Zhang *Journal of Inequalities and Applications* 2014, 2014:15
http://www.journalofinequalitiesandapplications.com/content/2014/1/15

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On complete monotonicity of the Riemann zeta function

Ruiming Zhang[*]

[*]Correspondence:
ruimingzhang@gmail.com
College of Science, Northwest A&F University, Yangling, Shaanxi 712100, P.R. China

Abstract

Under the assumption of the Riemann hypothesis for the Riemann zeta function and some Dirichlet *L*-series we demonstrate that certain products of the corresponding zeta functions are completely monotonic. This may provide a method to disprove a certain Riemann hypothesis numerically.

MSC: Primary 30E15; secondary 33D45

Keywords: Riemann zeta function; Dirichlet series; Riemann hypothesis; complete monotonic functions

1 Introduction

The Riemann zeta function $\zeta(s)$ can be defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \Re(s)>1, \tag{1.1}$$

and on the rest of the complex plane by analytic continuation. It is known that the extended $\zeta(s)$ is meromorphic with infinitely many zeros at $-2n$ for $n\in\mathbb{N}$ (a.k.a trivial zeros) and with infinitely many zeros within the vertical strip $0<\Re(s)<1$ (nontrivial zeros). The Riemann hypothesis for $\zeta(s)$ says that all nontrivial zeros are actually on the critical line $\Re(s)=\frac{1}{2}$.

For any complex number $z\in\mathbb{C}$, let $\Gamma(z)$ be Euler's Gamma function defined by [1–8]

$$\frac{1}{\Gamma(z)}=z\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)\left(1+\frac{1}{j}\right)^{-z}. \tag{1.2}$$

Then, the Riemann $\Xi(z)$ function [1–7]

$$\Xi(z)=-\frac{1+4z^2}{8}\pi^{-\frac{1+2iz}{4}}\Gamma\left(\frac{1+2iz}{4}\right)\zeta\left(\frac{1+2iz}{2}\right) \tag{1.3}$$

is an even entire function of order 1. The celebrated Riemann hypothesis is equivalent to the statement that $\Xi(z)$ has only real zeros.

Let $\chi(n)$ be a real primitive character with modulus m; the function $L(s,\chi)$ is defined by [3, 8]

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re(s) > 1. \tag{1.4}$$

Let

$$\alpha = \begin{cases} 0, & \chi(-1) = 1, \\ 1, & \chi(-1) = -1, \end{cases} \tag{1.5}$$

then

$$\Xi(z,\chi) = \left(\frac{\pi}{m}\right)^{-(1+2\alpha+2iz)/4} \Gamma\left(\frac{1+2\alpha+2iz}{4}\right) L\left(\frac{1+2iz}{2}, \chi\right) \tag{1.6}$$

is an even entire function of order 1. The Riemann hypothesis for $L(s,\chi)$ is equivalent to $\Xi(z,\chi)$ having only real zeros.

Given real numbers a, b with $a < b$ and an indefinite differentiable real valued function $f(x)$ on (a,b), $f(x)$ is called completely monotonic on (a,b) if $(-1)^m f^{(m)}(x) \geq 0$ for all $x \in (a,b)$ and $m = 0, 1, \ldots$. In this work, under the assumptions of the Riemann hypothesis for the Riemann zeta function and certain L-series, we apply the ideas from [8, 9] to prove that some products of these zeta functions are completely monotonic. This complete monotonicity may provide a method to disprove a certain Riemann hypothesis via numerical methods.

2 Main results

Lemma 1 *Given a non-increasing sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} |\lambda_n| < \infty, \tag{2.1}$$

then, the entire function

$$f(x) = \prod_{n=1}^{\infty} (1 - x\lambda_n) \tag{2.2}$$

is completely monotonic on $(-\infty, \lambda_1^{-1})$.

Proof It is a direct consequence of Theorem 1 of [8]. □

Assuming the Riemann hypothesis is true, we list all positive zeros of $\Xi(z)$ as

$$z_1 \leq z_2 \leq \cdots \leq z_n \leq \cdots, \tag{2.3}$$

and z_1 is approximately 14.1347. Then,

$$\Xi(z) = \Xi(0) \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_n^2}\right). \tag{2.4}$$

Thus,

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n^2}\right)=\frac{\Xi(\sqrt{z})}{\Xi(0)}, \tag{2.5}$$

$$\frac{\Xi(z^{\frac{1}{4}})\Xi(iz^{\frac{1}{4}})}{\Xi^2(0)}=\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n^4}\right), \tag{2.6}$$

and

$$\frac{\Xi(z^{\frac{1}{6}})\Xi(\rho z^{\frac{1}{6}})\Xi(\rho^2 z^{\frac{1}{6}})}{\Xi^3(0)}=\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n^6}\right) \tag{2.7}$$

for $0\leq \arg(z)<2\pi$, where $\rho=e^{\frac{2\pi i}{3}}$. In fact, for any positive integer $\ell>1$ and assume that ρ_ℓ is a primitive ℓth root of unity; then we have

$$\frac{\prod_{j=1}^{\ell}\Xi(\rho_\ell^j z^{\frac{1}{2\ell}})}{\Xi^\ell(0)}=\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n^{2\ell}}\right). \tag{2.8}$$

