and (2.3), respectively.

By (3.2), (3.4) we easily get

$$\operatorname{sgn} f'(x) = \operatorname{sgn} p \operatorname{sgn} g(x), \tag{3.6}$$

$$\operatorname{sgn} g'(x) = \operatorname{sgn} h(x). \tag{3.7}$$

Necessity. We first present two limit relations:

$$\lim_{x\to 0^+} x^4 f(x) = \frac{kp}{36}\left(p + \frac{12}{5(k+2)}\right), \tag{3.8}$$

$$\lim_{x\to(\pi/2)^-} f(x) = \begin{cases} \infty & \text{if } p > 0, \\ \frac{2}{k+2}(\frac{2}{\pi})^{kp} - 1 & \text{if } p < 0. \end{cases} \tag{3.9}$$

In fact, using power series extension yields

$$f(x) = \frac{kp}{36}\frac{kp + 2p + 12/5}{k+2}x^4 + o\bigl(x^4\bigr),$$

which implies the first limit relation (3.8). From the fact that $\lim_{x\to\pi/2^-}\tan x = \infty$, the second one (3.9) easily follows.

Now we can derive that the necessary condition of (1.13) holds for $x \in (0, \pi/2)$ from the simultaneous inequalities $\lim_{x\to 0^+} x^4 f(x) \geq 0$ and $\lim_{x\to(\pi/2)^-} f(x) \geq 0$. Solving for p yields $p > 0$ or

$$p \leq \min\left(-\frac{12}{5(k+2)}, -\frac{\ln(k+2) - \ln 2}{k(\ln\pi - \ln 2)}\right) = -\frac{\ln(k+2) - \ln 2}{k(\ln\pi - \ln 2)},$$

where the equality holds due to Lemma 3.

Sufficiency. We prove that the condition $p > 0$ or $p \leq -\frac{\ln(k+2)-\ln 2}{k(\ln\pi-\ln 2)}$ is sufficient. We divide the proof into three cases.

Case 1 $p > 0$. Clearly, $h(x) > 0$, then $g'(x) > 0$ and $g(x) > g(0^+) = 0$, which together with $\operatorname{sgn} p = 1$ yields $f'(x) > 0$ and $f(x) > f(0^+) = 0$.

Case 2 $p \leq -1$. By Lemma 1 it is easy to get

$$p + \frac{C}{kA + B} < p + 1 \leq 0,$$

which reveals that $h(x) < 0$, $g'(x) < 0$ and $g(x) < g(0^+) = 0$, which in combination with $\operatorname{sgn} p = -1$ implies $f'(x) > 0$ and $f(x) > f(0^+) = 0$.

Case 3 $-1 < p \leq -\frac{\ln(k+2)-\ln 2}{k(\ln\pi-\ln 2)}$. Lemma 1 reveals that $\frac{C}{kA+B}$ is increasing on $(0,\pi/2)$, so is the function $x \mapsto p + \frac{C}{kA+B} := \lambda(x)$. Since

$$\lambda(0^+) = p + \frac{12}{5(k+2)} < 0, \qquad \lambda\left(\frac{\pi}{2}^-\right) = p + 1 > 0,$$

there exists $x_1 \in (0,\pi/2)$ such that $\lambda(x) < 0$ for $x \in (0,x_1)$ and $\lambda(x) > 0$ for $x \in (x_1,\pi/2)$, and so is $g'(x)$. Therefore, $g(x) < g(0^+) = 0$ for $x \in (0,x_1)$ but $g(\pi/2^-) = 1$, which implies that there exists $x_0 \in (x_1,\pi/2)$ such that $g(x) < 0$ for $x \in (0,x_0)$ and $g(x) > 0$ for $x \in (x_0,\pi/2)$. Due to $\operatorname{sgn} p = -1$, it is deduced that $f'(x) > 0$ for $x \in (0,x_0)$ and $f'(x) < 0$ for $x \in (x_0,\pi/2)$, which reveals that f is increasing on $(0,x_0)$ and decreasing on $(x_0,\pi/2)$. It follows that

$$0 = f(0^+) < f(x) < f(x_0) = 0 \quad \text{for } x \in (0,x_0),$$
$$f(x_0) > f(x) > f(\pi/2^-) = \frac{2}{k+2}\left(\frac{2}{\pi}\right)^{kp} - 1 \geq 0 \quad \text{for } x \in (x_0,\pi/2),$$

that is, $f(x) > 0$ for $x \in (0,\pi/2)$.

