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Geodesic r-preinvex functions on Riemannian manifolds

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Abstract

In this article, we introduce a new class of functions called r-invexity and geodesic r-preinvexity functions on a Riemannian manifolds. Further, we establish the relationships between r-invexity and geodesic r-preinvexity on Riemannian manifolds. It is observed that a local minimum point for a scalar optimization problem is also a global minimum point under geodesic r-preinvexity on Riemannian manifolds. In the end, a mean value inequality is extended to a Cartan-Hadamard manifold. The results presented in this paper extend and generalize the results that have appeared in the literature.

MSC: 58E17; 90C26

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1 Introduction

Convexity is one of the most frequently used hypotheses in optimization theory. It is well known that a local minimum is also a global minimum for a convex function. A significant generalization of convex functions is that of an invex function introduced by Hanson [1]. Hanson's initial results inspired a great deal of subsequent work, which has greatly expanded the role and applications of invexity in non-linear optimization and other branches of pure and applied sciences.

Ben-Israel and Mond [2] introduced a new generalization of convex sets and convex functions, Craven [3] called them invex sets and preinvex functions, respectively. Jeyakumar [4] studied the properties of preinvex functions and their role in optimization and mathematical programming. Jeyakumar and Mond [5] introduced a new class of functions, namely V-invex functions, and established sufficient optimality criteria and duality results in the multiobjective programming problems. Antczak [6] introduced the concept of r-invexity and r-preinvexity in mathematical programming. Making a step forward Antczak [7] introduced the concept of $V-r$-invexity for differentiable multiobjective programming problems, which is a generalization of V-invex functions [5] and r-invex functions [6].

On the other hand, in the last few years, several important concepts of non-linear analysis and optimization problems have been extended from Euclidean space to a Riemannian manifolds. In general, a manifold is not a linear space, but naturally concepts and techniques from linear spaces to Riemannian manifold can be extended. Rapcsak [8] and Udriste [9] considered a generalization of convexity, called geodesic convexity, and extended many results of convex analysis and optimization theory to Riemannian manifolds.

The notion of invex functions on Riemannian manifolds was introduced by Pini [10] and Mititelu [11], and they investigated its generalization. Barani and Pouryayevali [12] introduced the geodesic invex set, geodesic η-invex function, and geodesic η-preinvex functions on a Riemannian manifold and found some interesting results. Further, Agarwal *et al.* [13] generalized the notion of geodesic η-preinvex functions to geodesic α-preinvex functions. Recently, Zhou and Huang [14] introduced the concept of roughly B-invex set and functions on Riemannian manifolds.

Motivated by work of Barani and Pouryayevali [12] and Antczak [6, 7], we introduce the concept of geodesic r-preinvex functions and r-invex functions on Riemannian manifolds, which is a generalization of preinvexity as defined in [6, 12]. Some relations between r-invex and geodesic r-preinvex functions are investigated. The existence conditions for global minima of these functions under proximal subdifferential of lower semicontinuity are also explored. In the end, a mean value inequality is also derived.

2 Preliminaries

In this section we recall some basic definitions and some basic results of Riemannian manifolds, for further study these materials are available in (*cf.* [15]).

Let M be a C^∞-manifold modeled on a Hilbert space H, either finite or infinite dimensional, endowed with a Riemannian metric g_p on a tangent space T_pM. The corresponding norm is denoted by $\| \|_p$ and the length of a piecewise C^1 curve $\gamma : [a,b] \to M$ is defined by

$$L(\gamma) = \int_a^b \left\| \gamma'(t) \right\|_{\gamma(t)} dt.$$

For any point $p, q \in M$, we define

$$d(p,q) = \inf\left\{ L(\gamma) | \gamma \text{ is a piecewise } C^1 \text{ curve joining } p \text{ and } q \right\},$$

