Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Large deviations for randomly weighted sums with dominantly varying tails and widely orthant dependent structure

Xiaodong Bai[1*], Lixing Song[2] and Tao Hu[3]

*Correspondence: baixd518@gmail.com
[1]School of Science, Dalian Nationalities University, Liaohe West Road, Dalian, 116600, China
Full list of author information is available at the end of the article

Abstract

We prove large deviation inequalities for the randomly weighted partial and random sums $S_n^\theta = \sum_{i=1}^n \theta_i X_i$, $n \geq 1$; $S_c^\theta(t) = \sum_{i=1}^{N(t)} (\theta_i X_i + c)$, $c \in R$, where $\{N(t), t \geq 0\}$ is a counting process, $\{\theta_i, i \geq 1\}$ is a sequence of positive random variables with two-sided bounds, and $\{X_i, i \geq 1\}$ is a sequence of non-identically distributed real-valued random variables, while the three random sources above are mutually independent. Special attention is paid to the distribution of dominated variation and the widely orthant dependence structure.

MSC: Primary 62E20; secondary 62H20; 62P05

Keywords: large deviations; randomly weighted sums; widely orthant dependence; dominated variation

1 Introduction

Let $\{X_k, k \geq 1\}$ be a sequence of real-valued random variables (r.v.s) with X_k's distribution function (d.f.) $F_k(x) = 1 - \overline{F}_k(x)$ and $\mu_k = EX_k = 0$ for every $k \geq 1$, and $\{\theta_k, k \geq 1\}$ be another sequence of positive random variables, satisfying $P_r(a \leq \theta_k \leq b) = 1$, $k \geq 1$, where $0 < a \leq b < \infty$. $\{N(t), t \geq 0\}$ denotes a counting process (that is, a non-negative, non-decreasing, and integer-valued stochastic process) with a finite mean function $\lambda(t)$ for $t \geq 0$ and $\lambda(t) \to \infty$ as $t \to \infty$. Besides, the three random sources above are mutually independent. Denote $S_n^\theta = \sum_{i=1}^n \theta_i X_i$, $n \geq 1$ and $S_c^\theta(t) = \sum_{i=1}^{N(t)} (\theta_i X_i + c)$, $t \geq 0$, $c \in R$. By convention, the summation over an empty set of indices produces a value of 0. In the present paper, we are interested in the probabilities of large deviations of $\{S_n^\theta\}$ and $\{S_c^\theta(t)\}$ in the situation that $\{X_k, k \geq 1\}$ are heavy-tailed and widely orthant dependent.

Since the theory of large deviations with heavy tails is widely used in insurance and finance, in recent decades, there have been a series of articles devoted to related problems. For more details, please refer to Embrechts *et al.* [1], Klüppelberg and Mikosch [2], Mikosch and Nagaev [3] and references therein. Recently Tang [4] extended the asymptotic behavior of large deviation probabilities of partial sums of heavy-tailed random variables to the case of negatively dependent ones. Under the assumption that random variables are non-identically distributed and extended negatively dependent, Liu [5] obtained a result similar to the one in the above paper, which was promoted to random sums in various situations later by Chen *et al.* [6]. Specially, Shen and Lin [7] investigated large deviations of randomly weighted partial sums with negatively dependent and consistently varying-tailed random variables, but, unfortunately, there are some flaws in their proofs.

In this paper, motivated by the work of Liu and Hu [8] and Chen *et al.* [6], on the one hand, we aim to prove that for each fixed $\gamma > 0$, there exist positive constant M_1 and M_2 such that the inequalities

$$P_r\big(S_n^\theta > x\big) \geq M_1\big(1 + o(1)\big) \sum_{k=1}^{n} P_r(\theta_k X_k > x) \tag{1.1}$$

and

$$P_r\big(S_n^\theta > x\big) \leq M_2\big(1 + o(1)\big) \sum_{k=1}^{n} P_r(\theta_k X_k > x) \tag{1.2}$$

hold uniformly for all $x \geq \gamma n$ as $n \to \infty$, respectively; on the other hand, for arbitrarily fixed $\gamma > c$ (c is an arbitrarily given real number), there are positive constant $\widetilde{M}_1$ and $\widetilde{M}_2$ such that

$$P_r\big(S_c^\theta(t) > x\big) \geq \widetilde{M}_1\big(1 + o(1)\big) E\left[\sum_{k=1}^{N(t)} P_r\big(\theta_k X_k > x - c\lambda(t)\big)\right] \tag{1.3}$$

and

$$P_r\big(S_c^\theta(t) > x\big) \leq \widetilde{M}_2\big(1 + o(1)\big) E\left[\sum_{k=1}^{N(t)} P_r\big(\theta_k X_k > x - c\lambda(t)\big)\right] \tag{1.4}$$

hold uniformly for all $x \geq \gamma\lambda(t)$ as $t \to \infty$, respectively.

The paper is organized as follows. Section 2 presents our main results after recalling some preliminaries. Sections 3 and 4 prove Theorems 2.1 and 2.2, respectively.

2 Main results

We say that a random variable X or its distribution function is heavy-tailed if $Ee^{tX} = \infty$ for all $t > 0$. An important class of heavy-tailed distributions is $\mathcal{D}$, which consists of all distributions with dominated variation in the sense that the relation $\limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) < \infty$ holds for some (hence for all) $0 < y < 1$. Recall the upper/lower Matuszewska index of distribution F, defined as $J_F^+ = -\lim_{y\to\infty}(\log \overline{F}_*(y)/\log y)$ and $J_F^- = -\lim_{y\to\infty}(\log \overline{F}^*(y)/\log y)$, where $\overline{F}_*(y) = \liminf_{x\to\infty} \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x)$ for any $y > 0$. From Lemma 3.5 of Tang and Tsitsiashvili [9], we know that if $F \in \mathcal{D}$, then $0 \leq J_F^- \leq J_F^+ < \infty$, and for arbitrary $p_1 < J_F^-$ and $p_2 > J_F^+$, there exist positive constant $\widetilde{C}_i$ and $\widetilde{D}_i$, $i = 1, 2$, such that

$$\overline{F}(y)/\overline{F}(x) \geq \widetilde{C}_1(x/y)^{p_1} \tag{2.1}$$

holds for all $x \geq y \geq \widetilde{D}_1$, and

$$\overline{F}(y)/\overline{F}(x) \leq \widetilde{C}_2(x/y)^{p_2} \tag{2.2}$$

holds for all $x \geq y \geq \widetilde{D}_2$. Hence, for any $p_1 < J_F^-$, we have

$$\overline{F}(x) = o\bigl(x^{-p_1}\bigr); \tag{2.3}$$

and for any $p_2 > J_F^+$,

$$x^{-p_2} = o\bigl(\overline{F}(x)\bigr). \tag{2.4}$$

Furthermore, if the distribution $F^+(x) = F(x)\mathbf{1}_{(x\geq 0)}$ has a finite mean, then $J_F^+ \geq 1$.

Now, we present some new dependence structures introduced in Wang *et al.* [10].