Corollary 2 *Under the Riemann hypothesis, let z_1 be the least positive zeros of $\Xi(z)$; then the function $\Xi(\sqrt{z})$ is completely monotonic for $z\in(-\infty,z_1^2)$, $\Xi(z^{\frac{1}{4}})\Xi(iz^{\frac{1}{4}})$ is completely monotonic for $z\in(-\infty,z_1^4)$, and $\Xi(z^{\frac{1}{6}})\Xi(\rho z^{\frac{1}{6}})\Xi(\rho^2 z^{\frac{1}{6}})$ is completely monotonic for $z\in(-\infty,z_1^6)$. Let ρ_ℓ be a primitive ℓth root of unity for some positive integer ℓ; then $\prod_{j=1}^{\ell}\Xi(\rho_\ell^j z^{\frac{1}{2\ell}})$ is completely monotonic for $z\in(-\infty,z_1^{2\ell})$.*

Proof Notice that $\Xi(0)$ is a positive constant, and the claims are obtained by applying Corollary 1 to equations (2.5)-(2.8). □

Assuming the Riemann hypothesis for $L(s,\chi)$, we list all the positive zeros for $\Xi(z,\chi)$ as [8]

$$z_1(\chi)\leq z_2(\chi)\leq\cdots\leq z_n(\chi)\leq\cdots. \tag{2.9}$$

Then

$$\Xi(z,\chi)=\Xi(0,\chi)\prod_{n=1}^{\infty}\left(1-\frac{z^2}{z_n(\chi)^2}\right). \tag{2.10}$$

Evidently,

$$\Xi(0,\chi)\neq 0, \tag{2.11}$$

otherwise $\Xi(z,\chi)\equiv 0$, which is clearly false. Thus,

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n(\chi)^2}\right)=\frac{\Xi(\sqrt{z},\chi)}{\Xi(0,\chi)} \tag{2.12}$$

for $0 \leq \arg(z) < 2\pi$. Furthermore,

$$\frac{\Xi(z^{\frac{1}{4}},\chi)\Xi(iz^{\frac{1}{4}},\chi)}{\Xi^2(0)} = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n^4(\chi)}\right) \tag{2.13}$$

and

$$\frac{\Xi(z^{\frac{1}{6}},\chi)\Xi(\rho z^{\frac{1}{6}},\chi)\Xi(\rho^2 z^{\frac{1}{6}},\chi)}{\Xi^3(0)} = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n^6(\chi)}\right) \tag{2.14}$$

for $0 \leq \arg(z) < 2\pi$, where $\rho = e^{\frac{2\pi i}{3}}$. Let ρ_ℓ be a primitive ℓth root of unity for some positive integer ℓ; then we have

$$\frac{\prod_{j=1}^{\ell}\Xi(\rho_\ell^j z^{\frac{1}{2\ell}},\chi)}{\Xi^\ell(0,\chi)} = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n^{2\ell}(\chi)}\right). \tag{2.15}$$

Corollary 3 *Assume that the Riemann hypothesis is true for $L(s,\chi)$ and $z_1(\chi)$ is the least positive zero of $\Xi(z,\chi)$; then the function $\frac{\Xi(\sqrt{z},\chi)}{\Xi(0,\chi)}$ is completely monotonic for $z \in (-\infty, z_1^2(\chi))$, $\frac{\Xi(z^{\frac{1}{4}},\chi)\Xi(iz^{\frac{1}{4}},\chi)}{\Xi^2(0)}$ is completely monotonic for $z \in (-\infty, z_1^4(\chi))$, and $\frac{\Xi(z^{\frac{1}{6}},\chi)\Xi(\rho z^{\frac{1}{6}},\chi)\Xi(\rho^2 z^{\frac{1}{6}},\chi)}{\Xi^3(0)}$ is completely monotonic for $z \in (-\infty, z_1^6(\chi))$. Let ρ_ℓ be a primitive ℓth root of unity for some positive integer ℓ, then $\prod_{j=1}^{\ell}\Xi(\rho_\ell^j z^{\frac{1}{2\ell}},\chi)$ is completely monotonic for $z \in (-\infty, z_1^{2\ell}(\chi))$.*

Proof These are consequences of Lemma 1 and equations (2.12)-(2.15). □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This research is partially supported by National Natural Science Foundation of China, grant No. 11371294.

Received: 29 September 2013 Accepted: 13 December 2013 Published: 09 Jan 2014

References
1. Abramowitz, M, Stegun, IA (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Tenth printing with corrections. Natl. Bur. of Standards, New York (1972)
2. Andrews, GE, Askey, R, Roy, R: Special Functions. Cambridge University Press, Cambridge (1999)
3. Davenport, H: Multiplicative Number Theory. Springer, New York (2000)
4. Erdélyi, A: Higher Transcendental Functions, Vol. I. Krieger, Malabar (1985)
5. Erdélyi, A: Higher Transcendental Functions, Vol. II. Krieger, Malabar (1985)
6. Erdélyi, A: Higher Transcendental Functions, Vol. III. Krieger, Malabar (1985)
7. Titchmarsh, EC: The Theory of Riemann Zeta Function, 2nd edn. Clarendon Press, New York (1987)
8. Zhang, R: Sums of zeros for certain special functions. Integral Transforms Spec. Funct. **21**(5), 351-365 (2010)
9. Ismail, MEH, Zhang, R: Completely monotonic Fredholm determinants. J. Approx. Theory (submitted)

10.1186/1029-242X-2014-15
Cite this article as: Zhang: On complete monotonicity of the Riemann zeta function. *Journal of Inequalities and Applications* 2014, 2014:15