then d is a distance which induces the original topology on M. We know that on every Riemannian manifold there exists exactly one covariant derivative called a Levi-Civita connection, denoted by $\nabla_X Y$, for any vector fields $X, Y \in TM$; we also recall that a geodesic is a C^∞-smooth path γ whose tangent is parallel along the path γ, that is, γ satisfies the equation $\nabla_{d\gamma(t)/dt}\, d\gamma(t)/dt = 0$. Any path γ joining p and q in M such that $L(\gamma) = d(p,q)$ is a geodesic and is called a minimal geodesic. The existence theorem for ordinary differential equation implies that for every $v \in TM$, there exist an open interval $J(v)$ containing 0 and exactly one geodesic $\gamma_v : J(v) \to M$ with $d\gamma_v(0)/dt = v$. This implies that there is an open neighborhood $\bar{T}M$ of the submanifold M of TM such that for every $\exp : \bar{T}M \to M$ is there is defined $\exp(v) = J_v(1)$ and the restriction of exp to a fiber T_pM in $\bar{T}M$ is denoted by $\exp_p$ for every $p \in M$. We use parallel transport of vectors along the geodesic. Recall that for a given curve $\gamma : I \to M$, a number $t_0 \in I$, and a vector $v_0 \in T_{\gamma(t_0)}M$, there exists exactly one parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = v_0$. Moreover, the mapping defined by $v_0 \mapsto V(t)$ is a linear isometry between the tangent spaces $T_{\gamma(t_0)}M$ and $T_{\gamma(t)}M$, for each $t \in I$. We denote this mapping by $P^t_{t_0,\gamma}$ and we call it the parallel translation from $T_{\gamma(t_0)}M$ to $T_{\gamma(t)}M$ along the curve γ.

If f is a differentiable map from the manifold M to manifold N, then df_x, denotes the differential of f at x. We also recall that a simply connected complete Riemannian manifold of non-positive sectional curvature is called a Cartan-Hadamard manifold.

3 Geodesic *r*-invex functions

In this section, we define geodesic r-invex functions and r-preinvex functions. Barani and Pouryayevali [12] define the invex sets as follows.

Definition 3.1 Let M be a Riemannian manifold and $\eta : M \times M \to TM$ such that for every $x, y \in M$, $\eta(x,y) \in T_yM$. A non-empty subset S of M is said to be a geodesic invex set with respect to η if for every $x, y \in S$, there exists a unique geodesic $\gamma_{x,y} : [0,1] \to M$ such that

$$\gamma_{x,y}(0) = y, \qquad \gamma'_{x,y}(0) = \eta(x,y), \qquad \gamma_{x,y}(t) \in S$$

for all $t \in [0,1]$.

Remark 3.1 [12] If we consider M to be a Cartan-Hadamard manifold (either infinite or finite dimensional), then on M there exists a natural map η playing the role of $x - y$ in the R^n. Indeed we define the function η as

$$\eta(x,y) = \gamma'_{x,y}(0)$$

for all $x, y \in M$. Here $\gamma_{x,y}$ is the unique minimal geodesic joining y to x (see [16, p.253]) as follows:

$$\gamma_{x,y}(t) = \exp_y\bigl(t \exp_y^{-1} x\bigr)$$

for all $t \in [0,1]$. Therefore, every geodesic convex set $S \subseteq M$ is a geodesic convex set with respect to η defined in above equation. The converse is not true in general.

Example 3.1 [12] Let M be a Cartan-Hadamard manifold and $x_0, y_0 \in M$, $x_0 \neq y_0$. Let $B(x_0, r_1) \cup B(y_0, r_2) = \phi$ for some $0 < r_1, r_2 < \frac{1}{2}d(x_0, y_0)$, where $B(x,r) = \{y \in M | d(x,y) < r\}$ is an open ball with center x and radius r. We define

$$S = B(x_0, r_1) \cup B(y_0, r_2),$$

then S is not a geodesic convex set because every geodesic curve passing through x_0 and y_0 does not completely lie in S. Now we define the function $\eta : M \times M \to TM$ such that

$$\eta(x,y) = \begin{cases} \exp_y^{-1} x & \text{if } x, y \in B(x_0, r_1) \text{ or } x, y \in B(x_0, r_2), \\ 0_y & \text{otherwise.} \end{cases}$$

For every $x, y \in M$, consider $\gamma : [0,1] \to M$ defined by

$$\gamma_{x,y}(t) = \exp\bigl(t\eta(x,y)\bigr)$$

for all $t \in [0,1]$.