Definition 2.1 We say that the r.v.s $\{\eta_n, n \geq 1\}$ are widely upper orthant dependent (WUOD) if there exists a finite real sequence $\{g_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, +\infty)$, $1 \leq i \leq n$,

$$P_r\left(\bigcap_{i=1}^{n}\{\eta_i > x_i\}\right) \leq g_U(n)\prod_{i=1}^{n} P_r(\eta_i > x_i); \tag{2.5}$$

we say that the r.v.s $\{\eta_n, n \geq 1\}$ are widely lower orthant dependent (WLOD) if there exists a finite real sequence $\{g_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, +\infty)$, $1 \leq i \leq n$,

$$P_r\left(\bigcap_{i=1}^{n}\{\eta_i \leq x_i\}\right) \leq g_L(n)\prod_{i=1}^{n} P_r(\eta_i \leq x_i); \tag{2.6}$$

if they are both WUOD and WLOD, then we say that the r.v.s $\{\eta_n, n \geq 1\}$ are widely orthant dependent (WOD). WUOD, WLOD, and WOD r.v.s are called, by a joint name, wide dependence (WD) r.v.s, and $g_U(n)$, $g_L(n)$, $n \geq 1$, are called dominating coefficients.

Wang *et al.* [10] also gave some examples of WD r.v.s with various dominating coefficients which show that WD r.v.s contain some common negatively dependent r.v.s, some positively dependent r.v.s and some others.

From the definitions of WD, the following proposition can be obtained directly (see, *e.g.*, Wang *et al.* [10]).

Proposition 2.1 (1) *Let* $\{\eta_n, n \geq 1\}$ *be WUOD (WLOD) with dominating coefficients* $g_U(n)$, $n \geq 1$ ($g_L(n)$, $n \geq 1$). *If* $\{f_n(\cdot), n \geq 1\}$ *are non-decreasing, then* $\{f_n(\eta_n), n \geq 1\}$ *are still WUOD (WLOD) with dominating coefficients* $g_U(n)$, $n \geq 1$ ($g_L(n)$, $n \geq 1$); *if* $\{f_n(\cdot), n \geq 1\}$ *are non-increasing, then* $\{f_n(\eta_n), n \geq 1\}$ *are WLOD (WUOD) with dominating coefficients* $g_U(n)$, $n \geq 1$ ($g_L(n)$, $n \geq 1$).

(2) *If* $\{\eta_n, n \geq 1\}$ *are non-negative and WUOD with dominating coefficients* $g_U(n)$, $n \geq 1$, *then for each* $n \geq 1$,

$$E\prod_{i=1}^{n}\eta_i \leq g_U(n)\prod_{i=1}^{n}E\eta_i.$$

In particular, if $\{\eta_n, n \geq 1\}$ *are WUOD with dominating coefficients* $g_U(n)$, $n \geq 1$, *then for each* $n \geq 1$ *and any* $s > 0$,

$$E\exp\left\{s\sum_{i=1}^{n}\eta_i\right\} \leq g_U(n)\prod_{i=1}^{n}E\exp\{s\eta_i\}.$$

For convenience, we introduce some notation. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(t) \asymp b(t)$ if $0 < \liminf_{t\to\infty} a(t)/b(t) \leq \limsup_{t\to\infty} a(t)/b(t) < \infty$. For two positive bivariate functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, we say that the asymptotic relation $a(n,x) \lesssim b(n,x)$ or $a(n,x) \gtrsim b(n,x)$ holds uniformly over all x in a nonempty set Δ, if $\limsup_{n\to\infty}\sup_{x\in\Delta} a(n,x)/b(n,x) \leq 1$ or $\liminf_{n\to\infty}\inf_{x\in\Delta} a(n,x)/b(n,x) \geq 1$. For a real number x, we write $x^+ = \max\{x,0\}$ and $x^- = -\min\{x,0\}$.

Before we state our main results, we will introduce some basic assumptions, to be used in this paper.

(A_1) There exist a real-valued random variable Y with its d.f. $F_Y(x) \in \mathcal{D}$, and some positive integer n_0, positive constants C_1 and T such that for all $n > n_0$,

$$\frac{1}{n}\sum_{i=1}^{n}\overline{F}_i(x) \geq C_1\overline{F}_Y(x), \tag{2.7}$$

holds uniformly for all $x \geq T$, and $E(Y^+)^s < \infty$ for some $s > 1$.

(A_2) There exist a real-valued random variable Z with its d.f. $F_Z(x) \in \mathcal{D}$, and positive constants C_2, C_3 and T such that for every $n \geq 1$,

$$\frac{1}{n}\sum_{i=1}^{n}\overline{F}_i(x) \leq C_2\overline{F}_Z(x) \tag{2.8}$$

holds uniformly for all $x \geq T$, and for n large enough,

$$\frac{1}{n}\sum_{i=1}^{n}F_i(x) \leq C_3F_Z(x) \tag{2.9}$$

holds uniformly for all $x \leq -T$; and $EZ < \infty$, $J^-_{F_Z} > 1$.

(A_3)

$$\limsup_{x\to\infty}\overline{F}_Z(x)/\overline{F}_Y(x) < \infty. \tag{2.10}$$

(A_4)

$$F_Z(-x) = o\big(\overline{F}_Y(x)\big). \tag{2.11}$$

Remark 2.1 According to (2.7), (2.8), and (2.10), we can see that the r.v. Z's and Y's right tails are weak equivalent, *i.e.*, $\overline{F}_Z(x) \asymp \overline{F}_Y(x)$. The assumption ($A_4$), which is equivalent to $F_Z(-x) = o(\overline{F}_Z(x))$, shows the r.v. Z's left tails are lighter than the r.v. Y's right tails. It is clear that all assumptions (A_1)-(A_4) are easily satisfied.

The main results of this paper are given below.

Theorem 2.1 *Let the random variables* $\{X_n, n \geq 1\}$ *introduced in Section* 1 *be WOD and, for some* $r > 1$, $E(X_n^-)^r < \infty$, $n \geq 1$. *If the assumptions* (A_1)-(A_4) *hold and there exists a positive number* $\beta < J_{F_Z}^- - 1$ *such that*

$$g_L(n) = o(n^\beta) \quad \text{and} \quad g_U(n) = o(n^\beta), \tag{2.12}$$

then (1.1) *and* (1.2) *hold, respectively.*

Theorem 2.2 *In addition to the conditions of Theorem* 2.1, *if one of the following two conditions is satisfied*:

(I) *when* $c \geq 0$, *we have for arbitrarily fixed* $\omega > 0$ *and some* $r > J_{F_Y}^+$

$$E\bigl[N(t)^r \mathbf{1}_{(N(t)>(1+\omega)\lambda(t))}\bigr] = O\bigl(\lambda(t)\bigr); \tag{2.13}$$

(II) *when* $c < 0$, *we have for all* $0 < \omega < 1$

$$\lim_{t\to\infty} \frac{P_r(N(t) \leq (1-\omega)\lambda(t))}{\overline{F}_Y(\lambda(t))} = 0, \tag{2.14}$$

then (1.3) *and* (1.4) *hold, respectively.*

Remark 2.2 According to (3.13), (3.15), and (3.16), we can take

$$M_1 = (C_1/C_2) \liminf_{x\to\infty} \bigl(\overline{F}_Y(ux/a)/\overline{F}_Z(x/b)\bigr)$$

and

$$M_2 = (C_2/C_1) \limsup_{x\to\infty} \bigl(\overline{F}_Z(\nu x/b)/\overline{F}_Y(x/a)\bigr)$$

in Theorem 2.1, respectively, where a, b, u, ν, C_1, and C_2 are some fixed positive constants. For the given distribution functions $\overline{F}_Z(x)$ and $\overline{F}_Y(x)$, we can obtain the sharp lower and upper bound M_1, M_2. Hence, though the above expressive forms are not nice-looking, causes no trouble for real applications. For $\widetilde{M}_1$ and $\widetilde{M}_2$, by the proof of Theorem 2.2, we can make a similar remark.

