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An application of nonstationary wavelets

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Abstract

Using nonstationary wavelets, we investigate the wavelet expansion in the standard Besov spaces. Especially, the nonstationary wavelets' characterization for Besov spaces is given.

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1 Introduction

In this paper, we shall study the nonstationary wavelet [1] expansion in the standard Besov spaces. Nonstationary wavelet systems are generally obtained from a sequence of nonstationary refinable functions.

Definition 1.1 A sequence of functions $\{\phi_{j-1}\}_{j\in N}$ in $L_2(R)$ is said to consist of nonstationary refinable functions if, for all $j \in N$,

$$\widehat{\phi}_{j-1}(\xi) = \widehat{a}_j\left(\frac{\xi}{2}\right)\widehat{\phi}_j\left(\frac{\xi}{2}\right), \quad \text{a.e. } \xi \in R,$$

where $\widehat{a}_j$ are 2π-periodic measurable functions, called refinement masks, or simply masks.

The classical Fourier transform is defined by $\widehat{f}(\xi) := \int_{R^n} f(x)e^{-ix\xi}\,dx$ for $f \in L_1$. The standard extension can be made to L_2 functions. Wavelet functions ψ^l_{j-1}, $j \in N$ and $l = 1, 2, \ldots, Z_j$ (quite often $Z_j = 3$), are generally obtained from nonstationary refinable functions by

$$\widehat{\psi}^l_{j-1}(\xi) = \widehat{b}^l_j\left(\frac{\xi}{2}\right)\widehat{\phi}_j\left(\frac{\xi}{2}\right), \quad j \in N, l = 1, 2, \ldots, Z_j,$$

where $\widehat{b}^l_j(\xi)$ are 2π-periodic measurable functions called *wavelet masks*. The masks $\widehat{a}_j$ and $\widehat{b}^l_j$ satisfy $|\widehat{a}_j(\xi)|^2 + \sum_{l=1}^{Z_j}|\widehat{b}^l_j(\xi)|^2 = 1$. When $Z_j = 3$, let $\widehat{b}^1_j(\xi) := e^{-i\xi}\overline{\widehat{a}_j(\xi+\pi)}$, $\widehat{b}^2_j(\xi) := 2^{-1}A_j(\xi) + e^{-i\xi}\overline{A_j(\xi)}$ and $\widehat{b}^3_j(\xi) := 2^{-1}A_j(\xi) - e^{-i\xi}\overline{A_j(\xi)}$, where $A_j(\xi) := 1 - |\widehat{a}_j(\xi)|^2 - |\widehat{a}_j(\xi+\pi)|^2$. Define $X(\phi_0; \{\psi^l_j\}_{j\in N_0, l=1,2,\ldots,Z_j}) := \{\phi_0(\cdot - k) : k \in Z\} \cup \{\psi^l_{j;j,k} : j \in N_0, l = 1, 2, \ldots, Z_j\}$. Then the following theorem holds.

Theorem 1.1 (i) (*Theorem* 1.3, [1]). *Let* $\widehat{a}_j(\xi) := \widehat{a}^I_{m_j,l_j}$ (*mask for the pseudo-spline of type I with order* (m_j, l_j)). *Then* $X(\phi_0; \{\psi^l_j\}_{j\in N_0, l=1,2,3})$ *is a compactly supported and* C^∞ *wavelet frame in* $H^s_2(R)$ *for arbitrary* $s > 0$;

(ii) (*Theorem* 1.4, [2]). *When* $\widehat{a}_j := \widehat{a}^I_{m_j,m_j}$, *then* $X(\phi_0; \{\psi_j\}_{j\in N_0})$ $(Z_j = 1)$ *is a compactly supported orthonormal basis in* $L_2(R)$, *and* ψ_j $(Z_j = 1)$ *has* m_{j+1} *vanishing moments. More precisely,* $\operatorname{Supp}\phi_j \subset [-L, L]$ *and* $\operatorname{Supp}\psi_j \subset [-L, L]$ *with uniform constant* $L > 0$.