Hence $\gamma_{x,y}(0) = y$, $\gamma'_{x,y}(0) = \eta(x,y)$. Barani and Pouryayevali [12] showed that S is a geodesic invex set with respect to η.

Let S be a geodesic convex subset of a finite dimensional Cartan-Hadamard manifold M and $x \in M$, then there exists a unique point $p_s(x) \in S$ such that for each $y \in S$, $d(x, p_s(x)) \leq d(x,y)$. The point $p_s(x)$ is called the projection of x onto S (see [16, p.262]).

Definition 3.2 [12] Let M be an n-dimensional Riemannian manifold and S be an open subset of M which is geodesic invex set with respect to $\eta : M \times M \to TM$. Let f be a real valued function such that $f : S \to R$. Then f is said to be an η-invex function with respect to η if

$$f(x) - f(y) \geq df_y\big(\eta(x,y)\big)$$

for all $x, y \in S$.

Definition 3.3 [12] Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic η-invex set with respect to $\eta : M \times M \to TM$. The function $f : S \to R$ is said to be geodesic η-preinvex if for any $x, y \in S$

$$f\big(\gamma_{x,y}(t)\big) \leq tf(x) + (1-t)f(y)$$

for all $t \in [0,1]$, where $\gamma_{x,y}$ is the unique geodesic defined in Definition 3.1. If the above inequality is strict, then f is called a strictly geodesic preinvex function.

Now we define an r-invex function and a geodesic r-preinvex function on M.

Definition 3.4 Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic invex set with respect to $\eta : M \times M \to TM$. Let f be a real differentiable function S. Then f is said to be r-invex with respect to η if

$$\begin{aligned} &\frac{1}{r}e^{rf(x)} - \frac{1}{r}e^{rf(y)} \geq e^{rf(y)} df_y\big(\eta(x,y)\big) \quad \text{if } r \neq 0, \\ &f(x) - f(y) \geq df_y\big(\eta(x,y)\big) \quad \text{if } r = 0. \end{aligned}$$

Definition 3.5 Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic invex set with respect to $\eta : M \times M \to TM$. The function $f : S \to R$ is said to be geodesic r-preinvex if for any $x, y \in S$, we have

$$f\big(\gamma_{x,y}(t)\big) \leq \begin{cases} \log(te^{rf(x)} + (1-t)e^{rf(y)})^{\frac{1}{r}} & \text{if } r \neq 0, \\ tf(x) + (1-t)f(y) & \text{if } r = 0. \end{cases}$$

If the above inequality is strict, then f is called a strictly geodesic r-preinvex function.

We give the following non-trivial example for a geodesic r-preinvex function that is yet not geodesic η-preinvex.

Example 3.2 Let $M = \{e^{i\theta} : 0 < \theta < 1\}$ and $f : M \to R$ defined by $f(e^{i\theta}) = \cos\theta$ with $x, y \in M$, $x = e^{i\alpha}$ and $y = e^{i\beta}$. If $\gamma_{x,y}(t) = e^{i((1-t)\beta + t\alpha)}$ then f is a geodesic r-preinvex function but not a geodesic η-preinvex function at $\alpha = \frac{\pi}{2}$, $\beta = \frac{\pi}{4}$, since $\cos[\frac{\pi}{4} + \frac{\pi}{4}t] > \frac{t}{\sqrt{2}}$ at $t = 0$.