3 Proof of Theorem 2.1

We start with a series of lemmas based on which Theorem 2.1 will be proved. The proofs of the following Lemmas 3.1 and 3.2 are straightforward and are therefore omitted.

Lemma 3.1 *If* $E(X^+)^s < \infty$ *for some* $s > 0$, *then the relation*

$$\lim_{n\to\infty} \sup_{x\geq\gamma n} n^s \overline{F}(x) = 0$$

holds for arbitrarily fixed $\gamma > 0$.

Lemma 3.2 *If $E(X_i^{\pm})^q < \infty$, $i \geq 1$, and $E(Z^{\pm})^q < \infty$ for some $q \geq 1$, and* (2.8), (2.9) *hold, then there exists positive constant $\hat{\mu}_q^{\pm} < \infty$ such that*

$$\sum_{i=1}^{n} E\bigl(X_i^{\pm}\bigr)^q \leq n\hat{\mu}_q^{\pm}$$

holds for any $n = 1, 2, \ldots$.

Lemma 3.3 *Let $\{X_n, n \geq 1\}$ be a sequence of real-valued and WUOD r.v.s with X_n's d.f. $F_n(x) = 1 - \overline{F}_n(x)$ and $\mu_n = EX_n = 0$ for every $n \geq 1$, and let $\{c_1, c_2, \ldots\}$ be a sequence of real numbers satisfying $0 < a \leq c_i \leq b < \infty$, $i \geq 1$. If the assumptions* (A_1)-(A_3) *hold and there exists a positive number $\beta > 0$ such that*

$$g_U(n) = O\bigl(n^{\beta}\bigr), \tag{3.1}$$

then there exists a constant $M_2 > 0$ such that the relation

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \sup_{\underline{c}_n\in[a,b]^n} \frac{P_r(\sum_{i=1}^{n} c_i X_i > x)}{\sum_{i=1}^{n} P_r(c_i X_i > x)} \leq M_2 \tag{3.2}$$

holds for arbitrarily fixed constant $\gamma > 0$, where $\underline{c}_n = (c_1, c_2, \ldots, c_n)$.

Proof For arbitrarily fixed $0 < \nu < 1$, let $\widetilde{X}_i = \min\{X_i, \nu x/c_i\}$, $i \geq 1$ and $\widetilde{S}_n = \sum_{i=1}^{n} c_i \widetilde{X}_i$. Using the standard truncation technique, we have

$$P_r\left(\sum_{i=1}^{n} c_i X_i > x\right) \leq \sum_{i=1}^{n} P_r(c_i X_i > \nu x) + P_r(\widetilde{S}_n > x). \tag{3.3}$$

Now, we deal with the second term on the right-hand side of (3.3). Let $c = c(n, x, \underline{c}_n) = \max\{-\log\sum_{k=1}^{n} P_r(c_k X_k > \nu x), 1\}$, then we can obtain $\limsup_{n\to\infty}\sup_{x\geq\gamma n}\sup_{\underline{c}_n\in[a,b]^n} c = \infty$ according to (2.8) and Lemma 3.1. Write $\mathbf{W} = P_r(\widetilde{S}_n > x)/\sum_{i=1}^{n} P_r(c_i X_i > \nu x)$. For a positive number $h = h(n, x, \underline{c}_n) > 0$, which we shall specify later, by Chebyshev's inequality and Proposition 2.1, we have

$$\begin{aligned}
\mathbf{W} &\leq g_U(n)\exp\left\{-hx + c + \sum_{i=1}^{n} \log Ee^{hc_i\widetilde{X}_i}\right\} \\
&\leq g_U(n)\exp\left\{-hx + c + \sum_{i=1}^{n} \log\left[\int_{-\infty}^{\nu x/c_i} \bigl(e^{hc_i y} - 1\bigr)\,dF_i(y)\right.\right. \\
&\quad \left.\left. + \bigl(e^{h\nu x} - 1\bigr)P_r(c_i X_i > \nu x) + 1\right]\right\} \\
&\leq g_U(n)\exp\left\{-hx + c + \sum_{i=1}^{n}\left[\int_{-\infty}^{\nu x/c_i} \bigl(e^{hc_i y} - 1\bigr)\,dF_i(y)\right.\right. \\
&\quad \left.\left. + \bigl(e^{h\nu x} - 1\bigr)P_r(c_i X_i > \nu x)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le g_U(n)\exp\Bigg\{-hx+c+\sum_{i=1}^{n}\int_{-\infty}^{0}\left(e^{hc_iy}-1\right)dF_i(y) \\
&\quad+\sum_{i=1}^{n}\left(\int_{0}^{\frac{\nu x}{c_ic^{\kappa}}}+\int_{\frac{\nu x}{c_ic^{\kappa}}}^{\frac{\nu x}{c_i}}\right)\left(e^{hc_iy}-1\right)dF_i(y)+\left(e^{h\nu x}-1\right)\sum_{i=1}^{n}P_r(c_iX_i>\nu x)\Bigg\} \\
&= g_U(n)\exp\Bigg\{-hx+c+I_1+I_2+I_3+\left(e^{h\nu x}-1\right)\sum_{i=1}^{n}P_r(c_iX_i>\nu x)\Bigg\},
\end{aligned}
\tag{3.4}
$$

where $\kappa > 1$ is an arbitrarily fixed constant. Take $h = (c - \kappa p_2 \log c)/\nu x$, where $p_2 > J^+_{F_Y}$. By (2.4) and (2.7),

$$
\begin{aligned}
\limsup_{n\to\infty}\sup_{x\ge\gamma n}\sup_{\underline{c}_n\in[a,b]^n} h &\le \limsup_{n\to\infty}\sup_{x\ge\gamma n}\frac{-\log C_1 n\overline{F}_Y(\nu x/a)}{\nu x} \\
&\le \limsup_{n\to\infty}\sup_{x\ge\gamma n}\frac{-\log C_1 n(\frac{\nu x}{a})^{-p_2}}{\nu x} = 0.
\end{aligned}
$$

For I_1, we have

$$
\begin{aligned}
\frac{I_1}{hn} &= \int_{-\infty}^{0}\left(-\frac{1}{n}\sum_{i=1}^{n}F_i(y)c_ie^{hc_iy}\right)dy \\
&\le b\left(\int_{-\infty}^{-T}+\int_{-T}^{0}\right)\frac{1}{n}\sum_{i=1}^{n}F_i(y)\left(1-e^{hc_iy}\right)dy-\frac{1}{n}\sum_{i=1}^{n}c_iEX_i^-.
\end{aligned}
\tag{3.5}
$$

