Chen and Ding *Journal of Inequalities and Applications* 2014, 2014:139
http://www.journalofinequalitiesandapplications.com/content/2014/1/139

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Vector-valued inequalities for the commutators of rough singular kernels

Yanping Chen[1*] and Yong Ding[2]

*Correspondence: yanpingch@126.com
[1]Department of Applied Mathematics, School of Mathematics and Physics, University of Science and Technology Beijing, Beijing, 100083, People's Republic of China
Full list of author information is available at the end of the article

Abstract
Vector-valued inequalities are considered for the commutator of the singular integral with rough kernel. The results obtained in this paper are substantial improvement and extension of some known results.
MSC: 42B20; 42B25

Keywords: singular integral; commutator; BMO; rough kernel; vector-valued inequalities

1 Introduction

The homogeneous singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy,$$

when $\Omega \in L^1(S^{n-1})$ satisfies the following conditions:

(a) Ω is a homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, *i.e.*,

$$\Omega(tx) = \Omega(x) \quad \text{for any } t > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.1}$$

(b) Ω has mean zero on S^{n-1}, the unit sphere in $\mathbb{R}^n$, *i.e.*,

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0. \tag{1.2}$$

Using a rotation method, Calderón and Zygmund [1] proved that T_Ω is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if Ω is odd or $\Omega \in L\log^+ L(S^{n-1})$. In [2], Grafakos and Stefanov gave a nice survey, which contains a thorough discussion of the history of the operator T_Ω.

For a function $b \in L_{\text{loc}}(\mathbb{R}^n)$, let A be a linear operator on some measurable function space. Then the commutator between A and b is defined by $[b,A]f(x) := b(x)Af(x) - A(bf)(x)$.

In 1976, Coifman *et al.* [3] obtained a characterization of L^p-boundedness of the commutators $[b,R_j]$ generated by the Reisz transforms R_j $(j = 1,\dots,n)$ and a BMO function b. As an application of this characterization, a decomposition theorem of the real Hardy space is given in this paper. Moreover, the authors in [3] proved also that if $\Omega \in \text{Lip}(S^{n-1})$, then the commutator $[b,T_\Omega]$ for T_Ω and a BMO function b is bounded on L^p for $1 < p < \infty$

which is defined by

$$[b, T_\Omega]f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \big(b(x) - b(y)\big) f(y)\,dy.$$

In the same paper, Coifman *et al.* [3] outlined a different approach, which is less direct but shows a close relationship between the weighted inequalities of the operator T_Ω and the weighted inequalities of the commutator $[b, T_\Omega]$. In 1993, Alvarez *et al.* [4] developed the idea of [3] and established a generalized boundedness criterion for the commutator of linear operators. The result of Alvarez *et al.* (see [4], Theorem 2.13) can be stated as follows.

Theorem A ([4]) *Let $1 < p < \infty$. If a linear operator T is bounded on $L^p(w)$ for all $w \in A_q$, $(1 < q < \infty)$, where A_q denotes the weight class of Muckenhoupt, then for $b \in BMO$, $\|[b, T]f\|_{L^p} \leq C\|b\|_{BMO}\|f\|_{L^p}$.*

Combining Theorem A with the well-known results by Duoandikoetxea [5] on the weighted L^p boundedness of the rough singular integral T_Ω, we know that if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then $[b, T_\Omega]$ is bounded on L^p for $1 < p < \infty$. However, it is not clear up to now whether the operator T_Ω with $\Omega \in L^1 \setminus \bigcup_{q>1} L^q(S^{n-1})$ is bounded on $L^p(w)$ for $1 < p < \infty$ and all $w \in A_r$ $(1 < r < \infty)$. Hence, if $\Omega \in L^1 \setminus \bigcup_{q>1} L^q(S^{n-1})$, the L^p boundedness of $[b, T_\Omega]$ cannot be deduced from Theorem A. In this case, Hu [6] used the refined Fourier estimate, the Littlewood-Paley decomposition, and the properties of Young functions and got the following result.

Theorem B ([6]) *Suppose that $\Omega \in L(\log^+ L)^2(S^{n-1})$ satisfying* (1.1) *and* (1.2). *Then, for $b \in BMO(\mathbb{R}^n)$ and $1 < p < \infty$, the commutator $[b, T_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{BMO}$.*

Recently, Chen and Ding [7] gave a sufficient condition which contains $\bigcup_{q>1} L^q(S^{n-1})$ such that the commutator of convolution operators is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This condition was introduced by Grafakos and Stefanov in [8], and it is defined by

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} \big|\Omega(y)\big| \left(\ln \frac{1}{|\xi \cdot y|}\right)^{1+\alpha} d\sigma(y) < \infty, \tag{1.3}$$

where $\alpha > 0$ is a fixed constant. Let $F_\alpha(S^{n-1})$ denote the space of all integrable functions Ω on S^{n-1} satisfying (1.3). The result in [7] can be stated as follows.

Theorem C *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1) *and* (1.2). *If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$, then $[b, T_\Omega]$ extends to a bounded operator from L^p into itself for $\frac{\alpha+1}{\alpha} < p < \alpha + 1$.*

The condition (1.3) above has been considered by many authors in the context of rough integral operators. One can consult [9–15] among numerous references for its development and applications. The examples in [8] show that there is the following relationship between $F_\alpha(S^{n-1})$ and $H^1(S^{n-1})$ (the Hardy space on S^{n-1}):

$$\bigcup_{q>1} L^q(S^{n-1}) \subset \bigcap_{\alpha>0} F_\alpha(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq \bigcup_{\alpha>0} F_\alpha(S^{n-1}).$$

On the other hand, for all $\tau \geq 0$, $L(\log^+ L)^{1+\tau}(S^{n-1}) \subset H^1(S^{n-1})$. So, for all $\tau \geq 0$, $\bigcap_{\alpha>0} F_\alpha(S^{n-1}) \nsubseteq L(\log^+ L)^{1+\tau}(S^{n-1})$.