Proposition 3.1 *If $f : S \to R$ is a geodesic r-preinvex function with respect to $\eta : S \times S \to TM$ and $y \in S$, then for any real number $\lambda \in R$, the level set $S_\lambda = \{x \mid x \in S, f(x) \leq \lambda\}$ is a geodesic invex set.*

Proof For any $x, y \in S_\lambda$ and $0 \leq t \leq 1$, we have $f(x) \leq \lambda$, $f(y) \leq \lambda$. Since f is geodesic r-preinvex function, then we have

$$f(\gamma_{x,y}(t)) \leq \log\left(te^{rf(x)} + (1-t)e^{rf(y)}\right)^{\frac{1}{r}}$$

or

$$\begin{aligned} e^{rf(\gamma_{x,y}(t))} &\leq te^{rf(x)} + (1-t)e^{rf(y)} \\ &\leq te^{r\lambda} + (1-t)e^{r\lambda}. \end{aligned}$$

Equivalently,

$$e^{rf(\gamma_{x,y}(t))} \leq e^{r\lambda}$$

or

$$f(\gamma_{x,y}(t)) \leq \lambda.$$

Therefore, $\gamma_{x,y}(t) \in S_\lambda$ for all $t \in [0,1]$, and the result is proved. □

4 Geodesic *r*-preinvexity and differentiability

In this section, we discuss property and condition (say condition (C)) introduced by Barani and Pouryayevali [12] on the function $\eta : M \times M \to TM$, which will be used in the subsequent analysis.

Pini [10] define the following property.

Definition 4.1 Let M be a Riemannian manifold and $\gamma : [0,1] \to M$ be a curve on M such that $\gamma_{x,y}(0) = y$ and $\gamma_{x,y}(1) = x$. Then $\gamma_{x,y}$ is said to possess the property (P) with respect to $y, x \in M$ if

$$\gamma'_{x,y}(s)(t-s) = \eta(\gamma_{x,y}(t), \gamma_{x,y}(s))$$

for all $s, t \in [0,1]$.

Pini [10] also proved the following conditions as follows:

$$(\mathrm{C}_1) \quad P^0_{s,\gamma_{x,y}}\left[\eta(y, \gamma_{x,y(s)})\right] = -s\eta(x,y),$$

$$(\mathrm{C}_2) \quad P^0_{s,\gamma_{x,y}}\left[\eta(x, \gamma_{x,y(s)})\right] = (1-s)\eta(x,y)$$

for all $s \in [0,1]$, which taken together are called condition (C).

Theorem 4.1 *Let M be a Riemannian manifold and S be an open subset of M which is a geodesic invex set with respect to $\eta : M \times M \to TM$. Let $f : S \to R$ be a differentiable and geodesic r-preinvex function on S. Then f is an r-invex function on S.*

Proof Since S is a geodesic invex set with respect to η, then for all $x, y \in S$, there exists a unique geodesic $\gamma_{x,y}(0) = y$, $\gamma'_{x,y}(0) = \eta(x,y)$, $\gamma_{x,y}(t) \in S$ for all $t \in [0,1]$. By the differentiability of f at $y \in M$, we have

$$df_y(\eta(x,y)) = \lim_{t\to 0} \frac{1}{t}\left[f(\gamma_{x,y}(t)) - f(y)\right],$$

and so

$$f(y) + df_y(\eta(x,y))t + O^2(t) = f(\gamma_{x,y}(t)).$$

But f is geodesic r-preinvex for $t \in (0,1]$, and we have

$$f(y) + df_y(\eta(x,y))t + O^2(t) \le \log\left(te^{rf(x)} + (1-t)e^{rf(y)}\right)^{\frac{1}{r}}$$

or

$$e^{rf(y)+rdf_y(\eta(x,y))t+ro^2(t)} - e^{rf(y)} \le t\left(e^{rf(x)} - e^{rf(y)}\right).$$

Dividing by t and taking the limit $t \to 0$, we get

$$e^{rf(y)} df_y(\eta(x,y)) \le \frac{1}{r}\left(e^{rf(x)} - e^{rf(y)}\right).$$

Hence, f is an r-invex function on S. □

Theorem 4.2 *Let M be a Riemannian manifold and S be an open subset of M, which is a geodesic invex set with respect to $\eta : M \times M \to TM$. Let $f : S \to R$ be a differentiable function, η satisfies the condition* (C), *then f is geodesic r-preinvex on S iff f is r-invex on S.*

Proof We know that for a geodesic invex set with respect to η for every $x, y \in S$, there exists a unique geodesic $\gamma_{x,y} : [0,1] \to M$ such that $\gamma_{x,y}(0) = y$, $\gamma'_{x,y}(0) = \eta(x,y)$, $\gamma_{x,y}(t) \in S$, for all $t \in [0,1]$.

