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Common tripled fixed point theorems for W-compatible mappings along with the CLR_g property in abstract metric spaces

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Abstract
In this paper, we introduce the concept of common limit in the range property for mappings $F: X \times X \times X \to X$ and $g: X \to X$, where (X, d) is an abstract metric space, and establish tripled coincidence point and common tripled fixed point theorems by using the common limit in the range property. Our theorems generalize and extend several well-known comparable results in the literature, in particular the results of Aydi *et al.* (Fixed Point Theory Appl. 2012:134, 2012). Also, we give examples to validate our main results which are not applied by the results in the literature.

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1 Introduction

The concept of K-metric spaces was reintroduced by Huang and Zhang under the name of cone metric spaces [1] which is the generalization of a metric space. The idea of cone metric spaces is to replace the codomain of a metric from the set of real numbers to an ordered Banach space. They reintroduced the definitions of convergent and Cauchy sequences in the sense of an interior point of the underlying cone. They also continued with results concerning the normal cones only. One of the main results of Huang and Zhang in [1] is fixed point theorems for contractive mappings in normal cone spaces. In fact, the fixed point theorem in cone metric spaces is appropriate only in the case when the underlying cone is non-normal and its interior is nonempty. Janković *et al.* [2] studied this topic and gave some examples showing that theorems from ordinary metric spaces cannot be applied in the setting of non-normal cone metric spaces. Many works for fixed point theorems in cone metric spaces appeared in [3–19].

In 2011, Abbas *et al.* [3] introduced the concept of w-compatible mappings and obtained a coupled coincidence point and a coupled point of coincidence for such mappings satisfying a contractive condition in cone metric spaces. Very recently, Aydi *et al.* [20] introduced the concept of W-compatible mappings for mappings $F: X \times X \times X \to X$ and $g: X \to X$, where (X, d) is an abstract metric space and established tripled coincidence point and common tripled fixed point theorems in these spaces.

On the other hand, Sintunavarat and Kumam [21] coined the idea of common limit range property for mappings $F: X \to X$ and $g: X \to X$, where (X, d) is a metric space (and fuzzy metric spaces) and proved the common fixed point theorems by using this property. Afterward, Jain *et al.* [22] extended this property for mappings $F : X \times X \to X$ and $g : X \to X$, where (X, d) is a metric space (and fuzzy metric spaces) and established a coupled fixed point theorem for mappings satisfying this property. Several common fixed point theorems have been proved by many researcher in the framework of many spaces via the common limit range property (see [23–26] and the references therein).

Starting from the background of coupled fixed points, the concept of tripled fixed points was introduced by Samet and Vetro [27] and Berinde and Borcut [28] and it was motivated by the fact that through the coupled fixed point technique we cannot give the solution of some problems in nonlinear analysis such as a system of the following form:

$$\begin{cases} x^2 + 2yz - 6x + 3 = 0, \\ y^2 + 2xz - 6y + 3 = 0, \\ z^2 + 2yx - 6z + 3 = 0. \end{cases}$$

In this paper, we introduce the concept of common limit in the range property for mappings $F : X \times X \times X \to X$ and $g : X \to X$, where (X, d) is an abstract metric space, and study the existence of tripled coincidence point and common tripled fixed point theorems for W-compatible mappings in cone metric spaces by using the common limit in the range property. We also furnish examples to demonstrate the validity of the results which are not applied by the results in the literature. It is worth mentioning that our results do not rely on the assumption of normality condition of the cone and thus fixed point results in this trend are still of interest and importance in some ways.

2 Preliminaries

In this paper, we denote from now on $\underbrace{X \times X \cdots X \times X}_{k \text{ terms}}$ by X^k where $k \in \mathbb{N}$ and X is a nonempty set. The following definitions and results will be needed in the sequel.

Definition 2.1 Let E be a real Banach space and 0_E be the zero element in E. A subset P of E is called a cone if it satisfies the following conditions:

(a) P is closed, nonempty and $P \neq \{0_E\}$,
(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$,
(c) $P \cap (-P) = \{0_E\}$.

Given a cone P of the real Banach space E, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in \operatorname{Int}(P)$, where $\operatorname{Int}(P)$ stands for the interior of P. Also we will use $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$.