Denote $g(n,h,y,\underline{c}_n) = (\sum_{i=1}^{n} F_i(y)(1-e^{hc_iy}))/n$ and $g_n(y) = \sup_{x\ge\gamma n}\sup_{\underline{c}_n\in[a,b]^n} g(n,h,y,\underline{c}_n)$. By (2.9), we see that there exists $N > 0$ such that for $n \ge N$ and all $y \le -T$, $|g_n(y)| \le C_3F_Z(y)$. And for any $[s,t] \subset (-\infty,-T]$, $|g_n(y)| \le \sup_{x\ge\gamma n}\sup_{\underline{c}_n\in[a,b]^n}(1-e^{bhs}) \to 0$, $n\to\infty$ for all $y \in [s,t]$. Hence,

$$
\begin{aligned}
\limsup_{n\to\infty}\sup_{x\ge\gamma n}\sup_{\underline{c}_n\in[a,b]^n}\int_{-\infty}^{-T} g(n,h,y,\underline{c}_n)\,dy &\le \limsup_{n\to\infty}\int_{-\infty}^{-T} g_n(y)\,dy \\
&= \int_{-\infty}^{-T}\lim_{n\to\infty} g_n(y)\,dy = 0.
\end{aligned}
\tag{3.6}
$$

From the definition of $g_n(y)$, we know that $|g_n(y)| \le 1$ for every n and all $y \in [-T,0)$, and $g_n(y) \to 0$, $n\to\infty$ for all $y\in[-T,0)$. So,

$$
\begin{aligned}
\limsup_{n\to\infty}\sup_{x\ge\gamma n}\sup_{\underline{c}_n\in[a,b]^n}\int_{-T}^{0} g(n,h,y,\underline{c}_n)\,dy &\le \limsup_{n\to\infty}\int_{-T}^{0} g_n(y)\,dy \\
&= \int_{-T}^{0}\lim_{n\to\infty} g_n(y)\,dy = 0.
\end{aligned}
\tag{3.7}
$$

Combining (3.6), (3.7) with (3.5), we obtain

$$
I_1 = \varphi nh - h\sum_{i=1}^{n} c_iEX_i^-,
\tag{3.8}
$$

where $\limsup_{n\to\infty}\sup_{x\geq\gamma n}\sup_{\underline{c}_n\in[a,b]^n}\varphi = 0$. For I_2 and I_3, we have

$$\begin{aligned} I_2 + I_3 &\leq \sum_{i=1}^{n}\left(\int_0^{\frac{\nu x}{c_i c^\kappa}} hc_i y e^{hc_i y}\,dF_i(y) + \int_{\frac{\nu x}{c_i c^\kappa}}^{\frac{\nu x}{c_i}} e^{hc_i y}\,dF_i(y)\right) \\ &\leq \sum_{i=1}^{n}\left(hc_i e^{\frac{h\nu x}{c^\kappa}} EX_i^+ + e^{h\nu x}\overline{F}_i\left(\frac{\nu x}{c_i c^\kappa}\right)\right), \end{aligned} \tag{3.9}$$

where $\limsup_{n\to\infty}\sup_{x\geq\gamma n}\sup_{\underline{c}_n\in[a,b]^n} x/c^\kappa \to \infty$ according to (2.4) and (2.7). Plugging (3.8) and (3.9) into (3.4) yields

$$\begin{aligned} \mathbf{W} &\leq g_U(n)\exp\left\{-hx + c + \varphi hn + hb\sum_{i=1}^{n} EX_i^+\left(e^{\frac{h\nu x}{c^\kappa}} - 1\right)\right. \\ &\qquad \left. + C_2 n\overline{F}_Z\left(\frac{\nu x}{bc^\kappa}\right)e^{h\nu x} + C_2 n\overline{F}_Z\left(\frac{\nu x}{b}\right)\left(e^{h\nu x} - 1\right)\right\} \\ &\leq g_U(n)\exp\left\{-hx + c + \left[\varphi + b\hat{\mu}_1^+\left(e^{hu\nu/c^\kappa} - 1\right)\right]hn\right. \\ &\qquad \left. + C_2\widetilde{C}_2 c^{\kappa p_2} n\overline{F}_Z\left(\frac{\nu x}{b}\right)e^{h\nu x} + C_2 n\overline{F}_Z\left(\frac{\nu x}{b}\right)e^{h\nu x}\right\} \\ &\leq g_U(n)\exp\left\{-hx + c + \left[\varphi + b\hat{\mu}_1^+\left(e^{hu\nu/c^\kappa} - 1\right)\right]hn\right. \\ &\qquad \left. + \hat{B}\left(c^{\kappa p_2} + 1\right)n\overline{F}_Z\left(\frac{\nu x}{b}\right)e^{h\nu x}\right\}, \end{aligned} \tag{3.10}$$

where in the first step we apply (2.8), in the second step we use Lemma 3.2 and (2.2), and $\hat{B}$ is an appropriately chosen positive number. By the value of h, we have

$$\limsup_{n\to\infty}\sup_{x\geq\gamma n}\sup_{\underline{c}_n\in[a,b]^n}\hat{B}\left(c^{\kappa p_2} + 1\right)n\overline{F}_Z\left(\frac{\nu x}{b}\right)e^{h\nu x} < \infty$$

and

$$\limsup_{n\to\infty}\sup_{x\geq\gamma n}\sup_{\underline{c}_n\in[a,b]^n}\frac{[\varphi + b\hat{\mu}_1^+(e^{hu\nu/c^\kappa} - 1)]hn + (\kappa p_2\log c)/\nu}{c} = 0.$$

Hence,

$$\mathbf{W} \leq g_U(n)n^{-\beta}\exp\left\{\beta\log n + \left(1 - \frac{1}{\nu}\right)c + o(c) + O(1)\right\}. \tag{3.11}$$

Using (2.3) and (2.8), we obtain

$$\limsup_{n\to\infty}\sup_{x\geq\gamma n}\sup_{\underline{c}_n\in[a,b]^n}\frac{\log n}{c} \leq \frac{1}{p_1 - 1},$$

where $1 < p_1 < J_{F_Z}^-$. Taking $0 < \nu < (p_1 - 1)/(\beta + p_1 - 1) < 1$, from (3.11), we have

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \sup_{\underline{c}_n\in[a,b]^n} \mathbf{W} \leq \limsup_{n\to\infty} \sup_{x\geq\gamma n} \sup_{\underline{c}_n\in[a,b]^n} g_U(n) n^{-\beta} \exp\left\{\left(\frac{\beta}{p_1-1} + 1 - \frac{1}{\nu}\right)c + o(c) + O(1)\right\} = 0. \tag{3.12}$$

Applying (2.7), (2.8), and (2.10), we know that there exists $M_2 > 0$ such that

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \sup_{\underline{c}_n\in[a,b]^n} \frac{\sum_{i=1}^n P_r(c_i X_i > \nu x)}{\sum_{i=1}^n P_r(c_i X_i > x)} \leq M_2. \tag{3.13}$$

Combining (3.12), (3.13) with (3.3), we can obtain (3.2). □

Lemma 3.4 *Let $\{X_n, n \geq 1\}$ be a sequence of real-valued and WOD r.v.s with X_n's d.f. $F_n(x) = 1 - \overline{F}_n(x)$ and $\mu_n = EX_n = 0$ for every $n \geq 1$, and let $\{c_1, c_2, \ldots\}$ be a sequence of real numbers satisfying $0 < a \leq c_i \leq b < \infty$, $i \geq 1$. If the assumptions* (A$_1$)-(A$_4$) *hold and there exists a positive number $\beta < J_{F_Z}^- - 1$ such that*

$$g_U(n) = o(n^\beta) \quad \text{and} \quad g_L(n) = o(n^\beta), \tag{3.14}$$

then there exists a constant $M_1 > 0$ such that the relation

$$\liminf_{n\to\infty} \inf_{x\geq\gamma n} \inf_{\underline{c}_n\in[a,b]^n} \frac{P_r(\sum_{i=1}^n c_i X_i > x)}{\sum_{i=1}^n P_r(c_i X_i > x)} \geq M_1 \tag{3.15}$$

holds for arbitrarily fixed constant $\gamma > 0$, where $\underline{c}_n = (c_1, c_2, \ldots, c_n)$.