Fix $t \in [0,1]$ and set $\bar{x} = \gamma_{x,y}(t)$, then by geodesic r-invexity of f on S, we have

$$\frac{1}{r}e^{rf(x)} - \frac{1}{r}e^{rf(\bar{x})} \ge e^{rf(\bar{x})} df_{\bar{x}}(\eta(x,\bar{x})), \tag{1}$$

$$\frac{1}{r}e^{rf(y)} - \frac{1}{r}e^{rf(\bar{x})} \ge e^{rf(\bar{x})} df_{\bar{x}}(\eta(y,\bar{x})). \tag{2}$$

On multiplying (1) by t and (2) by $(1-t)$, respectively, and then adding we get

$$t\frac{1}{r}e^{rf(x)} + (1-t)\frac{1}{r}e^{rf(y)} - \frac{1}{r}e^{rf(\bar{x})} \ge e^{rf(\bar{x})} df_{\bar{x}}\left[t\eta(x,\bar{x}) + (1-t)\eta(y,\bar{x})\right]. \tag{3}$$

By the condition (C), we have

$$t\eta(x,\bar{x}) + (1-t)\eta(y,\bar{x}) = t(1-t)P^t_{0,\gamma}\left[\eta(x,y)\right] - (1-t)tP^t_{0,\gamma}\left[\eta(x,y)\right] = 0. \tag{4}$$

This together with (3) implies

$$te^{rf(x)} + (1-t)e^{rf(y)} \ge e^{rf(\bar{x})}$$

or

$$f(\bar{x}) \geq \log\bigl(te^{rf(x)} + (1-t)e^{rf(y)}\bigr)^{\frac{1}{r}},$$

Hence, f is geodesic r-preinvex on S. □

5 Geodesic *r*-preinvexity and semi-continuity

In this section, we discuss geodesic r-preinvexity on Riemannian manifold under proximal subdifferential of a lower semi-continuous function. First, we recall the definition of a proximal subdifferentiable of a function defined on a Riemannian manifold in [12].

Definition 5.1 Let M be a Riemannian manifold and $f: M \to (-\infty, \infty]$ be a lower semi-continuous function. A point $\xi \in T_yM$ is said to be proximal subgradient of f at $y \in \operatorname{dom}(f)$, if there exist a positive number δ and σ such that

$$f(x) \geq f(y) + \langle \xi, \exp_y^{-1} x\rangle_y - \sigma d^2(x,y)$$

for all $x \in B(y,\delta)$, where $\operatorname{dom} f = \{x \in M : f(x) < \infty\}$. The set of all proximal subgradient of $y \in M$ is denoted by $\partial_p f(y)$.

Theorem 5.1 *Let M be a Riemannian manifold and S be an open subset of M, which is geodesic invex with respect to $\eta: M \times M \to TM$. Let $f: S \to R$ be geodesic r-preinvex, if $\bar{x} \in S$ is a local minimum of the problem*

(P) *Minimize $f(x)$*

subject to $x \in S$,

then $\bar{x}$ is a global minimum of (P).

Proof Let $\bar{x} \in S$ be a local minimum; then there exists a neighborhood $N_\epsilon(\bar{x})$ such that

$$f(\bar{x}) \leq f(x) \tag{5}$$

for all $x \in S \cap N_\epsilon(\bar{x})$.

If $\bar{x}$ is not a global minimum of f, then there exists a point $x^* \in S$ such that

$$f\bigl(x^*\bigr) < f(\bar{x})$$

or

$$e^{rf(x^*)} < e^{rf(\bar{x})}.$$

As S is a geodesic invex set with respect to η, there exists a unique geodesic γ such that $\gamma(0) = \bar{x}$, $\gamma'(0) = \eta(x^*, \bar{x})$, $\gamma(t) \in S$, for all $t \in [0,1]$.