The study of vector-valued inequalities for singular integrals with rough kernels has attracted much attention (for example, see [16]). In 2011, Tang and Wu [17] considered the vector-valued inequalities $(L^p(\ell^q), L^p(\ell^q))$, $(1 < p, q < \infty)$ of the commutator $[b, T_\Omega]$ with the kernel $\Omega \in L(\log^+ L)^2(S^{n-1})$ satisfying (1.1) and (1.2). In this paper, we consider the vector-valued inequalities for a class of commutators of singular integrals with $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 0$. Now we state our result as follows.

Theorem 1.1 *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1) *and* (1.2) *if $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$. Suppose that $1 < p, q < \infty$ satisfy*

(a) $2 \leq p, q < \infty$ *and* $p \cdot q < 2(\alpha + 1)$; *or*

(b) $2 < p < \infty$, $1 < q < 2$ *and* $p \cdot q' < 2(\alpha + 1)$; *or*

(c) $1 < p, q < 2$ *and* $p' \cdot q' < 2(\alpha + 1)$; *or*

(d) $1 < p < 2$, $2 < q < \infty$ *and* $p' \cdot q < 2(\alpha + 1)$.

Then $[b, T_\Omega]$ extends to a bounded operator from $L^p(\ell^q)$ into itself.

Corollary 1.2 *Let Ω be a function in $L^1(S^{n-1})$ satisfying* (1.1) *and* (1.2). *If $\Omega \in \bigcap_{\alpha>1} F_\alpha(S^{n-1})$, then $[b, T_\Omega]$ extends to a bounded operator from $L^p(\ell^q)$ into itself for $1 < p, q < \infty$.*

This paper is organized as follows. First, in Section 2, we give some definitions, which will be used in the proofs of the main results. In Section 3, we give some preliminary lemmas for the proof of Theorem 1.1. Then, in Section 4, we give the proof of Theorem 1.1. Throughout this paper, the letter C stands for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence. Moreover, the notations '$\vee$' and '$\wedge$' denote the Fourier transform and the inverse Fourier transform, respectively. As usual, for $p \geq 1$, $p' = p/(p-1)$ denotes the dual exponent of p.

We collect the notation to be used throughout this paper:

$$\|\{f_j\}\|_{L^p(\ell^q)} = \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p}; \qquad \|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.$$

2 Definitions

Firstly, we need to recall some definitions which will be used in the proof of Theorem 1.1.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function which is supported in the unit ball and satisfies $\varphi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$. The function $\psi(\xi) = \varphi(\frac{1}{2}) - \varphi(\xi)$ is supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ and satisfies the identity

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0.$$

We denote by Δ_j and G_j the convolution operators whose symbols are $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$, respectively.

The paraproduct of Bony [18] between two functions f, g is defined by

$$\pi_f(g) = \sum_{j\in\mathbb{Z}} (\Delta_j f)(G_j - 3g).$$

At least formally, we have the following Bony decomposition:

$$fg = \pi_f(g) + \pi_g(f) + R(f,g) \quad \text{with } R(f,g) = \sum_{i\in\mathbb{Z}} \sum_{|k-i|\le 2} (\Delta_i f)(\Delta_k g). \tag{2.1}$$

3 Key lemmas

Let us begin with some lemmas, which will be used in the proof of Theorem 1.1. The first one can be found in [17].

Lemma 3.1 *If $\phi \in \mathcal{S}(\mathbb{R}^n)$ with supp$(\phi) \subset x : 1/2 \le |x| \le 2$ and for $l \in \mathbb{Z}$, define the multiplier operator S_l by $S_l f(\xi) = \phi(2^{-l}\xi) f(\xi)$ and S_l^2 by $S_l^2 f = S_l(S_l)$. Then, for $b \in BMO(\mathbb{R}^n)$, for any positive integer k and $b \in BMO(\mathbb{R}^n)$, denote by $S_{l;b;k}$ (respectively $S^2_{l;b;k}$) the kth-order commutator of S_l (respectively S_l^2). Then, for $1 < p, q < \infty$, we have*

$$\text{(i)} \quad \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} |S_{l;b;k} f_j|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p} \le C\|b\|_{BMO} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p},$$

$$\text{(ii)} \quad \left\| \left(\sum_{j\in\mathbb{Z}} \left(\left| \sum_{l\in\mathbb{Z}} S_{l;b;k} f_{j,l} \right|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p} \le C\|b\|_{BMO} \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} |f_{j,l}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p},$$

$$\text{(iii)} \quad \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} |S^2_{l;b;k} f_j|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p} \le C\|b\|_{BMO} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p},$$

$$\text{(iv)} \quad \left\| \left(\sum_{j\in\mathbb{Z}} \left(\left| \sum_{l\in\mathbb{Z}} S^2_{l;b;k} f_{j,l} \right|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p} \le C\|b\|_{BMO} \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} |f_{j,l}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p},$$

where C is independent of j and l.

Lemma 3.2 ([19]) *Let $1 < p, q < \infty$, $\{(\sum_k |g_{k;j}|^2)^{1/2}\}_j \in L^p(\ell^q)$, and $\Omega \in L^1(S^{n-1})$. Denote $\sigma_k(x) = |x|^{-n}|\Omega(x')|\chi_{\{2^k<|x|\le 2^{k+1}\}}(x)$. Then*

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |\sigma_k * g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p} \le C\|\Omega\|_{L^1} \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |g_{k,j}|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p},$$

where C is independent of $\{g_{k,j}\}$.

Lemma 3.3 ([7]) *For the multiplier G_k ($k \in \mathbb{Z}$) defined in Section 2 and $b \in BMO(\mathbb{R}^n)$,*

$$\left| G_k b(x) - G_k b(y) \right| \le C \frac{|x-y|^\delta 2^{k\delta}}{\delta} \|b\|_{BMO} \quad \textit{for } 0 < \delta < 1,$$

where C is independent of k and δ.