In this paper, we use $H^s_2(R)$ to denote the classical L_2-Sobolev spaces with the smoothness parameter s. It is well known that Besov spaces contain a large number of fundamental spaces, such as Sobolev spaces, Hölder spaces, Lipschitz spaces etc. [3, 4]. They are frequently used in certain PDEs as the solution spaces. To extend the result of (i) in Theorem 1.1, we shall characterize Besov spaces by using nonstationary wavelets in this paper. It should be pointed out that Bittner and Urban [5] study the following standard Besov spaces: Let $0 < p, q \le \infty$, $s > 0$, and let $[s]$ stand for the largest integer less than or equal to s,

$$B^s_{p,q}(R^n) := \{f \in L_p(R^n) : |f|_{B^s_{p,q}(R^n)} < \infty\}.$$

Here, $|f|_{B^s_{p,q}(R^n)} := \|(2^{js}\omega^M_p(f, 2^{-j}))_{j\in Z}\|_{\ell_q}$ with $M \ge [s] + 1$ and $\omega^M_p(f, 2^{-j})$ denotes the Mth order smooth modulus of a function f, defined by $\sup_{|h|\le 2^{-j}} \|\Delta^M_h f(\cdot)\|_{L_p(R^n)}$ as usual. The classical difference operator Δ_h is defined by $\Delta_h f(\cdot) := f(\cdot + h) - f(\cdot)$, as well as $\Delta^M_h f = \Delta_h(\Delta^{M-1}_h f)$ for a positive integer $M > 1$. The Besov (quasi-)norm is given by $\|f\|_{B^s_{p,q}(R^n)} := \|f\|_{L_p(R^n)} + |f|_{B^s_{p,q}(R^n)}$ and the two integers $M, M' > s$ yield equivalent norms ([6], Remark 3.2.2).

Based on Hermite multiwavelets, Bittner and Urban characterize $B^s_{p,q}$ by using sequence norms,

$$\|a\|_{\ell_p} := \|(a_{j_0,k})_{k\in Z^n}\|_{\ell_p}, \qquad \|b\|_{\ell^s_{p,q}} := \|(2^{j(s+\frac{n}{2}-\frac{n}{p})}\|(b_{j,k})_{k\in Z^n}\|_{\ell_p})_{j\ge j_0}\|_{\ell_q}$$

for $a = (a_k)_{k\in Z^n} \in \ell_p$, $b = (b_{j,k})_{j\ge j_0,k\in Z^n} \in \ell^s_{p,q}$. However, due to the regularity restrictions of the Hermite splines, their characterization requires $\frac{1}{p} < s < \min\{3, 1 + \frac{1}{p}\}$ in the quadratic case and $1 + \frac{1}{p} < s < \min\{4, 2 + \frac{1}{p}\}$ in the cubic one (*e.g.* [5, 7, 8]). In [9], we remove that restriction of s by using the B-spline wavelets with weak duals as introduced in [10], but the supports of the wavelets become larger as s increases. So, the main result of this paper is to characterize Besov spaces via nonstationary wavelets because of their arbitrary smoothness and uniform support.

Let N, Z, and R be the set of positive integers, the set of integers, and the set of real numbers, respectively, as well as $N_0 := N \cup \{0\}$. Throughout this paper, we use $A \preceq B$ to abbreviate that A is bounded by a constant multiple of B, $A \succeq B$ is defined as $B \preceq A$ and $A \sim B$ means $A \preceq B$ and $B \preceq A$. Write

$$\langle f, g\rangle = \int_\Omega f(t)\overline{g(t)}\, dt$$

for $f \in L_p(\Omega)$, $g \in L_{p'}(\Omega)$ with Lebesgue measurable set $\Omega \subseteq R^n$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $1 \le p \le \infty$. For a Lebesgue measurable function f, the support of f means the set $\operatorname{Supp}(f) := \{x \in R : f(x) \ne 0\}$, which is well defined up to a set of measure 0. Define $f_{j,k}(\cdot) := 2^{\frac{j}{2}} f(2^j \cdot -k)$ throughout this paper.

Now, we state the Main Theorem of this paper.

Main Theorem *Let* $1 \le p \le 2$, $0 < q \le \infty$; ϕ_j *and* ψ_j *are from* (ii) *of Theorem* 1.1. *When* $f = \sum_k c_{0,k}\phi_{0;0,k} + \sum_{j=0}^{\infty}\sum_{k\in Z} d_{j,k}\psi_{j;j,k}$, *then*

$$\|f\|_{B^s_{p,q}(R)} \preceq \|c\|_{l_p} + \|d\|_{l^s_{p,q}}$$

for $s > 0$. *Moreover, when* $m = \inf_j m_{j+1}$, $c_{0,k} := \langle f, \phi_{0;0,k}\rangle$, $d_{j,k} := \langle f, \psi_{j;j,k}\rangle$ *with* $j \in N_0$ *and* $k \in Z$, *then*

$$\|c\|_{l_p} + \|d\|_{l^s_{p,q}} \preceq \|f\|_{B^s_{p,q}(R)}$$

for $f \in B^s_{p,q}(R)$, $0 < s < m$.

2 Proof of Main Theorem

This section is devoted to proving the Main Theorem. We begin with three lemmas for proving upper and lower bounds of the characterization.