If we choose $\epsilon > 0$ such that $d(r(t), \bar{x}) < \epsilon$, then $\gamma(t) \in N_\epsilon(\bar{x})$. From the geodesic r-preinvexity of f, we have

$$f\bigl(\gamma(t)\bigr) \leq \log\bigl(te^{r(x^*)} + (1-t)e^{r(\bar{x})}\bigr)^{\frac{1}{r}}.$$

Equivalently, we have

$$e^{rf(\gamma(t))} \leq te^{r(x^*)} + (1-t)e^{r(\bar{x})} < te^{r(\bar{x})} + (1-t)e^{r(\bar{x})}$$

or

$$e^{rf(\gamma(t))} < e^{rf(\bar{x})},$$

or

$$f(\gamma(t)) < f(\bar{x})$$

for all $t \in (0,1]$. Therefore, for each $\gamma(t) \in S \cap N_\epsilon(\bar{x}), f(\gamma(t)) < f(\bar{x})$, which is a contradiction to (5). Hence the result. □

Theorem 5.2 *Let M be a Cartan-Hadamard manifold and S be an open subset of M, which is geodesic r-preinvex with respect to $\eta : M \times M \to TM$ with $\eta(x,y) \neq 0$ for all $x \neq y$. Assume that $f : S \to (-\infty, \infty]$ is a lower semi-continuous geodesic r-preinvex function and $y \in \mathrm{dom}(f)$, $\xi \in \partial_p f(y)$. Then there exists a positive number δ such that*

$$e^{rf(x)} - e^{rf(y)} \geq e^{rf(y)} \langle \xi, \eta(x,y) \rangle_y$$

for all $x \in S \cap B(y,\delta)$.

Proof From the definition of $\partial_p f(y)$, there are positive numbers δ and σ such that

$$f(x) \geq f(y) + \langle \xi, \exp_y^{-1} x \rangle_y - \sigma d^2(x,y) \tag{6}$$

for all $x \in B(y,\delta)$.

Now, fix $x \in S \cap B(y,\delta)$. Since S is a geodesic invex set with respect to η, there exists a unique geodesic $\gamma_{x,y} : [0,1] \to M$ such that $\gamma_{x,y}(0) = y$, $\gamma'_{x,y}(0) = \eta(x,y)$, $\gamma_{x,y}(t) \in S$, for all $t \in [0,1]$.

Since M is a Cartan-Hadamard manifold, then $\gamma_{x,y}(t) = \exp_y(t\eta(x,y))$ for each $t \in [0,1]$ (see [4, p.253]). If we choose $t_0 = \frac{\delta}{\|\eta(x,y)\|_y}$, then $\exp_y(t\eta(x,y)) \in S \cap B(y,\delta)$ for all $t \in [0,t_0)$.

From the geodesic r-preinvexity of f, we get

$$f(\exp_y(t\eta(x,y))) \leq \log(te^{rf(x)} + (1-t)e^{rf(y)})^{\frac{1}{r}}$$

or

$$e^{rf(\exp_y(t\eta(x,y)))} \leq te^{rf(x)} + (1-t)e^{rf(y)}. \tag{7}$$

Using (6) for each $t \in (0,t_0)$, we get

$$\begin{aligned} f(\exp_y(t\eta(x,y))) &\geq f(y) + \langle \xi, \exp_y^{-1} \exp_y(t\eta(x,y)) \rangle_y - \sigma d^2(\exp_y(t\eta(x,y),y)) \\ &= f(y) + \langle \xi, t\eta(x,y) \rangle_y - \sigma d^2(\exp_y(t\eta(x,y),y)). \end{aligned}$$

Since M is a Cartan-Hadamard manifold, for each $t \in (0, t_0)$, we have

$$d^2(\exp_y(t\eta(x,y)), y)) = \|t\eta(x,y)\|_y^2 = t^2\|\eta(x,y)\|_y^2.$$

Thus we have

$$f(\exp_y(t\eta(x,y))) \geq f(y) + \langle \xi, t\eta(x,y)\rangle_y - \sigma t^2\|\eta(x,y)\|_y^2$$

or

$$e^{rf(\exp_y(t\eta(x,y)))} \geq e^{rf(y)}e^{\langle \xi, t\eta(x,y)\rangle_y - \sigma t^2\|\eta(x,y)\|_y^2}. \tag{8}$$