Proof It is sufficient to prove that

$$\liminf_{n\to\infty} \inf_{x\geq\gamma n} \inf_{\underline{c}_n\in[a,b]^n} \frac{P_r(\sum_{i=1}^n c_i X_i > x)}{\sum_{i=1}^n P_r(c_i X_i > ux)} \geq 1 \tag{3.16}$$

holds for arbitrarily fixed $u > 1$. We write $A_k = \{c_k X_k > ux\}$ and $B_k = \bigcap_{1\leq i\neq k\leq n} A_i^c$. Observing that $A_k \cap B_k$, $k = 1, 2, \ldots, n$, are mutually disjoint, we have

$$\begin{aligned} P_r\left(\sum_{i=1}^n c_i X_i > x\right) &\geq \sum_{k=1}^n \left[P_r(A_k) - P_r(A_k \cap B_k^c) - P_r\left(\sum_{i=1}^n c_i X_i \leq x, A_k \cap B_k\right)\right] \\ &\geq \sum_{k=1}^n \left[P_r(A_k) - \sum_{1\leq i\neq k\leq n} P_r(A_k \cap A_i) \right. \\ &\quad \left. - P_r\left(\sum_{i:1\leq i\leq n, i\neq k} c_i X_i \leq (1-u)x, A_k \cap B_k\right)\right] \\ &\geq \sum_{k=1}^n P_r(A_k)\left[1 - g_U(n) \sum_{k=1}^n P_r(A_k)\right] \\ &\quad - \sum_{k=1}^n P_r\left(\sum_{i:1\leq i\leq n, i\neq k} c_i X_i \leq (1-u)x, A_k \cap B_k\right) = J_1 - J_2, \end{aligned} \tag{3.17}$$

where at the third step we used Proposition 2.1.

For J_1, by (2.3), (2.8), and (3.14), we have for arbitrarily fixed $\gamma > 0$

$$\begin{aligned}
&\liminf_{n\to\infty} \inf_{x\geq\gamma n} \inf_{\underline{c}_n\in[a,b]^n} \frac{J_1}{\sum_{k=1}^n P_r(c_kX_k > ux)} \\
&\quad\geq 1 - \limsup_{n\to\infty} \sup_{x\geq\gamma n} g_U(n)nC_2\overline{F}_Z\left(\frac{ux}{b}\right) \\
&\quad\geq 1 - \limsup_{n\to\infty} C_2 g_U(n)n^{1-p_1}\left(\frac{b}{u\gamma}\right)^{p_1} = 1,
\end{aligned} \tag{3.18}$$

where $p_1 = \beta + 1$.

Now we deal with J_2. For fixed $0 < w < 1$, Let $W_k = -X_k$ and $\widetilde{W}_k = \min\{W_k, (wx)/c_k\}$, then

$$\begin{aligned}
J_2 &\leq \sum_{k=1}^n P_r\left(A_k \cap B_k, \bigcup_{i=1}^n \{c_iW_i > wx\}\right) \\
&\quad + \sum_{k=1}^n P_r\left(\sum_{i:1\leq i\leq n, i\neq k} c_iX_i \leq (1-u)x, \bigcap_{j=1}^n \{c_jW_j \leq wx\}\right) \\
&\leq \sum_{i=1}^n P_r(c_iW_i > wx) + \sum_{k=1}^n P_r\big(c_k\widetilde{W}_k \geq (u-1)x\big) = K_1 + K_2.
\end{aligned} \tag{3.19}$$

For K_1, by (2.7), (2.9)-(2.11), we have

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \sup_{\underline{c}_n\in[a,b]^n} \frac{K_1}{\sum_{i=1}^n P_r(c_iX_i > ux)} \leq \limsup_{x\to\infty} \frac{C_3F_Z(-\frac{w}{b}x)}{C_1\overline{F}_Y(\frac{u}{a}x)} = 0. \tag{3.20}$$

For K_2, using Chebyshev's inequality and Proposition 2.1 again, we have

$$\begin{aligned}
K_2 &\leq \sum_{k=1}^n g_L(n-1)\exp\left\{-h(u-1)x + \sum_{i:1\leq i\leq n, i\neq k} \log Ee^{hc_i\widetilde{W}_i}\right\} \\
&\leq \sum_{k=1}^n g_L(n-1)\exp\Bigg\{-h(u-1)x \\
&\quad + \sum_{i:1\leq i\leq n, i\neq k}\left[\int_{-\infty}^{\frac{wx}{c_i}} \big(e^{hc_iy}-1\big)\,dF_{W_i}(y) + \big(e^{hwx}-1\big)P_r(c_iW_i > wx)\right]\Bigg\} \\
&\leq \sum_{k=1}^n g_L(n-1)\exp\Bigg\{-h(u-1)x + \sum_{i:1\leq i\leq n, i\neq k}\Bigg[\int_{-\infty}^0 \big(e^{hc_iy}-1\big)\,dF_{W_i}(y) \\
&\quad + \int_0^{\frac{wx}{c_i}} \frac{e^{hc_iy}-1-hc_iy}{(c_iy)^s}(c_iy)^s\,dF_{W_i}(y) + hc_iEX_i^- \\
&\quad + \big(e^{hwx}-1\big)P_r(c_iW_i > wx)\Bigg]\Bigg\},
\end{aligned} \tag{3.21}$$

where $h = (\log(w^{s-1}x^s/n\hat{\mu}_s^- + 1))/wx$ and $s > 1$. Using similar techniques as in (3.8), we can obtain

$$\left|\sum_{i:1\leq i\leq n, i\neq k}\left[\int_{-\infty}^0 \big(e^{hc_iy}-1\big)\,dF_{W_i}(y) + hc_iEX_i^-\right]\right| \leq |\alpha|nh, \tag{3.22}$$

where $\alpha \to 0$ holds uniformly for $x \geq \gamma n$, as $n \to \infty$. By (2.9) and (2.11),

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{x\geq\gamma n} \sup_{\underline{c}_n\in[a,b]^n} \sum_{i:1\leq i\leq n, i\neq k} \left(e^{hwx}-1\right) P_r(c_i W_i > wx) \\
&\quad \leq \limsup_{x\to\infty} \frac{w^{s-1}}{\hat{\mu}_s^-} C_3 x^s \overline{F}_Y\left(\frac{b}{w}x\right) \to 0. \qquad (3.23)
\end{aligned}$$