The cone P in the normed space $(E, \|\cdot\|)$ is called normal whenever there is a number $k > 0$ such that for all $x, y \in E$, $0_E \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$. The least positive number satisfying this norm inequality is called the normal constant of P. In 2008, Rezapour and Hamlbarani [11] showed that there are no normal cones with a normal constant $k < 1$.

In what follows we always suppose that E is a real Banach space with a cone P satisfying $\operatorname{Int}(P) \neq \emptyset$ (such cones are called *solid*).

Definition 2.2 ([1, 29]) Let X be a nonempty set. Suppose that $d : X \times X \to E$ satisfies the following conditions:

(d1) $0_E \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$,

(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(d3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric or a K-metric on X and (X,d) is called a cone metric space or a K-metric space.

Remark 2.3 The concept of K-metric space is more general than the concept of metric space, because each metric space is a K-metric space where $X = \mathbb{R}$ with the usual norm and a cone $P = [0,\infty)$.

Definition 2.4 ([1]) Let X be a K-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

(C1) a Cauchy sequence if and only if for each $c \in E$ with $c \gg 0_E$ there is some $k \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq k$;
(C2) a convergent sequence if and only if for each $c \in E$ with $c \gg 0_E$ there is some $k \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq k$, where $x \in X$. This limit is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Remark 2.5 Every convergent sequence in a K-metric space X is a Cauchy sequence but the converse is not true.

Definition 2.6 A K-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 2.7 ([27]) Let X be a nonempty set. An element $(x,y,z) \in X^3$ is called a tripled fixed point of a given mapping $F : X^3 \to X$ if $x = F(x,y,z)$, $y = F(y,z,x)$ and $z = F(z,x,y)$.

Berinde and Borcut [28] defined differently the notion of a tripled fixed point in the case of ordered sets in order to keep true the mixed monotone property.

Definition 2.8 ([20]) Let X be a nonempty set. An element $(x,y,z) \in X^3$ is called

(i) a tripled coincidence point of mappings $F : X^3 \to X$ and $g : X \to X$ if $gx = F(x,y,z)$, $gy = F(y,x,z)$ and $gz = F(z,x,y)$. In this case (gx, gy, gz) is called a tripled point of coincidence;
(ii) a common tripled fixed point of mappings $F : X^3 \to X$ and $g : X \to X$ if $x = gx = F(x,y,z)$, $y = gy = F(y,z,x)$ and $z = gz = F(z,x,y)$.

Example 2.9 Let $X = \mathbb{R}$. We define $F : X^3 \to X$ and $g : X \to X$ as follows:

$$F(x,y,z) = \left(\frac{2x+2y}{\pi}\right)\sin(2z) \quad \text{and} \quad gx = 1 + \pi - 4x$$

for all $x, y, z \in X$. Then $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ is a tripled coincidence point of F and g, and $(1,1,1)$ is a tripled point of coincidence.

Definition 2.10 ([20]) Let X be a nonempty set. Mappings $F : X^3 \to X$ and $g : X \to X$ are called W-compatible if

$$F(gx, gy, gz) = g\big(F(x,y,z)\big)$$

whenever $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,y,x) = gz$.

Example 2.11 Let $X = [0,1)$. Define $F : X^3 \to X$ and $g : X \to X$ as follows:

$$F(x,y,z) = \frac{x^2 + y^2 + z^2}{12} \quad \text{and} \quad gx = \frac{x}{4}$$

for all $x,y,z \in X$. One can show that (x,y,z) is a tripled coincidence point of F and g if and only if $x = y = z = 0$. Since $F(g0,g0,g0) = g(F(0,0,0))$, we get that F and g are W-compatible.

3 Main results

Now, we introduce the following concepts.

Definition 3.1 Let (X,d) be a K-metric space with a cone P having nonempty interior (normal or non-normal). Mappings $F : X^3 \to X$ and $g : X \to X$ are said to satisfy the $E.A.$ property if there exist sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that

$$\begin{aligned}
&\lim_{n\to\infty} F(x_n,y_n,z_n) = \lim_{n\to\infty} g(x_n) = x, \\
&\lim_{n\to\infty} F(y_n,z_n,x_n) = \lim_{n\to\infty} g(y_n) = y, \\
&\lim_{n\to\infty} F(z_n,x_n,y_n) = \lim_{n\to\infty} g(z_n) = z
\end{aligned}$$

for some $x,y,z \in X$.