Thus from (7) and (8), we have

$$te^{rf(x)} + (1-t)e^{rf(y)} \geq e^{rf(y)}e^{\langle \xi, t\eta(x,y)\rangle_y - \sigma t^2\|\eta(x,y)\|_y^2}.$$

By further calculation we arrive at

$$e^{rf(x)} - e^{rf(y)} \geq e^{rf(y)}\frac{1}{t}\left[e^{\langle \xi, t\eta(x,y)\rangle_y - \sigma t^2\|\eta(x,y)\|_y^2} - 1\right],$$

taking the limit $t \to 0$

$$e^{rf(x)} - e^{rf(y)} \geq e^{rf(y)}\langle \xi, \eta(x,y)\rangle_y.$$

This proves the theorem completely. □

6 Mean value inequality

In this section, we introduce a mean value inequality for Cartan-Hadamard manifold which is an extension of the result proved by Antczak [17] and Barani and Pouryayevali [12].

Definition 6.1 [12] Let S be a non-empty subset of a Riemannian manifold M, which is a geodesic η-invex set with respect to $\eta : M \times M \to TM$, and let x and u be two arbitrary points of S. Let $\gamma : [0,1] \to M$ be the unique geodesic such that $\gamma(0) = u$, $\gamma'(0) = \eta(x,u)$, $\gamma(t) \in S$, for all $t \in [0,1]$.

A set P_{uv} is said to be a closed η-path joining the points u and $v = \gamma(1)$, if

$$P_{uv} = \{y : y = \gamma(t), t \in [0,1]\}.$$

An open η-path joining the point u and v is a set of the form

$$P_{uv}^0 = \{y : y = \gamma(t), t \in (0,1)\}.$$

If $u = v$ we set $P_{uv}^0 = \phi$.

Theorem 6.1 (Mean value inequality) *Let M be a Cartan-Hadamard manifold and S be an open subset of M, which is a geodesic invex set with respect to $\eta : M \times M \to TM$ such that*

$\eta(a,b) \neq 0$ for all $a, b \in S$, $a \neq b$. Let $\gamma_{b,a}(t) = \exp_a(t\eta(b,a))$ for every $a, b \in S$, $t \in [0,1]$ and $c = \gamma_{b,a}(1)$. Then a necessary and sufficient condition for a function $f : S \to R$ to be geodesic r-preinvex is that the inequality

$$e^{rf(x)} \leq e^{rf(a)} + \frac{e^{rf(b)} - e^{rf(a)}}{\langle \eta(b,a), \eta(b,a)\rangle_a} \langle \exp_a^{-1} x, \eta(b,a)\rangle_a \tag{9}$$

is true for all $x \in P_{ca}$.

Proof Let $f : S \to R$ be a geodesic preinvex function, $a, b \in S$ and $x \in P_{ca}$. If $x = a$ or $x = c$ then (9) is true trivially. If $x \in P_{ca}$, then $x = \exp(t\eta(b,a))$, for some $t \in (0,1)$. From the geodesic η-invexity of S, we have $x \in S$ and

$$t = \frac{\langle \exp_a^{-1} x, \eta(b,a)\rangle_a}{\langle \eta(b,a), \eta(b,a)\rangle_a}.$$

Since f is geodesic preinvex on S, it follows that

$$f(x) = f\bigl(\exp_a\bigl(t\eta(b,a)\bigr)\bigr) \leq \log\bigl(te^{rf(b)} + (1-t)e^{rf(a)}\bigr)^{\frac{1}{r}}$$

or

$$\begin{aligned} e^{rf(x)} &\leq te^{rf(b)} + (1-t)e^{rf(a)} \\ &= e^{rf(a)} + t\bigl(e^{rf(b)} - e^{rf(a)}\bigr). \end{aligned}$$