Take sufficiently large n such that $|\alpha| \leq (u-1)\gamma/2$. Combining (3.21)-(3.23) and observing the monotonicity of $0 \leq (e^{hc_i y} - 1 - hc_i y)/(c_i y)^s$ for all $y > 0$, we have

$$\begin{aligned}
K_2 &\leq \sum_{k=1}^n g_L(n-1) n^{-\beta} \exp\Biggl\{\beta \log n - h(u-1)x + |\alpha| nh \\
&\quad + \frac{e^{hwx} - 1 - hwx}{(wx)^s} \sum_{i:1\leq i\leq n, i\neq k} c_i^s E\left(X_i^-\right)^s + o(1)\Biggr\} \\
&\leq \sum_{k=1}^n g_L(n-1) n^{-\beta} \exp\Biggl\{\log\left(\frac{x}{\gamma}\right)^\beta + \log\left(\frac{w^{s-1}x^s}{n\hat{\mu}_s^-} + 1\right)^{-\frac{u-1}{2w}} \\
&\quad + \frac{e^{hwx}-1}{(wx)^s} b^s n \hat{\mu}_s^- + o(1)\Biggr\} \leq \widetilde{C} n x^{\beta - \frac{(s-1)(u-1)}{2w}}, \qquad (3.24)
\end{aligned}$$

where $\widetilde{C}$ is some positive constant. For fixed $u > 1$, we take $0 < w < 1$ such that $\frac{(s-1)(u-1)}{2w} > \beta + J_{F_Y}^+$. By (2.4) and (2.7), we have

$$\limsup_{n\to\infty} \sup_{x\geq\gamma n} \sup_{\underline{c}_n\in[a,b]^n} \frac{K_2}{\sum_{k=1}^n P_r(c_k X_k > ux)} \leq \limsup_{x\to\infty} \frac{\widetilde{C} x^{\beta - \frac{(s-1)(u-1)}{2w}}}{C_1 \overline{F}_Y(\frac{u}{a}x)} = 0. \qquad (3.25)$$

Combing (3.17)-(3.20) with (3.25), we can obtain (3.16). This ends the proof of Lemma 3.4. □

Proof of Theorem 2.1 By Lemma 3.3, for arbitrarily fixed $\gamma > 0$, we have uniformly for $x \geq \gamma n$

$$\begin{aligned}
P_r\left(\sum_{k=1}^n \theta_k X_k > x\right) &= E\left[P_r\left(\sum_{k=1}^n \theta_k X_k > x | \theta_1, \theta_2, \ldots, \theta_n\right)\right] \\
&\lesssim M_2 E\left[\sum_{k=1}^n P_r(\theta_k X_k > x | \theta_1, \theta_2, \ldots, \theta_n)\right] \\
&= M_2 \sum_{k=1}^n P_r(\theta_k X_k > x). \qquad (3.26)
\end{aligned}$$

According to Lemma 3.4 and using a similar method of proof as in (3.26), we can obtain the remainder of Theorem 2.1. □

4 Proof of Theorem 2.2

For proving Theorems 2.2, we first give two lemmas.

Lemma 4.1 *Let* $\{X_n, n \geq 1\}$ *be a sequence of real-valued and WUOD r.v.s with* X_n*'s d.f.* $F_n(x) = 1 - \overline{F}_n(x)$ *and* $0 < EX_n^+ < 0$ *for every* $n \geq 1$, *and Let* $\{\theta_i, i \geq 1\}$ *be a sequence of non-negative r.v.s satisfying* $P_r(a \leq \theta_i \leq b) = 1$, $i \geq 1$, $0 < a \leq b < \infty$ *and independent of* $\{X_n, n \geq 1\}$. *If* (2.8) *and* $EZ^+ < \infty$ *hold and* $g_U(n) = O(n^\beta)$ *for some positive number* β, *then for every fixed* $u > 0$, *there is* $\hat{D} = \hat{D}(u) > 0$ *such that*

$$P_r\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \leq \sum_{i=1}^{n} P_r(\theta_i X_i > ux) + \hat{D}\left(\frac{n}{x}\right)^{\frac{1}{u}} n^\beta \tag{4.1}$$

holds for large n *and all* $x > 0$.

Proof Using the techniques similar to Lemma 3.3 with some obvious modifications, we can prove the lemma. □

Combining Lemma 2.1 of Chen *et al.* [6] with Lemma 3.1 of Ng *et al.* [11], we can obtain the following lemma.

Lemma 4.2 *If a non-negative random process* $\{\zeta(t), t \geq 0\}$ *satisfies* $E\zeta(t) \to 1$, $t \to \infty$, *then* (i)-(iv) *are mutually equivalent*:

(i) $\zeta(t) \xrightarrow{P_r} 1$, *as* $t \to \infty$;
(ii) *for every fixed* $\theta > 0$, $E\zeta(t)\mathbf{1}_{\{\zeta(t)-1>\theta\}} = o(1)$;
(iii) *for every fixed* $\theta > 0$, $E\zeta(t)\mathbf{1}_{\{|\zeta(t)-1|>\theta\}} = o(1)$;
(iv) *for every fixed* $0 < \theta < 1$, $P_r(1 - \zeta(t) \geq \theta) = o(1)$.

By Lemma 4.2 and (2.13), we know that

$$\frac{N(t)}{\lambda(t)} \xrightarrow{P_r} 1. \tag{4.2}$$

Proof of Theorem 2.2 Now, we prove Theorem 2.2 under condition (I). Using Theorem 2.1 and (2.8), we obtain for any fixed $\sigma > 0$ the result that there exists a positive integral number N such that when $n \geq N$, for sufficiently large x,

$$P_r\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \leq (M_2 + \sigma) \sum_{i=1}^{n} P_r(\theta_i X_i > x) \leq (M_2 + \sigma) C_2 n \overline{F}_Z\left(\frac{x}{b}\right). \tag{4.3}$$

It is clear that for every $n = 1, 2, \ldots, N$ and all large x,

$$P_r\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \leq \sum_{i=1}^{n} P_r(\theta_i X_i > x/N) \leq C_2 n \overline{F}_Z\left(\frac{x}{Nb}\right). \tag{4.4}$$

Hence, by (4.3) and (4.4), there exists some positive number D, for every $n = 1, 2, \ldots,$ and all sufficiently large x,

$$P_r\left(\sum_{i=1}^{n} \theta_i X_i > x\right) \leq D n \overline{F}_Z(x). \tag{4.5}$$

Take $0 < \omega < 1$ such that $c(1+\omega) < \gamma$. Throughout this proof, we suppose that $x \in [\gamma\lambda(t), \infty)$. Consider the following decomposition:

$$\begin{aligned} P_r\big(S_c^\theta(t) > x\big) &= \Bigg(\sum_{n<(1-\omega)\lambda(t)} + \sum_{(1-\omega)\lambda(t)\le n\le(1+\omega)\lambda(t)} + \sum_{n>(1+\omega)\lambda(t)}\Bigg) P_r\big(S_n^\theta > x - nc\big)P_r\big(N(t)=n\big) \\ &= L_1 + L_2 + L_3. \end{aligned} \tag{4.6}$$

Firstly, we deal with L_1. For sufficiently large t, by (4.5), we have

$$\begin{aligned} L_1 &\le \sum_{n<(1-\omega)\lambda(t)} P_r\big(S_n^\theta > x-(1-\omega)c\lambda(t)\big)P_r\big(N(t)=n\big) \\ &\le D\overline{F}_Z\big(x-(1-\omega)c\lambda(t)\big)\sum_{n<(1-\omega)\lambda(t)} nP_r\big(N(t)=n\big). \end{aligned}$$