This completes the proof. □

Theorem 2 *For fixed $k \geq 1$, the reversed inequality of* (1.13), *that is,*

$$\frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} < 1, \tag{3.10}$$

holds for $x \in (0,\pi/2)$ if and only if $-\frac{12}{5(k+2)} \leq p < 0$.

Proof Necessity. If inequality (3.10) holds for $x \in (0,\pi/2)$, then we have

$$\lim_{x\to 0^+} \frac{f(x)}{x^4} = \frac{kp}{36}\left(p + \frac{12}{5(k+2)}\right) \leq 0.$$

Solving the inequality for p yields $-\frac{12}{5(k+2)} \leq p < 0$.

Sufficiency. We prove that the condition $-\frac{12}{5(k+2)} \leq p < 0$ is sufficient. It suffices to show that $f(x) < 0$ for $x \in (0,\pi/2)$. By Lemma 1 it is easy to get

$$p + \frac{C}{kA+B} \geq p + \frac{12}{5(k+2)} \geq 0,$$

which reveals that $h(x) > 0$, $g'(x) > 0$ and $g(x) > g(0^+) = 0$. In combination with $\operatorname{sgn} p = -1$, it implies $f'(x) < 0$. Thus, $f(x) < f(0^+) = 0$, which proves the sufficiency and the proof is completed. □

Theorem 3 *For fixed $k \geq 1$, inequality* (1.14) *holds for $x \in (0,\infty)$ if and only if $p > 0$ or $p \leq -\frac{12}{5(k+2)}$.*

Proof Let

$$u(x) = \frac{2}{k+2}\left(\frac{\sinh x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tanh x}{x}\right)^{p} - 1. \tag{3.11}$$

Then inequality (1.14) is equivalent to $u(x) > 0$. Differentiation leads to

$$u'(x) = -\frac{kp}{2(k+2)}\frac{\sinh 2x - 2x}{x^2\cosh^2 x}\left(\frac{\tanh x}{x}\right)^{p-1} v(x), \tag{3.12}$$

where

$$v(x) = 1 - 4\frac{\sinh x - x\cosh x}{2x - \sinh 2x}\left(\frac{\sinh x}{x}\right)^{kp-p}(\cosh x)^{p+1}. \tag{3.13}$$

Differentiation again gives

$$v'(x) = \frac{2\cosh^p x(\frac{\sinh x}{x})^{kp-p}}{x\sinh x(x - \cosh x\sinh x)^2} w(x), \tag{3.14}$$

where

$$\begin{aligned} w(x) &= \cosh x(\sinh x - x\cosh x)^2(x - \cosh x\sinh x)kp \\ &\quad + (\sinh x - x\cosh x)(x - \cosh x\sinh x)^2 p \\ &\quad + x\sinh^2 x\bigl(2x^2\cosh x - x\sinh x - \cosh x\sinh^2 x\bigr) \\ &= kpE(x) + pF(x) + G(x) = (kE + F)\left(p + \frac{G}{kE+F}\right), \end{aligned} \tag{3.15}$$

where $E(x)$, $F(x)$ and $G(x)$ are defined as in (2.5), (2.6) and (2.7), respectively.

By (3.12) and (3.14) we easily get

$$\operatorname{sgn} u'(x) = -\operatorname{sgn}\frac{k}{k+2}\operatorname{sgn} p\ \operatorname{sgn} v(x), \tag{3.16}$$

$$\operatorname{sgn} v'(x) = \operatorname{sgn} w(x). \tag{3.17}$$

Necessity. If inequality (1.14) holds for $x \in (0,\infty)$, then we have $\lim_{x\to 0^+} x^{-4}u(x) \geq 0$. Expanding $u(x)$ in power series gives

$$u(x) = \frac{k}{36}p\left(p + \frac{12}{5p(k+2)}\right)x^4 + o\bigl(x^4\bigr).$$

Hence we get

$$\lim_{x\to 0^+} x^{-4}u(x) = \frac{k}{36}p\left(p + \frac{12}{5(k+2)}\right) \geq 0.$$

Solving the inequality for p yields $p > 0$ or $p \leq -\frac{12}{5(k+2)}$.