Lemma 3.4 ([20]) *For any $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$, the following properties hold:*

(i) $\Delta_j \Delta_i u \equiv 0$ *if* $|j-i| \ge 2$,

(ii) $\Delta_j(G_{i-3}\Delta_i u) \equiv 0$ *if* $|j-i| \ge 4$.

If we replace Δ_j with S_j, the above inequalities also hold.

4 Proof of Theorem 1.1

Recall that

$$[b, T_\Omega]f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n}\big(b(x)-b(y)\big)f(y)\,dy.$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $0 \le \phi \le 1$, $\operatorname{supp}\phi \subset \{1/2 \le |\xi| \le 2\}$ and $\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1$ for $|\xi| \ne 0$. Define the multiplier S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi)$. Set

$$\sigma_j(x) = \frac{\Omega(x')}{|x|^n}\chi_{\{2^j \le |x| < 2^{j+1}\}}(x)$$

for $j \in \mathbb{Z}$. Set

$$m_j(\xi) = \widehat{\sigma_j}(\xi), \qquad m_j^l(\xi) = m_j(\xi)\phi\big(2^{j-l}\xi\big).$$

Define the operator T_j and T_j^l by

$$\widehat{T_j f}(\xi) = m_j(\xi)\widehat{f}(\xi), \qquad \widehat{T_j^l f}(\xi) = m_j^l(\xi)\widehat{f}(\xi).$$

Denote by $[b, T_j]$ and $[b, T_j^l]$ the commutator of T_j and T_j^l, respectively. Define the operator V_l by

$$V_l h(x) = \sum_{j\in\mathbb{Z}} \big[b, S_{l-j}T_j^l S_{l-j}^2\big]h(x).$$

Then we know

$$[b, T_\Omega]h(x) = \sum_{l\in\mathbb{Z}} V_l h(x).$$

Then by the Minkowski inequality, we have, for $1 < p, q < \infty$,

$$\bigg\|\bigg(\sum_{s\in\mathbb{Z}}\big|[b, T_\Omega]f_s\big|^q\bigg)^{1/q}\bigg\|_{L^p} \le \sum_{l\in\mathbb{Z}}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\bigg)^{1/q}\bigg\|_{L^p}.$$

So, to prove Theorem 1.1, it suffices to prove that

$$\sum_{l\in\mathbb{Z}}\bigg(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\bigg)^{1/q}\bigg\|_{L^p} \le C\bigg(\sum_{s\in\mathbb{Z}}|f_s|^q\bigg)^{1/q}\bigg\|_{L^p}. \tag{4.1}$$

It is well known that for some constant $0 < \beta < 1$ and any fixed constant $0 < \upsilon < 1$ (see [7] and [15]),

$$\|V_l f\|_{L^2} \le C\|b\|_{BMO}2^{\beta l}\|\Omega\|_{L^1}\|f\|_{L^2}, \quad l \le 1,$$

and

$$\|V_l f\|_{L^2} \le C\|b\|_{BMO}\log^{(-\alpha-1)\nu+1}\big(2+2^l\big)\|f\|_{L^2}, \quad l \ge 2$$

which gives that

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^2 \right)^{1/2} \right\|_{L^2} \le C\|b\|_{BMO} 2^{\beta l} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^2 \right)^{1/2} \right\|_{L^2}, \quad l \le 1, \tag{4.2}$$

and

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^2 \right)^{1/2} \right\|_{L^2} \le C\|b\|_{BMO} \log^{(-\alpha-1)\beta+1}\left(2+2^l\right) \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^2 \right)^{1/2} \right\|_{L^2}, \quad l \ge 2. \tag{4.3}$$

If we can prove that, for any $1 < p, q < \infty$, $0 < \delta < 1$,

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \right\|_{L^p} \le C \max\left\{2, \frac{2^{\delta l}}{\delta}\right\} \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p}, \tag{4.4}$$

where C is independent of l and δ, we may finish the proof of Theorem 1.1. The proof of (4.4) will be postponed. Now, we will use (4.2), (4.3), and (4.4) to prove Theorem 1.1. Since

$$\begin{aligned} \sum_{l\in\mathbb{Z}} \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \Bigg\|_{L^p} &\le \sum_{l\le 1} \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \Bigg\|_{L^p} \\ &\quad + \sum_{l\ge 2} \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \Bigg\|_{L^p} \\ &:= I_1 + I_2, \end{aligned}$$

we will estimate I_1 and I_2, respectively. We first estimate I_1. For $l \le 1$, taking $q = 2$ in (4.4), then interpolating between (4.2) and (4.4), there exists a constant $0 < \theta_1 < 1$ such that for $1 < p < \infty$,

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^2 \right)^{1/2} \right\|_{L^p} \le C 2^{\theta_1 \beta l} \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^2 \right)^{1/2} \right\|_{L^p}. \tag{4.5}$$

For $l \le 1$ and any fixed $1 < p < \infty$, interpolating between (4.4) and (4.5), there exists a constant $0 < \theta_2 < 1$ such that for $1 < q < \infty$,

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \right\|_{L^p} \le C 2^{\theta_1 \theta_2 \beta l} \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p}.$$

Therefore we get, for $1 < p, q < \infty$,

$$\begin{aligned} \sum_{l\le 1} \left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \right\|_{L^p} &\le \sum_{l\le 1} 2^{\theta_1\theta_2\beta l} \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p} \\ &\le C \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p}. \end{aligned}$$

Next, we will estimate I_2 for (a), (b), (c), and (d), respectively. For $2 \leq l < \infty$, taking $\delta = 1/l$ in (4.4), we get, for any $1 < p, q < \infty$,

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \right\|_{L^p} \leq Cl \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p}. \tag{4.6}$$

Taking $q = 2$ in (4.6) gives that for any $1 < r < \infty$, we have

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^2 \right)^{1/2} \right\|_{L^r} \leq Cl \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^2 \right)^{1/2} \right\|_{L^r}. \tag{4.7}$$