Lemma 2.1 *Let* $p \in (\frac{2}{3}, 2]$, $q \in (0,\infty]$, $s > 0$ *be arbitrary, then* $\|\phi_0\|_{B^s_{p,q}} \le C$ *and* $\|\psi_j\|_{B^s_{p,q}} \le C$ *with a uniform constant* $C > 0$ *for* $j \in N_0$.

Proof First, we will show $\|\phi_j\|_p \le C$ with a uniform constant C for all j. By Lemma 2.1, Theorem 2.8 in [2], and Theorem 2.1 in [11], ϕ_j are all compactly supported and $\operatorname{Supp}\phi_j \subset [-L, L]$ for a uniform constant $L > 0$. Therefore, $\operatorname{Supp}\psi_j \subset [-L, L]$ for all j because of $\widehat{\psi}_{j-1}(\xi) := \widehat{b^1_j}(\frac{\xi}{2})\widehat{\phi_j}(\frac{\xi}{2}) = e^{-i\xi}\overline{\widehat{a_j}(\xi+\pi)}\widehat{\phi_j}(\frac{\xi}{2})$. Note that $\|\phi_j\|_{L_2} \le 1$ (Lemma 2.2 in [2]). This with

$$\begin{aligned}
&\left[\int |\phi_j(x)|^p\,dx\right]^{\frac{1}{p}} \\
&\quad \le \left\{\left[\int \left(\frac{1}{(1+x^2)^p}\right)^{\frac{2}{2-p}}dx\right]\left[\int \left((1+x^2)^p|\phi_j(x)|^p\right)^{\frac{2}{p}}dx\right]^{\frac{p}{2}}\right\}^{\frac{1}{p}} \\
&\quad \le C_{p,2}\|\phi_j\|_{L_2}
\end{aligned} \tag{2.1}$$

for $\frac{2}{3} < p \le 2$ leads to $\|\phi_j\|_{L_p} \le C$ for all $j \in N_0$. $\|\psi_j\|_{L_p} \le C$ holds similarly.

Second, let $[s] + 1 = M$, by (2.1),

$$\begin{aligned}
&\|\phi_0\|_{B^s_{p,q}} \\
&\quad = \|\phi_0\|_p + \left\|\left(2^{ls}\omega^M_p(\phi_0, 2^{-l})\right)_{l\in Z}\right\|_{l_q} \\
&\quad \preceq \|\phi_0\|_p + \left\|2^{l(s-M)}\right\|_{l_q}\left\|\phi_0^{(M)}\right\|_p \\
&\quad \preceq \|\phi_0\|_p + \left\|\phi_0^{(M)}\right\|_2 \le \|\phi_0\|_{H^M_2}.
\end{aligned}$$

Note that

$$\|\phi_0\|^2_{H^M_2} \preceq \sum_k \left|\langle\phi_0, \phi_{0;0,k}\rangle\right|^2 + \sum_{j=0}^{\infty}\sum_k 2^{2js}\left|\langle\phi_0, \psi_{j;j,k}\rangle\right|^2 = 1$$

because of Corollary 3.3 in [1] and the orthonormality property of $\{\phi_{0;0,k}\}_k$, $\{\psi_{j;j,k}\rangle\}_{j,k}$. This with (2.1) leads to

$$\|\phi_0\|_{B^s_{p,q}} \le C,$$

as well as $\|\psi_j\|_{B^s_{p,q}} \le C$ for $j \in N_0$ with a uniform constant $C > 0$. Thus, the result holds. □

The second lemma comes from [5], Lemma 3.4 and the third one comes from [6], (3.2.26).

Lemma 2.2 *Suppose $\varphi \in B^\sigma_{p,\infty}(R^n)$ is compactly supported with $0 < p, q \le \infty$ and $0 < s < \sigma$. Then*

$$\Big\|\sum_{k\in Z^n} c_k\varphi_{j,k}(\cdot)\Big\|_{B^s_{p,q}(R^n)} \preceq 2^{(\sigma+\frac{n}{2}-\frac{n}{p})j}\|c\|_{\ell_p}, \qquad \Big\|\sum_{j=j_0}^{\infty}\sum_{k\in Z^n} d_{j,k}\varphi_{j,k}(\cdot)\Big\|_{B^s_{p,q}(R^n)} \preceq \|d\|_{\ell^s_{p,q}}.$$

Note that the constants are uniform because of $\|\varphi\|_{B^\sigma_{p,\infty}(R^n)} \le C$.