Sufficiency. We prove that the condition $p > 0$ or $p \leq -\frac{12}{5(k+2)}$ is sufficient for (1.14) to hold.

If $p > 0$, then $w(x) < 0$ due to $E, F, G < 0$. Hence, from (3.17) we have $v'(x) < 0$ and $v(x) < \lim_{x\to 0^+} v(x) = 0$. It is derived by (3.16) that $u'(x) > 0$, and so $u(x) > \lim_{x\to 0^+} u(x) = 0$.

If $p \leq -\frac{12}{5(k+2)}$, then by Lemma 2 we have

$$p + \frac{G}{kE + F} \leq -\frac{12}{5(k+2)} + \frac{G}{kE + F} < 0$$

and

$$w(x) = (kE + F)\left(p + \frac{G}{kE + F}\right) > 0.$$

From (3.17) we have $v'(x) > 0$ and $v(x) > \lim_{x \to 0^+} v(x) = 0$. It follows by (3.16) that $u'(x) > 0$, which implies that $u(x) > \lim_{x \to 0^+} u(x) = 0$.

This completes the proof. □

Remark 2 For $k \geq 1$, since $\lim_{x \to \infty} u(x) = \infty$ for $p \neq 0$ and $\lim_{x \to \infty} u(x) = 0$ for $p = 0$, there does not exist p such that the reverse inequality of (1.14) holds for all $x > 0$. But we can show that there exists $x_0 \in (0, \infty)$ such that $u(x) < 0$, that is, the reverse inequality of (1.14) holds for $-\frac{12}{5(k+2)} < p < 0$. The details of the proof are omitted.

Theorem 4 *For fixed $k < -2$, the reverse of* (1.14), *that is,*

$$\frac{2}{k+2}\left(\frac{\sinh x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tanh x}{x}\right)^{p} < 1, \tag{3.18}$$

holds for $x \in (0, \infty)$ if and only if $p < 0$ or $p \geq -\frac{12}{5(k+2)}$.

Proof Necessity. If inequality (3.18) holds for $x \in (0, \infty)$, then we have

$$\lim_{x \to 0^+} \frac{u(x)}{x^4} = \frac{k}{36} p \left(p + \frac{12}{5(k+2)}\right) \leq 0.$$

Solving the inequality for p yields $p < 0$ or $p \geq -\frac{12}{5(k+2)}$.

Sufficiency. We prove that the condition $p < 0$ or $p \geq -\frac{12}{5(k+2)}$ is sufficient for (3.18) to hold.

If $p < 0$, then $w(x) = (kE + F)(p + \frac{G}{kE+F}) < 0$ due to $kE + F > 0$ and $G < 0$. Hence, from (3.17) we have $v'(x) < 0$ and $v(x) < \lim_{x \to 0^+} v(x) = 0$. It is derived by (3.16) that $u'(x) < 0$, and so $u(x) < \lim_{x \to 0^+} u(x) = 0$.

If $p \geq -\frac{12}{5(k+2)}$, then by Lemma 2 we have

$$p + \frac{G}{kE + F} \geq p + \frac{12}{5(k+2)} > 0$$

and

$$w(x) = (kE + F)\left(p + \frac{G}{kE + F}\right) > 0.$$

From (3.17) we have $v'(x) > 0$ and $v(x) > \lim_{x \to 0^+} v(x) = 0$. It follows by (3.16) that $u'(x) < 0$, which implies that $u(x) < \lim_{x \to 0^+} u(x) = 0$.

This completes the proof. □

4 Applications

4.1 Huygens-type inequalities

Letting $k = 1$ in Theorems 1 and 2, we have the following proposition.