Using the value of t we get

$$e^{rf(x)} \leq e^{rf(a)} + \frac{e^{rf(b)} - e^{rf(a)}}{\langle \eta(b,a), \eta(b,a)\rangle_a} \langle \exp_a^{-1} x, \eta(b,a)\rangle_a.$$

For sufficiency suppose that the mean value inequality (9) is true. Let $a, b \in S$ and $x = \exp_a(t\eta(b,a))$, for some $t \in [0,1]$. Then $x \in S$, and we have $f(x) = f(\exp_a(t\eta(b,a)))$, from (9)

$$\begin{aligned} e^{rf(x)} &\leq e^{rf(a)} + \frac{e^{rf(b)} - e^{rf(a)}}{\langle \eta(b,a), \eta(b,a)\rangle_a} \langle \exp_a^{-1} x, \eta(b,a)\rangle_a \\ &= e^{rf(a)} + \frac{e^{rf(b)} - e^{rf(a)}}{\langle \eta(b,a), \eta(b,a)\rangle_a} \bigl\langle \exp_a^{-1}\bigl(\exp_a\bigl(t\eta(b,a)\bigr)\bigr), \eta(b,a)\bigr\rangle_a \\ &= te^{rf(b)} + (1-t)e^{rf(a)} \end{aligned}$$

or

$$f(x) \leq \log\bigl(te^{rf(b)} + (1-t)e^{rf(a)}\bigr)^{\frac{1}{r}}.$$

Equivalently,

$$f\bigl(\exp_a\bigl(t\eta(b,a)\bigr)\bigr) \leq \log\bigl(te^{rf(b)} + (1-t)e^{rf(a)}\bigr)^{\frac{1}{r}},$$

which shows that f is geodesic r-preinvex function on S. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Hanson, MA: On sufficiency of the Kuhn-Tucker conditions. J. Math. Anal. Appl. **80**, 545-550 (1981)
2. Ben-Israel, B, Mond, B: What is the invexity. J. Aust. Math. Soc. B **28**, 1-9 (1986)
3. Craven, BD: Invex functions and constrained local minima. Bull. Aust. Math. Soc. **24**, 357-366 (1981)
4. Jeyakumar, V: Strong and weak invexity in mathematical programming. Math. Oper. Res. **55**, 109-125 (1985)
5. Jeyakumar, V, Mond, B: On generalized convex mathematical programming. J. Aust. Math. Soc. B **34**, 43-53 (1992)
6. Antczak, T: r-Preinvexity and r-invexity in mathematical programming. Comput. Math. Appl. **50**, 551-566 (2005)
7. Antczak, T: V-r-Invexity in multiobjective programming. J. Appl. Anal. **11**, 63-80 (2005)
8. Rapcsak, T: Smooth Nonlinear Optimization in R^n. Kluwer Academic, Dordrecht (1997)
9. Udriste, C: Convex Functions and Optimization Methods on Riemannian Manifolds. Math. Appl. Kluwer Academic, Dordrecht (1994)
10. Pini, R: Convexity along curves and invexity. Optimization **29**, 301-309 (1994)
11. Mititelu, S: Generalized invexity and vector optimization on differential manifolds. Differ. Geom. Dyn. Syst. **3**, 21-31 (2001)
12. Barani, A, Pouryayevali, MR: Invex sets and preinvex functions on Riemannian manifolds. J. Math. Anal. Appl. **328**, 767-779 (2007)
13. Agarwal, RP, Ahmad, I, Iqbal, A, Ali, S: Generalized invex sets and preinvex functions on Riemannian manifolds. Taiwan. J. Math. **16**(5), 1719-1732 (2012)
14. Zhou, L-W, Huang, N-J: Roughly geodesic B-invex and optimization problem on Hadamard manifolds. Taiwan. J. Math. **17**(3), 833-855 (2013)
15. Lang, S: Fundamentals of Differential Geometry. Graduate Texts in Mathematics. Springer, New York (1999)
16. Ferreira, OP, Oliveira, PR: Proximal point algorithm on Riemannian manifolds. Optimization **51**, 257-270 (2002)
17. Antczak, T: Mean value in invexity analysis. Nonlinear Anal. **60**, 1471-1484 (2005)

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