We first treat the case (a) : $2 \leq p, q < \infty$ and $p \cdot q < 2(\alpha + 1)$. Now, for any $p \geq 2$, we take r sufficiently large such that $r > p$ in (4.7). Using the Riesz-Thorin interpolation theorem between (4.3) and (4.7), we have that for any $l \geq 2$,

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^2 \right)^{1/2} \right\|_{L^p} \leq C\|b\|_{BMO} l^{1-\theta} \log^{((-\alpha-1)\nu+1)\theta}\left(2 + 2^l\right) \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^2 \right)^{1/2} \right\|_{L^p},$$

where $\theta = \frac{2(r-p)}{p(r-2)}$. We can see that if $r \mapsto \infty$, then θ goes to $2/p$ and $\log^{((-\alpha-1)\nu+1)\theta}(2 + 2^l)$ goes to $\log^{((-\alpha-1)\nu+1)2/p}(2 + 2^l)$. Therefore, we get

$$\begin{aligned} \left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^2 \right)^{1/2} \right\|_{L^p} &\leq C\|b\|_{BMO} l^{1-2/p} \log^{((-\alpha-1)\nu+1)\frac{2}{p}}\left(2 + 2^l\right) \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^2 \right)^{1/2} \right\|_{L^p} \\ &\leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p}} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^2 \right)^{1/2} \right\|_{L^p}. \end{aligned} \tag{4.8}$$

On the other hand, fix p, for any $2 \leq q < \infty$, (4.6) also means that for any λ sufficiently large such that $\lambda > q$,

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^\lambda \right)^{1/\lambda} \right\|_{L^p} \leq Cl \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^\lambda \right)^{1/\lambda} \right\|_{L^p}. \tag{4.9}$$

Using the Riesz-Thorin interpolation theorem between (4.8) and (4.9), we have that

$$\begin{aligned} \left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \right\|_{L^p} &\leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p}\theta_1} l^{1-\theta_1} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p} \\ &\leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p}\theta_1} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p}, \end{aligned}$$

where $\theta_1 = \frac{2(\lambda-q)}{q(\lambda-2)}$. We can see that if $\lambda \mapsto \infty$, then θ_1 goes to $2/q$ and $l^{1-(\alpha+1)\nu\frac{2}{p}\theta_1}$ goes to $l^{1-(\alpha+1)\nu\frac{2}{p}\frac{2}{q}}$. This gives that for any fixed $0 < \upsilon < 1$,

$$\left\| \left(\sum_{s\in\mathbb{Z}} |V_l f_s|^q \right)^{1/q} \right\|_{L^p} \leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p}\frac{2}{q}} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p}.$$

Thus, by the inequality above, we have, for $p \cdot q < 2(\alpha + 1)$,

$$\begin{aligned}\sum_{l\geq 2}\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\right)^{1/q}\right\|_{L^p} &\leq C\left(\sum_{l\geq 2} l^{1-(\alpha+1)\nu\frac{2}{p}\frac{2}{q}}\right)\|b\|_{BMO}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p}\\ &\leq C\|b\|_{BMO}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p}.\end{aligned}$$

Next, for the case (b) : $2 \leq p < \infty$, $1 < q < 2$, and $p \cdot q' < 2(\alpha + 1)$. For any $p \geq 2$, we have

$$\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^2\right)^{1/2}\right\|_{L^p} \leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p}}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^2\right)^{1/2}\right\|_{L^p}. \tag{4.10}$$

Similarly, fix p, for $1 < q < 2$, (4.6) also means that for any λ sufficiently small such that $1 < \lambda < q$,

$$\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^\lambda\right)^{1/\lambda}\right\|_{L^p} \leq Cl\|\Omega\|_{L^1}\|b\|_{BMO}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^\lambda\right)^{1/\lambda}\right\|_{L^p}. \tag{4.11}$$

Using the Riesz-Thorin interpolation theorem between (4.10) and (4.11), we have

$$\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\right)^{1/q}\right\|_{L^p} \leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p}\theta_1}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p},$$

where $\theta_1 = \frac{2(\lambda-q)}{q(\lambda-2)}$. We can see that if $\lambda \mapsto 1$, then θ_1 goes to $2/q'$ and $l^{1-(\alpha+1)\nu\frac{2}{p}\theta_1}$ goes to $l^{1-(\alpha+1)\nu\frac{2}{p}\frac{2}{q'}}$. This gives that for any fixed $0 < \upsilon < 1$,

$$\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\right)^{1/q}\right\|_{L^p} \leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p}\frac{2}{q'}}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p}.$$

Thus, for $2 \leq p < \infty$, $1 < q < 2$, and $p \cdot q' < 2(\alpha + 1)$, we have

$$\sum_{l\geq 2}\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\right)^{1/q}\right\|_{L^p} \leq C\|b\|_{BMO}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p}.$$

Now, for the case (c) : $1 < p, q < 2$ and $p \cdot q' < 2(\alpha + 1)$. For any $1 < p < 2$, we take r sufficiently small such that $1 < r < p$ in (4.7). Using the Riesz-Thorin interpolation theorem between (4.3) and (4.7), we have that for any $l \geq 2$,

$$\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^2\right)^{1/2}\right\|_{L^p} \leq C\|b\|_{BMO} l^{1-\theta}\log^{((-\alpha-1)\nu+1)\theta}\left(2 + 2^l\right)\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^2\right)^{1/2}\right\|_{L^p}, \tag{4.12}$$

where $\theta = \frac{2(r-p)}{p(r-2)}$. We can see that if $r \mapsto 1$, then θ goes to $2/p'$ and $\log^{((-\alpha-1)\nu+1)\theta}(2+2^l)$ goes to $\log^{((-\alpha-1)\nu+1)2/p'}(2+2^l)$. Therefore, we get

$$\begin{aligned}\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^2\right)^{1/2}\right\|_{L^p} &\leq C\|b\|_{BMO} l^{1-2/p}\log^{((-\alpha-1)\nu+1)\frac{2}{p'}}\left(2+2^l\right)\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^2\right)^{1/2}\right\|_{L^p} \\ &\leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p'}}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^2\right)^{1/2}\right\|_{L^p}. \end{aligned} \tag{4.13}$$