For convenience, write $H = E[\sum_{i=1}^{N(t)} P_r(\theta_i X_i > x - c\lambda(t))]$. According to $x-(1-\omega)c\lambda(t) \asymp x - c\lambda(t)$, (2.7), (2.10), and (iv) of Lemma 4.2, we have

$$\begin{aligned} &\limsup_{t\to\infty}\sup_{x\ge\gamma\lambda(t)}\frac{L_1}{H} \\ &\quad\le \limsup_{t\to\infty}\sup_{x\ge\gamma\lambda(t)}\frac{D\overline{F}_Z(x-(1-\omega)c\lambda(t))\sum_{n<(1-\omega)\lambda(t)} nP_r(N(t)=n)}{C_1\overline{F}_Y(\frac{x-c\lambda(t)}{a})E[N(t)\mathbf{1}_{\{N(t)>n_0\}}]} \\ &\quad\le \widetilde{D}\limsup_{t\to\infty}\frac{\sum_{n<(1-\omega)\lambda(t)} nP_r(N(t)=n)}{EN(t)-n_0} \le \widetilde{D}\limsup_{t\to\infty}\frac{E[\frac{N(t)}{\lambda(t)}\mathbf{1}_{\{N(t)<(1-\omega)\lambda(t)\}}]}{1-\frac{n_0}{\lambda(t)}} \\ &\quad\le \widetilde{D}(1-\omega)\lim_{t\to\infty}P_r\bigg(\frac{N(t)}{\lambda(t)} < 1-\omega\bigg) = 0, \end{aligned} \tag{4.7}$$

where $\widetilde{D}$ is some positive constant.

Secondly, we deal with L_2. On the one hand, by Theorem 2.1, for arbitrary $\varepsilon_1 > 0$ and sufficiently large t,

$$\begin{aligned} L_2 &\ge (M_1-\varepsilon_1)\sum_{(1-\omega)\lambda(t)\le n\le(1+\omega)\lambda(t)}\sum_{i=1}^{n} P_r(\theta_i X_i > x - nc)P_r\big(N(t)=n\big) \\ &\ge (M_1-\varepsilon_1)C_1\overline{F}_Y\bigg(\frac{x-(1-\omega)c\lambda(t)}{a}\bigg)\sum_{(1-\omega)\lambda(t)\le n\le(1+\omega)\lambda(t)} nP_r\big(N(t)=n\big). \end{aligned} \tag{4.8}$$

By (2.8), we have $H \le C_2\overline{F}_Z((x-c\lambda(t))/b)\lambda(t)$. Using (2.10) and (iii) of Lemma 4.2, we know that there is a positive constant $\widetilde{M}_1$ such that

$$\liminf_{t\to\infty}\inf_{x\ge\gamma\lambda(t)}\frac{L_2}{H} \ge \widetilde{M}_1\lim_{t\to\infty}E\bigg[\frac{N(t)}{\lambda(t)}\mathbf{1}_{\{|\frac{N(t)}{\lambda(t)}-1|\le\omega\}}\bigg] = \widetilde{M}_1. \tag{4.9}$$

On the other hand, using similar techniques as in (4.8), for any $\varepsilon_2 > 0$ and sufficiently large t, we have

$$L_2 \le (M_2+\varepsilon_2)C_2\overline{F}_Z\bigg(\frac{x-(1+\omega)c\lambda(t)}{b}\bigg)\sum_{(1-\omega)\lambda(t)\le n\le(1+\omega)\lambda(t)} nP_r\big(N(t)=n\big). \tag{4.10}$$

By (2.7), we have

$$H \geq C_1 \overline{F}_Y\left(\frac{x - c\lambda(t)}{a}\right) E\big[N(t)\mathbf{1}_{\{N(t)>n_0\}}\big]. \tag{4.11}$$

Then, by (4.10) and (4.11), there exists a positive number $\widetilde{M}_2$ such that

$$\limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{L_2}{H} \leq \widetilde{M}_2 \lim_{t\to\infty} \frac{E[\frac{N(t)}{\lambda(t)}\mathbf{1}_{\{|\frac{N(t)}{\lambda(t)}-1|\leq\omega\}}]}{1 - \frac{n_0}{\lambda(t)}} = \widetilde{M}_2. \tag{4.12}$$

Finally, we deal with L_3. Taking $0 < \upsilon < 1$ and splitting L_3 into two parts, we obtain

$$\begin{aligned} L_3 &= \left(\sum_{(1+\omega)\lambda(t)<n\leq(1-\upsilon)x/c} + \sum_{n>\max\{(1+\omega)\lambda(t),(1-\upsilon)x/c\}}\right) P_r\left(\sum_{i=1}^{n} \theta_i X_i > x - nc\right) P_r\big(N(t) = n\big) \\ &= R_1 + R_2, \end{aligned} \tag{4.13}$$

where R_1 is understood as 0 in case $(1+\omega)\lambda(t) > (1-\upsilon)x/c$. For R_1, taking $u = 1/p$ in (4.1) and letting $p > J^+_{F_Y}$, we have

$$\begin{aligned} R_1 &\leq \sum_{(1+\omega)\lambda(t)<n\leq(1-\upsilon)x/c} P_r\left(\sum_{i=1}^{n} \theta_i X_i > \upsilon x\right) P_r\big(N(t) = n\big) \\ &\leq \sum_{(1+\omega)\lambda(t)<n\leq(1-\upsilon)x/c} \left(\sum_{i=1}^{n} P_r(\theta_i X_i > \upsilon x/p) + \hat{D}(\upsilon x)^{-p} n^{p+\beta}\right) P_r\big(N(t) = n\big) \\ &\leq C_2 \overline{F}_Z\left(\frac{\upsilon x}{bp}\right) E\big[N(t)\mathbf{1}_{\{N(t)>(1+\omega)\lambda(t)\}}\big] + \hat{D}(\upsilon x)^{-p} E\big[N(t)^{p+\beta}\mathbf{1}_{\{N(t)>(1+\omega)\lambda(t)\}}\big]. \end{aligned} \tag{4.14}$$

Hence, according to (ii) of Lemma 4.2, (2.13), and (2.4), we have

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{R_1}{H} \\ &\quad\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{C_2 \overline{F}_Z(\frac{\upsilon x}{bp}) E[\frac{N(t)}{\lambda(t)}\mathbf{1}_{\{N(t)>(1+\omega)\lambda(t)\}}]}{C_1 \overline{F}_Y(\frac{x-c\lambda(t)}{a})(1 - \frac{n_0}{\lambda(t)})} \\ &\qquad+ \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{\hat{D}(\upsilon x)^{-p}}{C_1 \overline{F}_Y(\frac{x-c\lambda(t)}{a})} \times \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{E[N(t)^{p+\beta}\mathbf{1}_{\{N(t)>(1+\omega)\lambda(t)\}}]/\lambda(t)}{(1 - \frac{n_0}{\lambda(t)})} \\ &\quad= 0. \end{aligned} \tag{4.15}$$

For R_2, we have

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{R_2}{H} \\ &\quad\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{\sum_{n>\max\{(1+\omega)\lambda(t),(1-\upsilon)x/c\}} \frac{n^p}{((1-\upsilon)x/c)^p} P_r(N(t) = n)}{C_1 \overline{F}_Y(\frac{x-c\lambda(t)}{a}) E[N(t)\mathbf{1}_{\{N(t)>n_0\}}]} \\ &\quad\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{((1-\upsilon)x/c)^{-p} E[N(t)^p\mathbf{1}_{\{N(t)>(1+\omega)\lambda(t)\}}]/\lambda(t)}{C_1 \overline{F}_Y(\frac{x-c\lambda(t)}{a})(1 - \frac{n_0}{\lambda(t)})} = 0. \end{aligned} \tag{4.16}$$

Combing (4.6), (4.7), (4.9), (4.12), (4.13), and (4.15) with (4.16), we finish the proof under condition (I).