Proposition 1 *For $x \in (0, \pi/2)$, the double inequality*

$$\frac{2}{3}\left(\frac{\sin x}{x}\right)^p + \frac{1}{3}\left(\frac{\tan x}{x}\right)^p > 1 > \frac{2}{3}\left(\frac{\sin x}{x}\right)^q + \frac{1}{3}\left(\frac{\tan x}{x}\right)^q \tag{4.1}$$

holds if and only if $p > 0$ or $p \le -\frac{\ln 3 - \ln 2}{\ln \pi - \ln 2} \approx -0.898$ and $-4/5 \le q < 0$.

Let $M_r(a, b; w)$ denote the rth weighted power mean of positive numbers $a, b > 0$ defined by

$$M_r(a, b; w) := \left(wa^r + (1 - w)b^r\right)^{1/r} \quad \text{if } r \neq 0 \text{ and } M_0(a, b; w) = a^w b^{1-w}, \tag{4.2}$$

where $w \in (0, 1)$.

Since

$$\frac{2}{3}\left(\frac{\sin x}{x}\right)^p + \frac{1}{3}\left(\frac{\tan x}{x}\right)^p = \frac{\frac{2}{3} + \frac{1}{3}(\cos x)^{-p}}{(\frac{\sin x}{x})^{-p}},$$

by Proposition 1 the inequality

$$\frac{\sin x}{x} > \left(\frac{2}{3} + \frac{1}{3}(\cos x)^{-p}\right)^{-1/p} = M_{-p}\left(1, \cos x; \frac{2}{3}\right)$$

holds for $x \in (0, \pi/2)$ if and only if $-p \le 4/5$. Similarly, its reversed inequality holds if and only if $-p \ge \frac{\ln 3 - \ln 2}{\ln \pi - \ln 2}$. The facts can be stated as a corollary.

Corollary 1 *Let $M_r(a, b; w)$ be defined by* (4.2). *Then, for $x \in (0, \pi/2)$, the inequalities*

$$M_\alpha\left(1, \cos x; \frac{2}{3}\right) < \frac{\sin x}{x} < M_\beta\left(1, \cos x; \frac{2}{3}\right) \tag{4.3}$$

hold if and only if $\alpha \le 4/5$ and $\beta \ge \frac{\ln 3 - \ln 2}{\ln \pi - \ln 2} \approx -0.898$.

Remark 3 The Cusa-Huygens inequality [20] refers to

$$\frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3}\cos x \tag{4.4}$$

holds for $x \in (0, \pi/2)$, which is equivalent to the second inequality in (1.7). As an improvement and generalization, Corollary 1 was proved in [22] by Yang. Here we provide a new proof.

Remark 4 Let $a > b > 0$ and let $x = \arcsin \frac{a-b}{a+b} \in (0, \pi/2)$. Then $\sin x / x = P/A$, $\cos x = G/A$ and inequalities (4.3) can be rewritten as

$$M_\alpha\left(A, G; \frac{2}{3}\right) < P < M_\beta\left(A, G; \frac{2}{3}\right), \tag{4.5}$$

where P is the first Seiffert mean [23] defined by

$$P = P(a,b) = \frac{a-b}{2\arcsin\frac{a-b}{a+b}},$$

A and G denote the arithmetic and geometric means of a and b, respectively.

Let $x = \arctan\frac{a-b}{a+b}$. Then $\sin x/x = T/Q$, $\cos x = A/Q$, and inequalities (4.3) can be rewritten as

$$M_\alpha\left(Q,A;\frac{2}{3}\right) < T < M_\beta\left(Q,A;\frac{2}{3}\right), \tag{4.6}$$

where T is the second Seiffert mean [24] defined by

$$T = T(a,b) = \frac{a-b}{2\arctan\frac{a-b}{a+b}},$$

Q denotes the quadratic mean of a and b.

Obviously, by Corollary 2, the two double inequalities (4.5) (see [22]) and (4.6) hold if and only if $\alpha \le 4/5$ and $\beta \ge \frac{\ln 3-\ln 2}{\ln\pi-\ln 2} \approx -0.898$, (4.6) seems to be a new inequality.

In the same way, taking $k = 1$ in Theorem 3, we get the following.