Then, using the previous argument, for any fixed $1 < p < 2$ and $1 < q < 2$, we get

$$\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\right)^{1/q}\right\|_{L^p} \leq C\|b\|_{BMO} l^{1-(\alpha+1)\nu\frac{2}{p'}\frac{2}{q'}}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p}. \tag{4.14}$$

Thus if $p' \cdot q' < 2(\alpha+1)$, then

$$\sum_{l\geq 2}\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\right)^{1/q}\right\|_{L^p} \leq C\|b\|_{BMO}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p}.$$

Finally, for the case (d) : $1 < p < 2$, $2 \leq q < \infty$, and $p' \cdot q' < 2(\alpha+1)$, using the previous argument, we get

$$\sum_{l\geq 2}\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\right)^{1/q}\right\|_{L^p} \leq C\|b\|_{BMO}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p}.$$

Therefore, we prove that

$$I_2 \leq C\|b\|_{BMO}\left\|\left(\sum_{s\in\mathbb{Z}}|f_s|^q\right)^{1/q}\right\|_{L^p}$$

for four cases.

Now, we turn our attention to proving (4.4). Since $T_j^l S_{l-j} = T_j S_{l-j}^2$ for any $j, l \in \mathbb{Z}$, we may write

$$\left[b, S_{l-j}^2 T_j^l S_{l-j}\right]f = \left[b, S_{l-j}^2\right]\left(T_j S_{l-j}^2 f\right) + S_{l-j}^2[b, T_j]\left(S_{l-j}^2 f\right) + S_{l-j}^2 T_j\left(\left[b, S_{l-j}^2\right]f\right).$$

Thus,

$$\begin{aligned}&\left\|\left(\sum_{s\in\mathbb{Z}}|V_l f_s|^q\right)^{1/q}\right\|_{L^p} \\ &\quad\leq \left\|\left(\sum_{s\in\mathbb{Z}}\left|\sum_{j\in\mathbb{Z}}\left[b, S_{l-j}^2\right]\left(T_j S_{l-j}^2 f_s\right)\right|^q\right)^{1/q}\right\|_{L^p} + \left\|\left(\sum_{s\in\mathbb{Z}}\left|\sum_{j\in\mathbb{Z}}S_{l-j}^2 T_j\left(\left[b, S_{l-j}^2\right]f_s\right)\right|^q\right)^{1/q}\right\|_{L^p} \\ &\qquad + \left\|\left(\sum_{s\in\mathbb{Z}}\left|\sum_{j\in\mathbb{Z}}S_{l-j}^2[b, T_j]\left(S_{l-j}^2 f_s\right)\right|^q\right)^{1/q}\right\|_{L^p} \\ &\quad := L_1 + L_2 + L_3. \end{aligned} \tag{4.15}$$

Below we shall estimate L_i for $i = 1, 2, 3$, respectively. As regards L_1, by Lemma 3.1 and Lemma 3.2, we have, for $1 < p < \infty$,

$$\begin{aligned} L_1 &\le C\left\| \left(\sum_{s\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \left| T_j S_{l-j}^2 f_s \right|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p} \\ &\le C\|b\|_{BMO}\|\Omega\|_{L^1} \left\| \left(\sum_{s\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \left| S_{l-j}^2 f_s \right|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p} \\ &\le C\|b\|_{BMO}\|\Omega\|_{L^1} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p}. \end{aligned}$$

Similarly, we get

$$L_2 \le C\|b\|_{BMO}\|\Omega\|_{L^1} \left\| \left(\sum_{s\in\mathbb{Z}} |f_s|^q \right)^{1/q} \right\|_{L^p}.$$

Hence, by (4.15), to show (4.4) it remains to give the estimate of L_3. We will apply Bony paraproduct to do this. By (2.1),

$$fg = \pi_f(g) + \pi_g(f) + R(f, g).$$

We have

$$\begin{aligned} &b(x)\left(T_j S_{l-j}^2 f_s\right)(x) \\ &\quad = \pi_{(T_j S_{l-j}^2 f_s)}(b)(x) + R\left(b, T_j S_{l-j}^2 f_s\right)(x) + \pi_b\left(T_j S_{l-j}^2 f_s\right)(x) \end{aligned}$$

and

$$b S_{l-j}^2 f_s(x) = \pi_{(S_{l-j}^2 f_s)}(b)(x) + R\left(b, S_{l-j}^2 f_s\right)(x) + \pi_b\left(S_{l-j}^2 f_s\right)(x).$$

Then we get

$$\begin{aligned} &[b, T_j] S_{l-j}^2 f_s(x) \\ &\quad = b(x)\left(T_j S_{l-j}^2 f_s\right)(x) - T_j\left(b S_{l-j}^2 f_s\right)(x) \\ &\quad = \left[\pi_{(T_j S_{l-j}^2 f_s)}(b)(x) - T_j\left(\pi_{(S_{l-j}^2 f_s)}(b)\right)(x)\right] + \left[R\left(b, T_j S_{l-j}^2 f_s\right)(x) - T_j\left(R\left(b, S_{l-j}^2 f_s\right)\right)(x)\right] \\ &\qquad + \left[\pi_b\left(T_j S_{l-j}^2 f_s\right)(x) - T_j\left(\pi_b\left(S_{l-j}^2 f_s\right)\right)(x)\right]. \end{aligned}$$