Definition 3.2 Let (X,d) be a K-metric space with a cone P having nonempty interior (normal or non-normal). Mappings $F : X^3 \to X$ and $g : X \to X$ are said to satisfy the CLR_g property if there exist sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that

$$\begin{aligned}
&\lim_{n\to\infty} F(x_n,y_n,z_n) = \lim_{n\to\infty} g(x_n) = gx, \\
&\lim_{n\to\infty} F(y_n,z_n,x_n) = \lim_{n\to\infty} g(y_n) = gy, \\
&\lim_{n\to\infty} F(z_n,x_n,y_n) = \lim_{n\to\infty} g(z_n) = gz
\end{aligned}$$

for some $x,y,z \in X$.

Theorem 3.3 *Let (X,d) be a K-metric space with a cone P having nonempty interior (normal or non-normal), and let $F : X^3 \to X$ and $g : X \to X$ be mappings satisfying the CLR_g property. Suppose that for any $x,y,z,u,v,w \in X$, the following condition holds:*

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr) \\
&\quad \preceq a_1 d\bigl(F(x,y,z),gx\bigr) + a_2 d\bigl(F(y,z,x),gy\bigr) + a_3 d\bigl(F(z,x,y),gz\bigr) \\
&\qquad + a_4 d\bigl(F(u,v,w),gu\bigr) + a_5 d\bigl(F(v,w,u),gv\bigr) + a_6 d\bigl(F(w,u,v),gw\bigr) \\
&\qquad + a_7 d\bigl(F(u,v,w),gx\bigr) + a_8 d\bigl(F(v,w,u),gy\bigr) + a_9 d\bigl(F(w,u,v),gz\bigr) \\
&\qquad + a_{10} d\bigl(F(x,y,z),gu\bigr) + a_{11} d\bigl(F(y,z,x),gv\bigr) + a_{12} d\bigl(F(z,x,y),gw\bigr) \\
&\qquad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw),
\end{aligned}$$

where a_i, $i = 1,\ldots,15$, are nonnegative real numbers such that $\sum_{i=1}^{15} a_i < 1$. Then F and g have a tripled coincidence point.

Proof Since F and g satisfy the CLR_g property, there exist sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = gx, \tag{1}$$

$$\lim_{n\to\infty} F(y_n, z_n, x_n) = \lim_{n\to\infty} g(y_n) = gy, \tag{2}$$

$$\lim_{n\to\infty} F(z_n, x_n, y_n) = \lim_{n\to\infty} g(z_n) = gz \tag{3}$$

for some $x, y, z \in X$.

Now, we prove that $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$. Note that for each $n \in \mathbb{N}$, we have

$$d\bigl(F(x, y, z), gx\bigr) \preceq d\bigl(F(x, y, z), F(x_n, y_n, z_n)\bigr) + d\bigl(F(x_n, y_n, z_n), gx\bigr). \tag{4}$$

On the other hand, applying the given contractive condition and using the triangular inequality, we obtain that

$$\begin{aligned}
&d\bigl(F(x, y, z), F(x_n, y_n, z_n)\bigr) \\
&\quad\preceq a_1 d\bigl(F(x, y, z), gx\bigr) + a_2 d\bigl(F(y, z, x), gy\bigr) + a_3 d\bigl(F(z, x, y), gz\bigr) \\
&\qquad + a_4 d\bigl(F(x_n, y_n, z_n), gx_n\bigr) + a_5 d\bigl(F(y_n, z_n, x_n), gy_n\bigr) + a_6 d\bigl(F(z_n, x_n, y_n), gz_n\bigr) \\
&\qquad + a_7 d\bigl(F(x_n, y_n, z_n), gx\bigr) + a_8 d\bigl(F(y_n, z_n, x_n), gy\bigr) + a_9 d\bigl(F(z_n, x_n, y_n), gz\bigr) \\
&\qquad + a_{10} d\bigl(F(x, y, z), gx_n\bigr) + a_{11} d\bigl(F(y, z, x), gy_n\bigr) + a_{12} d\bigl(F(z, x, y), gz_n\bigr) \\
&\qquad + a_{13} d(gx, gx_n) + a_{14} d(gy, gy_n) + a_{15} d(gz, gz_n) \\
&\quad\preceq a_1 d\bigl(F(x, y, z), gx\bigr) + a_2 d\bigl(F(y, z, x), gy\bigr) + a_3 d\bigl(F(z, x, y), gz\bigr) \\
&\qquad + a_4 \bigl[d\bigl(F(x_n, y_n, z_n), gx\bigr) + d(gx, gx_n)\bigr] + a_5 \bigl[d\bigl(F(y_n, z_n, x_n), gy\bigr) + d(gy, gy_n)\bigr] \\
&\qquad + a_6 \bigl[d\bigl(F(z_n, x_n, y_n), gz\bigr) + d(gz, gz_n)\bigr] + a_7 d\bigl(F(x_n, y_n, z_n), gx\bigr) \\
&\qquad + a_8 d\bigl(F(y_n, z_n, x_n), gy\bigr) + a_9 d\bigl(F(z_n, x_n, y_n), gz\bigr) + a_{10} \bigl[d\bigl(F(x, y, z), gx\bigr) + d(gx, gx_n)\bigr] \\
&\qquad + a_{11} \bigl[d\bigl(F(y, z, x), gy\bigr) + d(gy, gy_n)\bigr] + a_{12} \bigl[d\bigl(F(z, x, y), gz\bigr) + d(gz, gz_n)\bigr] \\
&\qquad + a_{13} d(gx, gx_n) + a_{14} d(gy, gy_n) + a_{15} d(gz, gz_n)
\end{aligned}$$