Proposition 2 *For $x \in (0,\infty)$, the inequality*

$$\frac{2}{3}\left(\frac{\sinh x}{x}\right)^p + \frac{1}{3}\left(\frac{\tanh x}{x}\right)^p > 1 \tag{4.7}$$

holds if and only if $p > 0$ or $p \le -\frac{4}{5}$.

Similar to Corollary 1, we have the following.

Corollary 2 *Let $M_r(a,b;w)$ be defined by* (4.2). *Then, for $x \in (0,\infty)$, the inequalities*

$$M_\alpha\left(1,\cosh x;\frac{2}{3}\right) < \frac{\sinh x}{x} < M_\beta\left(1,\cosh x;\frac{2}{3}\right) \tag{4.8}$$

hold if and only if $\alpha \le 0$ and $\beta \ge 4/5$.

Remark 5 Let $a > b > 0$ and $x = \ln\sqrt{a/b}$. Then $\sinh x/x = L/G$, $\cosh x = A/G$, and (4.8) can be rewritten as

$$M_\alpha\left(G,A;\frac{2}{3}\right) < L < M_\beta\left(G,A;\frac{2}{3}\right), \tag{4.9}$$

where L is the logarithmic means of a and b defined by

$$L = L(a,b) = \frac{a-b}{\ln a - \ln b}.$$

Making use of $x = \operatorname{arcsinh}\frac{b-a}{a+b}$ yields $\sinh x/x = NS/A$ and $\cosh x = Q/A$, where NS is the Nueman-Sándor mean defined by

$$NS = NS(a,b) = \frac{a-b}{2\operatorname{arcsinh}\frac{a-b}{a+b}}.$$

Thus, (4.8) is equivalent to

$$M_\alpha\left(A, Q; \frac{2}{3}\right) < NS < M_\beta\left(A, Q; \frac{2}{3}\right). \tag{4.10}$$

Corollary 2 implies that inequalities (4.9) and (4.10) hold if and only if $\alpha \le 0$ and $\beta \ge 4/5$. The second inequality in (4.10) is a new inequality.

Remark 6 It should be pointed out that all inequalities involving $\sin x/x$ and $\cos x$ or $\sinh x/x$ and $\cosh x$ in this paper can be rewritten as the equivalent inequalities for bivariate means mentioned previously. In what follows we no longer mention this.

4.2 Wilker-Zhu-type inequalities

Letting $k = 2$ in Theorems 1 and 2, we have the following.

Proposition 3 *For $x \in (0, \pi/2)$, the double inequality*

$$\left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^{p} > 2 > \left(\frac{\sin x}{x}\right)^{2q} + \left(\frac{\tan x}{x}\right)^{q} \tag{4.11}$$

holds if and only if $p > 0$ or $p \le -\frac{\ln 2}{2(\ln\pi - \ln 2)} \approx -0.767$ and $-3/5 \le q < 0$.

Note that

$$\frac{(\frac{\sin x}{x})^{2p} + (\frac{\tan x}{x})^{p} - 2}{(\frac{\sin x}{x})^{p} + \frac{\sqrt{8+\cos^{-2p}x} + \cos^{-p}x}{2}} = \left(\frac{x}{\sin x}\right)^{-p} - \frac{\sqrt{8+\cos^{-2p}x} - \cos^{-p}x}{2}.$$

By Proposition 3 the inequality

$$\frac{x}{\sin x} > \left(\frac{\sqrt{8+\cos^{-2p}x} - \cos^{-p}x}{2}\right)^{-1/p}$$

or

$$\frac{\sin x}{x} < \left(\frac{\sqrt{8+\cos^{-2p}x} + \cos^{-p}x}{4}\right)^{-1/p} := H_{-p}(\cos x)$$

holds for $x \in (0, \pi/2)$ if and only if $-p \ge \frac{\ln 2}{2(\ln\pi - \ln 2)}$, where H_r is defined on $(0, \infty)$ by

$$H_r(t) = \left(\frac{\sqrt{8+t^{2r}} + t^r}{4}\right)^{1/r} \quad \text{if } r \ne 0 \text{ and } H_0(t) = \sqrt[3]{t}. \tag{4.12}$$

Likewise, its reversed inequality holds if and only if $-p \le 3/5$. This result can be stated as a corollary.