Thus

$$\begin{aligned} L_3 &\le \left\| \left(\sum_{s\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} S_{l-j}^2 \left[\pi_{(T_j S_{l-j}^2 f_s)}(b) - T_j\left(\pi_{(S_{l-j}^2 f_s)}(b)\right)\right] \right|^q \right)^{1/q} \right\|_{L^p} \\ &\quad + \left\| \left(\sum_{s\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} S_{l-j}^2 \left[R\left(b, T_j S_{l-j}^2 f_s\right) - T_j\left(R\left(b, S_{l-j}^2 f_s\right)\right)\right] \right|^q \right)^{1/q} \right\|_{L^p} \end{aligned}$$

$$\begin{aligned}
&+\left\|\left(\sum_{s\in\mathbb{Z}}\left|\sum_{j\in\mathbb{Z}}S^2_{l-j}\left[\pi_b\left(T_jS^2_{l-j}f_s\right)-T_j\left(\pi_b\left(S^2_{l-j}f_s\right)\right)\right]\right|^q\right)^{1/q}\right\|_{L^p}\\
&:=M_1+M_2+M_3. \qquad (4.16)
\end{aligned}$$

(a) *The estimate of* M_1. Recall that $\pi_g(f)=\sum_{j\in\mathbb{Z}}(\Delta_jf)(G_{j-3}g)$. For M_1, by Lemma 3.4(i), we know $\Delta_iS_kg=0$ for $g\in\mathcal{S}'(\mathbb{R}^n)$ when $|i-k|\geq 3$. Then

$$\begin{aligned}
&\pi_{(T_jS^2_{l-j}f_s)}(b)(x)-T_j\left(\pi_{(S^2_{l-j}f_s)}(b)\right)(x)\\
&\quad=\sum_{|i-(l-j)|\leq 2}\left\{\Delta_i\left(T_jS^2_{l-j}f_s\right)(x)(G_{i-3}b)(x)-T_j\left[\left(\Delta_iS^2_{l-j}f_s\right)(G_{i-3}b)\right](x)\right\}\\
&\quad=\sum_{|i-(l-j)|\leq 2}[G_{i-3}b,T_j]\left(\Delta_iS^2_{l-j}f_s\right)(x). \qquad (4.17)
\end{aligned}$$

Then we get

$$M_1\leq\sum_{|k|\leq 2}\left\|\left(\sum_{s\in\mathbb{Z}}\left|\sum_{j\in\mathbb{Z}}S^2_{l-j}\left([G_{l-j+k-3}b,T_j]\left(\Delta_{l-j+k}S^2_{l-j}f_s\right)\right)\right|^q\right)^{1/q}\right\|_{L^p}. \qquad (4.18)$$

Without loss of generality, we may assume $k=0$. By Lemma 3.1, we get

$$M_1\leq C\left\|\left(\sum_{s\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}}\left|[G_{l-j-3}b,T_j]\left(\Delta_{l-j}S^2_{l-j}f_s\right)\right|^2\right)^{q/2}\right)^{1/q}\right\|_{L^p}. \qquad (4.19)$$

Note that

$$\begin{aligned}
&\left|[G_{l-j-3}b,T_j]\left(\Delta_{l-j}S^2_{l-j}f_s\right)(x)\right|\\
&\quad=\left|\int_{2^j\leq|x-y|<2^{j+1}}\frac{\Omega(x-y)}{|x-y|^n}\left(G_{l-j-3}b(x)-G_{l-j-3}b(y)\right)\Delta_{l-j}S^2_{l-j}f_s(y)\,dy\right|\\
&\quad\leq C\int_{2^j\leq|x-y|<2^{j+1}}\frac{|\Omega(x-y)|}{|x-y|^n}\left|G_{l-j-3}b(x)-G_{l-j-3}b(y)\right|\left|\Delta_{l-j}S^2_{l-j}f_s(y)\right|\,dy.
\end{aligned}$$

By Lemma 3.3, we have, for any $0<\delta<1$,

$$\begin{aligned}
&\left|[G_{l-j-3}b,T_j]\Delta_{l-j}S^2_{l-j}f_s(x)\right|\\
&\quad\leq C2^{(l-j-3)\delta}\frac{|x-y|^\delta}{\delta}\|b\|_{BMO}\int_{2^j\leq|x-y|<2^{j+1}}\frac{|\Omega(x-y)|}{|x-y|^n}\left|\Delta_{l-j}S^2_{l-j}f_s(y)\right|\,dy\\
&\quad\leq C\frac{2^{l\delta}}{\delta}\|b\|_{BMO}\int_{2^j\leq|x-y|<2^{j+1}}\frac{|\Omega(x-y)|}{|x-y|^n}\left|\Delta_{l-j}S^2_{l-j}f_s(y)\right|\,dy\\
&\quad=C\frac{2^{l\delta}}{\delta}\|b\|_{BMO}T_{|\Omega|,j}\left(\left|\Delta_{l-j}S^2_{l-j}f_s\right|\right)(x), \qquad (4.20)
\end{aligned}$$

where

$$T_{|\Omega|,j}f_s(x)=\int_{2^j\leq|x-y|<2^{j+1}}\frac{|\Omega(x-y)|}{|x-y|^n}f_s(y)\,dy,$$

and C is independent of δ and l. Then, by (4.19), (4.20) and applying Lemma 3.2 and Lemma 3.1, we have that for $1 < p < \infty$,

$$\begin{aligned}
M_1 &\le C\frac{2^{l\delta}}{\delta}\|b\|_{BMO}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}\bigg(\sum_{j\in\mathbb{Z}}\big|T_{|\Omega|,j,d}\big(\big|\Delta_{l-j}S^2_{l-j}f_s\big|\big)\big|^2\bigg)^{q/2}\bigg)^{1/q}\bigg\|_{L^p} \\
&\le C\|\Omega\|_{L^1}\frac{2^{l\delta}}{\delta}\|b\|_{BMO}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}\bigg(\sum_{j\in\mathbb{Z}}\big|\Delta_{l-j}S^2_{l-j}f_s\big|^2\bigg)^{q/2}\bigg)^{1/q}\bigg\|_{L^p} \\
&\le C\|\Omega\|_{L^1}\frac{2^{l\delta}}{\delta}\|b\|_{BMO}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}\bigg(\sum_{j\in\mathbb{Z}}\big|S^2_{l-j}f_s\big|^2\bigg)^{q/2}\bigg)^{1/q}\bigg\|_{L^p} \\
&\le C\|\Omega\|_{L^1}\frac{2^{l\delta}}{\delta}\|b\|_{BMO}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}|f_s|^q\bigg)^{1/q}\bigg\|_{L^p}, \qquad (4.21)
\end{aligned}$$

where C is independent of l and δ.