Lemma 2.3 *If $f \in L_p(R)$, $1 \le p \le \infty$, define σ as a closed interval of R and let P_{m-1} be the set of m-order polynomials, then*

$$\inf_{P\in\Pi_{m-1}} \|f - P\|_{L_p(\sigma_{j,k})} \preceq \omega^m_p\big(f, 2^{-j}, \sigma_{j,k}\big),$$

where $\sigma_{j,k} := 2^{-j}(\sigma + k)$ and $\omega^m_p(f, 2^{-j}, \sigma_{j,k}) := \sup_{|h|\le 2^{-j}} \|\Delta^M_h f(\cdot)\|_{L_p((\sigma_{j,k})_{h,M})}$ with $(\sigma_{j,k})_{h,M} := \{x \in \sigma_{j,k}, x + lh \in \sigma, l = 1, 2, \dots, M\}$.

Now, we are in the position to show the Main Theorem.

By Lemma 2.1, $\|\phi_0\|_{B^s_{p,q}} \le C$ and $\|\psi_j\|_{B^s_{p,q}} \le C$. This, with Lemma 2.2, shows that

$$\begin{aligned}
&\Big\|\sum_k c_{0,k}\phi_{0;0,k} + \sum_{j=0}^{\infty}\sum_{k\in Z} d_{j,k}\psi_{j;j,k}\Big\|_{B^s_{p,q}(R)} \\
&\quad\preceq \Big\|\sum_k c_{0,k}\phi_{0;0,k}\Big\|_{B^s_{p,q}(R)} + \Big\|\sum_{j=0}^{\infty}\sum_{k\in Z} d_{j,k}\psi_{j;j,k}\Big\|_{B^s_{p,q}(R)} \\
&\quad\preceq \|c\|_{\ell_p} + \|d\|_{\ell^s_{p,q}}.
\end{aligned}$$

To prove the lower bound, one finds that $|\langle f, \phi_{0;0,k}\rangle| \le \|f\|_{L_p(\sigma_{0,k})}$,

$$\sum_k \big|\langle f, \phi_{0;0,k}\rangle\big|^p \preceq \sum_k \int_{\sigma_{0,k}} \big|f(x)\big|^p\,dx \preceq \|f\|^p_p,$$

where $\sigma_{0,k} := \operatorname{Supp}\phi_{0;0,k}$. Let $\sigma_{j,k} := \operatorname{Supp}\psi_{j;j,k}$. This with $m := \inf_j m_{j+1}$ shows that

$$\begin{aligned}
&\big|\langle f, \psi_{j;j,k}\rangle\big|^p \\
&\quad= \inf_{P\in P_{m-1}} \big|\langle f - P, \psi_{j;j,k}\rangle\big|^p
\end{aligned}$$

$$\begin{aligned}&\preceq \inf_{P\in P_{m-1}} \|f-P\|^p_{L_p(\sigma_{j,k})} 2^{\frac{jp}{2}-\frac{jp}{p'}} \\ &\leq 2^{\frac{jp}{2}-\frac{jp}{p'}} \omega_p^m\big(f,2^{-j},\sigma_{j,k}\big),\end{aligned}$$

where the equality comes from the m_{j+1} vanishing moments of ψ_j by Theorem 1.1 and the second inequality holds due to Lemma 2.3. Then $\|(d_{j,k})_k\|_{l_p} \preceq 2^{\frac{j}{2}-\frac{j}{p'}} \omega_p^m(f,2^{-j})$ by the same proof as of (3.3) in [8]. Therefore,

$$\|d\|_{l^s_{p,q}} \preceq \big\| \big(2^{s+\frac{j}{2}-\frac{j}{p'}} 2^{\frac{j}{2}-\frac{j}{p'}} \omega_p^m\big(f,2^{-j}\big)\big)_{j\in Z}\big\|_{l_q} \leq \|f\|_{B^s_{p,q}(R)}.$$

Remark 2.1 In conclusion, we have a characterization of Besov spaces by

$$\|f\|_{B^s_{p,q}} \sim \|c\|_{l_p} + \|d\|_{l^s_{p,q}}$$

with $c_{0,k} := \langle f,\phi_{0;0,k}\rangle$, $d_{j,k} := \langle f,\psi_{j;j,k}\rangle$, $1 \leq p \leq 2$, $0 < q \leq \infty$, $0 < s < m$, and $f \in B^s_{p,q}(R)$.

Remark 2.2 Some questions are left to be considered. Note that we assume $1 \leq p \leq 2$ in our Main Theorem. Then a natural question is to study the case for $p > 2$. Another one is to discuss whether or not the wavelet frames of (i) in Theorem 1.1 can characterize Besov spaces. The last question is to relax the restriction $s \in (0,m)$.

Competing interests

The author declares that he has no competing interests.

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