Corollary 3 *Let $H_r(t)$ be defined by* (4.12). *Then, for $x \in (0, \pi/2)$, the inequalities*

$$H_\alpha(\cos x) < \frac{\sin x}{x} < H_\beta(\cos x) \tag{4.13}$$

are true if and only if $\alpha \le 3/5$ and $\beta \ge \frac{\ln 2}{2(\ln \pi - \ln 2)} \approx 0.767$.

Taking $k = 2$ in Theorem 3, we have the following.

Proposition 4 *For $x \in (0, \infty)$, the inequality*

$$\left(\frac{\sinh x}{x}\right)^{2p} + \left(\frac{\tanh x}{x}\right)^{p} > 2$$

holds if and only if $p > 0$ or $p \le -3/5$.

In a similar way, we get Corollary 4.

Corollary 4 *Let $H_r(t)$ be defined by* (4.12). *Then, for $x \in (0, \infty)$, the inequalities*

$$H_\alpha(\cosh x) < \frac{\sinh x}{x} < H_\beta(\cosh x) \tag{4.14}$$

are true if and only if $\alpha \le 0$ and $\beta \ge 3/5$.

Now we give a generalization of inequalities (1.4) given by Zhu [15].

Proposition 5 *For fixed $k \ge 1$, both chains of inequalities*

$$\begin{aligned} \frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} &\ge \frac{k}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{2}{k+2}\left(\frac{\tan x}{x}\right)^{p} \\ &> \frac{2}{k+2}\left(\frac{x}{\sin x}\right)^{kp} + \frac{k}{k+2}\left(\frac{x}{\tan x}\right)^{p} > 1, \end{aligned} \tag{4.15}$$

$$\begin{aligned} \frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} &> \frac{2}{k+2}\left(\frac{x}{\tan x}\right)^{p} + \frac{k}{k+2}\left(\frac{x}{\sin x}\right)^{kp} \\ &\ge \frac{2}{k+2}\left(\frac{x}{\sin x}\right)^{kp} + \frac{k}{k+2}\left(\frac{x}{\tan x}\right)^{p} > 1 \end{aligned} \tag{4.16}$$

hold for $x \in (0, \pi/2)$ if and only if $k \ge 2$ and $p \ge \frac{\ln(k+2) - \ln 2}{k(\ln \pi - \ln 2)}$.

Proof The first inequality in (4.15) is equivalent to

$$\begin{aligned} &\frac{2}{k+2}\left(\frac{\sin x}{x}\right)^{kp} + \frac{k}{k+2}\left(\frac{\tan x}{x}\right)^{p} - \frac{k}{k+2}\left(\frac{\sin x}{x}\right)^{kp} - \frac{2}{k+2}\left(\frac{\tan x}{x}\right)^{p} \\ &\quad = \frac{k-2}{k+2}\left(\left(\frac{\tan x}{x}\right)^{p} - \left(\frac{\sin x}{x}\right)^{kp}\right) > 0. \end{aligned}$$

Due to $\frac{\tan x}{x} > 1$ and $\frac{\sin x}{x} < 1$, it holds for $x \in (0, \pi/2)$ if and only if

$$(k, p) \in \{k \ge 2, p > 0\} \cup \{1 \le k \le 2, p < 0\} := \Omega_1.$$

The second one is equivalent to

$$\frac{\frac{k}{k+2}(\frac{\sin x}{x})^{kp}+\frac{2}{k+2}(\frac{\tan x}{x})^{p}}{\frac{2}{k+2}(\frac{x}{\sin x})^{kp}+\frac{k}{k+2}(\frac{x}{\tan x})^{p}}>1,$$

which can be simplified to

$$\left(\frac{\sin x}{x}\right)^{kp}\left(\frac{\tan x}{x}\right)^{p}=\left(\left(\frac{\sin x}{x}\right)^{k+1}\frac{1}{\cos x}\right)^{p}>1.$$

It is true for $x\in(0,\pi/2)$ if and only if $(k,p)\in\{k+1\geq 3,p\geq 0\}:=\Omega_2$.