(b) *The estimate of* M_2. By Lemma 3.4(i), we know for $|k| \le 2$, $\Delta_{i+k}S_{l-j}g = 0$ for $g \in \mathcal{S}'(\mathbb{R}^n)$ when $|i-(l-j)| \ge 5$. Thus

$$\begin{aligned}
&R\big(b, T_jS^2_{l-j}f_s\big) - T_j\big(R\big(b, S^2_{l-j}f_s\big)\big)(x) \\
&\quad= \sum_{i\in\mathbb{Z}}\sum_{|k|\le 2}(\Delta_i b)(x)\big(T_j\Delta_{i+k}S^2_{l-j}f_s\big)(x) - T_j\bigg(\sum_{i\in\mathbb{Z}}\sum_{|k|\le 2}(\Delta_i b)\big(\Delta_{i+k}S^2_{l-j}f_s\big)\bigg)(x) \\
&\quad= \sum_{k=-2}^{2}\sum_{|i-(l-j)|\le 4}\big((\Delta_i b)(x)\big(T_j\Delta_{i+k}S^2_{l-j}f_s\big)(x) - T_j\big((\Delta_i b)\big(\Delta_{i+k}S^2_{l-j}f_s\big)\big)(x)\big) \\
&\quad= \sum_{k=-2}^{2}\sum_{|i-(l-j)|\le 4}[\Delta_i b, T_j]\big(\Delta_{i+k}S^2_{l-j}f_s\big)(x).
\end{aligned}$$

Then we get

$$M_2 \le \sum_{|k|\le 6}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}\bigg|\sum_{j\in\mathbb{Z}}S^2_{l-j}[\Delta_{l-j+k}b, T_j]\big(\Delta_{l-j+k}S^2_{l-j}f_s\big)\bigg|^q\bigg)^{1/q}\bigg\|_{L^p}.$$

Without loss of generality, we may assume $k = 0$. By the equality above and using Lemma 3.1, $\sup_{i\in\mathbb{Z}}\|\Delta_i(b)\|_{L^\infty} \le C\|b\|_{BMO}$ (see [21]) and Lemma 3.2, we have, for $1 < p < \infty$,

$$\begin{aligned}
M_2 &\le C\sup_{i\in\mathbb{Z}}\big\|\Delta_i(b)\big\|_{L^\infty}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}\bigg(\sum_{j\in\mathbb{Z}}\big|T_{|\Omega|,j}\big(\big|\Delta_{l-j}S^2_{l-j}f_s\big|\big)\big|^2\bigg)^{q/2}\bigg)^{1/q}\bigg\|_{L^p} \\
&\le C\|b\|_{BMO}\|\Omega\|_{L^1}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}\bigg(\sum_{j\in\mathbb{Z}}\big|\Delta_{l-j}S^2_{l-j}f_s\big|^2\bigg)^{q/2}\bigg)^{1/r}\bigg\|_{L^p} \\
&\le C\|b\|_{BMO}\|\Omega\|_{L^1}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}\bigg(\sum_{j\in\mathbb{Z}}\big|S^2_{l-j}f_s\big|^2\bigg)^{q/2}\bigg)^{1/q}\bigg\|_{L^p} \\
&\le C\|b\|_{BMO}\|\Omega\|_{L^1}\bigg\|\bigg(\sum_{s\in\mathbb{Z}}|f_s|^q\bigg)^{1/q}\bigg\|_{L^p}. \qquad (4.22)
\end{aligned}$$

(c) *The estimate of* M_3. Finally, we give the estimate of M_3. By Lemma 3.4(ii), we know $S_j(\Delta_i g G_{i-3}h) = 0$ for $g, h \in \mathcal{S}'(\mathbb{R}^n)$ if $|j-i| \geq 5$. We get

$$\begin{aligned}
&S^2_{l-j}\bigl(\pi_b\bigl(T_j S^2_{l-j} f_s\bigr) - T_j\bigl(\pi_b\bigl(S^2_{l-j} f_s\bigr)\bigr)\bigr) \\
&\quad = S^2_{l-j}\Biggl(\sum_{i\in\mathbb{Z}}(\Delta_i b)\bigl(G_{i-3} T_j S^2_{l-j} f_s\bigr) - T_j\Biggl(\sum_{i\in\mathbb{Z}}(\Delta_i b)\bigl(G_{i-3} S^2_{l-j} f_s\bigr)\Biggr)\Biggr)(x) \\
&\quad = \sum_{|i-(l-j)|\leq 4}\bigl\{S^2_{l-j}\bigl((\Delta_i b)\bigl(G_{i-3} T_j S^2_{l-j} f_s\bigr)\bigr)(x) - S^2_{l-j} T_j\bigl((\Delta_i b)\bigl(G_{i-3} S^2_{l-j} f_s\bigr)\bigr)(x)\bigr\}.
\end{aligned}$$

Thus, by Lemma 3.1, $\sup_{i\in\mathbb{Z}} \|\Delta_i(b)\|_{L^\infty} \leq C\|b\|_{BMO}$, and Lemma 3.2, we get, for $1 < p < \infty$,