Finally, we prove Theorem 2.2 under condition (II). Without loss of generality, we assume $c < \gamma < 0$. We still take $0 < \omega < 1$ such that $c(1+\omega) < \gamma < 0$ and use the decomposition (4.6). For L_1, we take $\gamma_1 > 0$ and divide the interval $[\gamma\lambda(t), \infty)$ into two parts, which are $[\gamma\lambda(t), \gamma_1\lambda(t))$ and $[\gamma_1\lambda(t), \infty)$. When $x \in [\gamma_1\lambda(t), \infty)$, we have

$$\overline{F}_Z\big(x - c\lambda(t)\big) \geq \overline{F}_Z\big((1 - c/\gamma_1)x\big) \asymp \overline{F}_Z(x). \tag{4.17}$$

When $x \in [\gamma\lambda(t), \gamma_1\lambda(t))$, we have

$$\overline{F}_Y\big(x - c\lambda(t)\big) \geq \overline{F}_Y\big((\gamma_1 - c)\lambda(t)\big) \asymp \overline{F}_Y\big(\lambda(t)\big). \tag{4.18}$$

Applying (2.14), (4.5), (4.17), and (4.18), we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{L_1}{H} \\ &\quad\leq \limsup_{t\to\infty} \sup_{x\geq\gamma_1\lambda(t)} \frac{D(1-\omega)\overline{F}_Z(x)P_r(N(t) \leq (1-\omega)\lambda(t))}{C_1\overline{F}_Y(\frac{x-c\lambda(t)}{a})(1-\frac{n_0}{\lambda(t)})} \\ &\qquad+ \limsup_{t\to\infty} \sup_{\gamma\lambda(t)\leq x<\gamma_1\lambda(t)} \frac{D(1-\omega)P_r(N(t) \leq (1-\omega)\lambda(t))}{C_1\overline{F}_Y(\frac{x-c\lambda(t)}{a})(1-\frac{n_0}{\lambda(t)})} = 0. \end{aligned} \tag{4.19}$$

For L_3, since $x - cn \geq \gamma\lambda(t) - cn \geq (\gamma(1+\omega)^{-1} - c)n$, then according to Theorem 2.1, (2.10), and Lemma 4.2, we have

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{L_3}{H} \\ &\quad\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{M_2\sum_{n>(1+\omega)\lambda(t)}\sum_{i=1}^n P_r(\theta_iX_i > x - cn)P_r(N(t) = n)}{C_1\overline{F}_Y(\frac{x-c\lambda(t)}{a})E[N(t)\mathbf{1}_{\{N(t)>n_0\}}]} \\ &\quad\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{M_2C_2\overline{F}_Z(\frac{x-c\lambda(t)}{b})E[\frac{N(t)}{\lambda(t)}\mathbf{1}_{\{N(t)>(1+\omega)\lambda(t)\}}]}{C_1\overline{F}_Y(\frac{x-c\lambda(t)}{a})(1-\frac{n_0}{\lambda(t)})} = 0. \end{aligned} \tag{4.20}$$

For L_2, observing $\lambda(t) \leq (\gamma - c)^{-1}(x - c\lambda(t))$ and $c < 0$, by Theorem 2.1, (2.14), and (iii) of Lemma 4.2, we know that, on the one hand, there exists a positive number $\widetilde{M}_1$ such that

$$\begin{aligned} &\liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{L_2}{H} \\ &\quad\geq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{M_1\sum_{(1-\omega)\lambda(t)\leq n\leq(1+\omega)\lambda(t)}\sum_{i=1}^n P_r(\theta_iX_i > x - cn)P_r(N(t) = n)}{C_2\overline{F}_Z(\frac{x-c\lambda(t)}{b})\lambda(t)} \\ &\quad\geq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{M_1C_1\overline{F}_Y(\frac{x-c(1+\omega)\lambda(t)}{a})E[\frac{N(t)}{\lambda(t)}\mathbf{1}_{\{|\frac{N(t)}{\lambda(t)}-1|\leq\omega\}}]}{C_2\overline{F}_Z(\frac{x-c\lambda(t)}{b})} \\ &\quad\geq \liminf_{t\to\infty} \inf_{x\geq\gamma\lambda(t)} \frac{M_1C_1\overline{F}_Y(a^{-1}(1-c\omega(\gamma-c)^{-1})(x-c\lambda(t)))E[\frac{N(t)}{\lambda(t)}\mathbf{1}_{\{|\frac{N(t)}{\lambda(t)}-1|\leq\omega\}}]}{C_2\overline{F}_Z(\frac{x-c\lambda(t)}{b})} \\ &\quad\geq \widetilde{M}_1; \end{aligned} \tag{4.21}$$

on the other hand, there exists a positive number $\widetilde{M}_2$ such that

$$\begin{aligned}
&\limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{L_2}{H} \\
&\quad\leq \limsup_{t\to\infty} \sup_{x\geq\gamma\lambda(t)} \frac{M_2 C_2 \overline{F}_Z(b^{-1}(1+c\omega(\gamma-c)^{-1})(x-c\lambda(t)))E[\frac{N(t)}{\lambda(t)}\mathbf{1}_{\{|\frac{N(t)}{\lambda(t)}-1|\leq\omega\}}]}{C_1\overline{F}_Y(\frac{x-c\lambda(t)}{a})(1-\frac{n_0}{\lambda(t)})} \\
&\quad\leq \widetilde{M}_2. \qquad (4.22)
\end{aligned}$$

Combing (4.19)-(4.22), we finish the proof under condition (II). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]School of Science, Dalian Nationalities University, Liaohe West Road, Dalian, 116600, China. [2]School of Mathematical Sciences, Dalian University of Technology, Lingong Road, Dalian, 116024, China. [3]School of Mathematical Sciences, Capital Normal University, Xisanhuan North Road, Beijing, 100048, China.

Acknowledgements
The authors would like to thank the two referees for careful reading of our manuscript and for helpful and valuable comments and suggestions, which helped us improve the earlier version of the paper. The research of the authors was supported by the Natural Science Foundation of the Inner Mongolia Autonomous Region (No. 2013MS0101), the National Natural Sciences Foundation of China (No. 11201317), and the Beijing Municipal Education Commission Foundation (No. KM201210028005).

Received: 11 May 2013 Accepted: 25 March 2014 Published: 04 Apr 2014

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10.1186/1029-242X-2014-140
Cite this article as: Bai et al.: **Large deviations for randomly weighted sums with dominantly varying tails and widely orthant dependent structure.** *Journal of Inequalities and Applications* 2014, 2014:140