By Theorem 1, the third one in (4.15) holds for $x\in(0,\pi/2)$ if and only if

$$(k,p)\in\{k\geq 1,-p>0\}\cup\left\{k\geq 1,-p\leq-\frac{\ln(k+2)-\ln 2}{k(\ln\pi-\ln 2)}\right\}:=\Omega_3.$$

Hence, inequalities (4.15) hold for $x\in(0,\pi/2)$ if and only if

$$(k,p)\in\Omega_1\cap\Omega_2\cap\Omega_3=\left\{k\geq 2,p\geq\frac{\ln(k+2)-\ln 2}{k(\ln\pi-\ln 2)}\right\},$$

which proves (4.15).

In the same way, we can prove (4.16), the details are omitted. □

Letting $k=2$ in Proposition 5, we have the following.

Corollary 5 *For $x\in(0,\pi/2)$, inequality* (1.4) *holds if and only if $p\geq\frac{\ln 2}{2(\ln\pi-\ln 2)}\approx 0.767$.*

Similarly, using Theorem 3 we easily prove the following proposition.

Proposition 6 *For fixed $k\geq 1$, the inequalities*

$$\frac{k}{k+2}\left(\frac{\sinh x}{x}\right)^{kp}+\frac{2}{k+2}\left(\frac{\tanh x}{x}\right)^{p}>\frac{2}{k+2}\left(\frac{x}{\sinh x}\right)^{kp}+\frac{k}{k+2}\left(\frac{x}{\tanh x}\right)^{p}>1 \quad (4.17)$$

hold for $x\in(0,\infty)$ if and only if $k\geq 2$ and $p\geq\frac{12}{5(k+2)}$.

Letting $k=2$ in Proposition 6, we have the following.

Corollary 6 *For $x\in(0,\infty)$, inequality* (1.5) *holds if and only if $p\geq 3/5$.*

Remark 7 Clearly, Corollaries 5 and 6 offer another method for solving the problems posed by Zhu in [16].

4.3 Other Wilker-type inequalities

Taking $k=3,4$ in Theorems 1 and 2, we obtain the following.

Proposition 7 *For $x\in(0,\pi/2)$, the inequality*

$$\frac{2}{5}\left(\frac{\sin x}{x}\right)^{3p}+\frac{3}{5}\left(\frac{\tan x}{x}\right)^{p}>1 \quad (4.18)$$

holds if and only if $p > 0$ *or* $p \leq -\frac{\ln 5 - \ln 2}{3(\ln \pi - \ln 2)} \approx -0.676$. *It is reversed if and only if* $-12/25 \leq p < 0$.

Proposition 8 *For* $x \in (0, \pi/2)$, *the inequality*

$$\frac{1}{3}\left(\frac{\sin x}{x}\right)^{4p} + \frac{2}{3}\left(\frac{\tan x}{x}\right)^{p} > 1 \tag{4.19}$$

holds if and only if $p > 0$ *or* $p \leq -\frac{\ln 3}{4(\ln \pi - \ln 2)} \approx -0.608$. *It is reversed if and only if* $-2/5 \leq p < 0$.

Putting $k = -3, -4$ in Theorem 3, we get the following.

Proposition 9 *For* $x \in (0, \infty)$, *the inequality*

$$\left(\frac{\tanh x}{x}\right)^{p} < \frac{2}{3}\left(\frac{x}{\sinh x}\right)^{3p} + \frac{1}{3} \tag{4.20}$$

holds if and only if $p < 0$ *or* $p \geq 12/5$.

Proposition 10 *For* $x \in (0, \pi/2)$, *the inequality*

$$2\left(\frac{\tanh x}{x}\right)^{p} < \left(\frac{x}{\sinh x}\right)^{4p} + 1 \tag{4.21}$$

holds if and only if $p < 0$ *or* $p \geq 6/5$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Z-HY carried out the proof of the Wilker-type inequality and drafted the manuscript. Y-MC provided the main idea and carried out the proof of the hyperbolic version of Wilker-type inequality. All authors read and approved the final manuscript.

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