$$\begin{aligned}
M_3 &\leq C \sup_{i\in\mathbb{Z}} \bigl\|\Delta_i(b)\bigr\|_{L^\infty} \Biggl\| \Biggl(\sum_{s\in\mathbb{Z}}\Biggl(\sum_{j\in\mathbb{Z}}\bigl|T_{|\Omega|,j}\bigl(\bigl|G_{l-j} S^2_{l-j} f_s\bigr|\bigr)\bigr|^2\Biggr)^{q/2}\Biggr)^{1/q}\Biggr\|_{L^p} \\
&\leq C\|b\|_{BMO}\|\Omega\|_{L^1} \Biggl\| \Biggl(\sum_{s\in\mathbb{Z}}\Biggl(\sum_{j\in\mathbb{Z}}\bigl|G_{l-j} S^2_{l-j} f_s\bigr|^2\Biggr)^{q/2}\Biggr)^{1/q}\Biggr\|_{L^p} \\
&\leq C\|b\|_{BMO}\|\Omega\|_{L^1} \Biggl\| \Biggl(\sum_{s\in\mathbb{Z}}\Biggl(\sum_{j\in\mathbb{Z}}\bigl|S^2_{l-j} f_s\bigr|^2\Biggr)^{q/2}\Biggr)^{1/q}\Biggr\|_{L^p} \\
&\leq C\|b\|_{BMO}\|\Omega\|_{L^1} \Biggl\| \Biggl(\sum_{s\in\mathbb{Z}} |f_s|^q\Biggr)^{1/q}\Biggr\|_{L^p}. \qquad (4.23)
\end{aligned}$$

By (4.16), (4.21)-(4.23), we get

$$L_3 \leq C \max\biggl\{2, \frac{2^{\delta l}}{\delta}\biggr\} \|b\|_{BMO}\|\Omega\|_{L^1} \Biggl\| \Biggl(\sum_{s\in\mathbb{Z}} |f_s|^q\Biggr)^{1/q}\Biggr\|_{L^p} \quad \text{for } l \in \mathbb{Z},$$

where C is independent of δ and l. This establishes the proof of (4.4).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YC carried out the vector-valued inequalities for the commutators of singular integral operator studies and drafted the manuscript. YD participated in the study of Littlewood-Paley theory. All authors read and approved the final manuscript.

Author details
[1]Department of Applied Mathematics, School of Mathematics and Physics, University of Science and Technology Beijing, Beijing, 100083, People's Republic of China. [2]School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems (BNU), Ministry of Education, Beijing, 100875, People's Republic of China.

Acknowledgements
The authors would like to express their deep gratitude to the referee for giving many valuable suggestions. The research was supported by NCET of China (Grant nos.: NCET-11-0574), the Fundamental Research Funds for the Central Universities (Grant nos.: FRF-TP-12-006B), NSF of China (Grant nos.: 10901017, 11371057) and SRFDP of China (Grant nos.: 20130003110003).

Received: 25 September 2013 Accepted: 20 March 2014 Published: 04 Apr 2014

References
1. Calderón, A, Zygmund, A: On singular integral operators. Am. J. Math. **78**, 289-309 (1956)
2. Grafakos, L, Stefanov, A: Convolution Calderón-Zygmund singular integral operators with rough kernels. In: Analysis of Divergence, Orono, ME, 1997. Appl. Numer. Harmon. Anal., pp. 119-143. Birkhäuser, Boston (1999)
3. Coifman, R, Rochberg, R, Weiss, G: Factorization theorems for Hardy spaces in several variables. Ann. Math. **103**, 611-635 (1976)

4. Alvarez, J, Bagby, R, Kurtz, D, Pérez, C: Weighted estimates for commutators of linear operators. Stud. Math. **104**, 195-209 (1993)
5. Duoandikoetxea, J: Weighted norm inequalities for homogeneous singular integrals. Trans. Am. Math. Soc. **336**, 869-880 (1993)
6. Hu, G: $L^p(\mathbb{R}^n)$ boundedness for the commutator of a homogeneous singular integral operator. Stud. Math. **154**, 13-27 (2003)
7. Chen, Y, Ding, Y: L^p bounds for the commutator of singular integral and Maximal singular integrals with rough kernel. Trans. Am. Math. Soc. To appear
8. Grafakos, L, Stefanov, A: L^p bounds for singular integrals and maximal singular integrals with rough kernels. Indiana Univ. Math. J. **47**, 455-469 (1998)
9. Al-Qassem, H, Al-Salman, A: Rough Marcinkiewicz integral operators. Int. J. Math. Sci. **27**, 495-503 (2001)
10. Al-Salman, A, Pan, Y: Singular integrals with rough kernels. Can. Math. Bull. **47**, 3-11 (2004)
11. Chen, D, Lu, S: L^p boundedness of parabolic Littlewood-Paley operator with rough kernel belonging to $F_\alpha(S^{n-1})$. Acta Math. Sci. **31**, 343-350 (2011)
12. Chen, J, Fan, D, Pan, Y: A note on a Marcinkiewicz integral operator. Math. Nachr. **227**, 33-42 (2001)
13. Cheng, L-C, Pan, Y: L^p bounds for singular integrals associated to surface of revolution. J. Math. Anal. Appl. **265**, 163-169 (2002)
14. Fan, D, Guo, K, Pan, Y: A note of a rough singular integral operator. Math. Inequal. Appl. **2**, 73-81 (1999)
15. Hu, G: $L^2(\mathbb{R}^n)$ boundedness for the commutators of convolution operators. Nagoya Math. J. **163**, 55-70 (2001)
16. Hofmann, S: Weighted norm inequalities and vector valued inequalities for certain rough operators. Indiana Univ. Math. J. **42**, 1-14 (1993)
17. Tang, L, Wu, H: Vector-valued inequalities for the commutators of singular integrals with rough kernels. J. Korean Math. Soc. **48**, 703-725 (2011)
18. Bony, J-M: Calcul symbolique et propagation des singularites pour les equations aux derivees partielles non lineaires. Ann. Sci. Éc. Norm. Super. **14**, 209-246 (1981)
19. Chen, Y, Ding, Y: Rough singular integrals on Triebel-Lizorkin space and Besov space. J. Math. Anal. Appl. **347**, 493-501 (2008)
20. Danchin, R: Fourier analysis methods for PDE's. Lecture notes (2005)
21. Grafakos, L: Classical and Modern Fourier Analysis. Pearson Education, Upper Saddle River (2004)

10.1186/1029-242X-2014-139
Cite this article as: Chen and Ding: Vector-valued inequalities for the commutators of rough singular kernels. *Journal of Inequalities and Applications* 2014, 2014:139