for all $n \in \mathbb{N}$. Combining the above inequality with (4), we have

$$\begin{aligned}
&d\bigl(F(x, y, z), gx\bigr) \\
&\quad\preceq a_1 d\bigl(F(x, y, z), gx\bigr) + a_2 d\bigl(F(y, z, x), gy\bigr) + a_3 d\bigl(F(z, x, y), gz\bigr) \\
&\qquad + a_4 \bigl[d\bigl(F(x_n, y_n, z_n), gx\bigr) + d(gx, gx_n)\bigr] + a_5 \bigl[d\bigl(F(y_n, z_n, x_n), gy\bigr) + d(gy, gy_n)\bigr] \\
&\qquad + a_6 \bigl[d\bigl(F(z_n, x_n, y_n), gz\bigr) + d(gz, gz_n)\bigr] + a_7 d\bigl(F(x_n, y_n, z_n), gx\bigr) \\
&\qquad + a_8 d\bigl(F(y_n, z_n, x_n), gy\bigr) + a_9 d\bigl(F(z_n, x_n, y_n), gz\bigr) + a_{10} \bigl[d\bigl(F(x, y, z), gx\bigr) + d(gx, gx_n)\bigr] \\
&\qquad + a_{11} \bigl[d\bigl(F(y, z, x), gy\bigr) + d(gy, gy_n)\bigr] + a_{12} \bigl[d\bigl(F(z, x, y), gz\bigr) + d(gz, gz_n)\bigr] \\
&\qquad + a_{13} d(gx, gx_n) + a_{14} d(gy, gy_n) + a_{15} d(gz, gz_n) + d\bigl(F(x_n, y_n, z_n), gx\bigr)
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
&(1 - a_1 - a_{10})d\bigl(F(x,y,z),gx\bigr) - (a_2 + a_{11})d\bigl(F(y,z,x),gy\bigr) - (a_3 + a_{12})d\bigl(F(z,x,y),gz\bigr) \\
&\quad \preceq (1 + a_4 + a_7)d\bigl(F(x_n,y_n,z_n),gx\bigr) + (a_5 + a_8)d\bigl(F(y_n,z_n,x_n),gy\bigr) \\
&\qquad + (a_6 + a_9)d\bigl(F(z_n,x_n,y_n),gz\bigr) + (a_4 + a_{10} + a_{13})d(gx,gx_n) \\
&\qquad + (a_5 + a_{11} + a_{14})d(gy,gy_n) + (a_6 + a_{12} + a_{15})d(gz,gz_n)
\end{aligned} \tag{5}$$

for all $n \in \mathbb{N}$. Similarly, we obtain

$$\begin{aligned}
&(1 - a_1 - a_{10})d\bigl(F(y,z,x),gy\bigr) - (a_2 + a_{11})d\bigl(F(z,x,y),gz\bigr) - (a_3 + a_{12})d\bigl(F(x,y,z),gx\bigr) \\
&\quad \preceq (1 + a_4 + a_7)d\bigl(F(y_n,z_n,x_n),gy\bigr) + (a_5 + a_8)d\bigl(F(z_n,x_n,y_n),gz\bigr) \\
&\qquad + (a_6 + a_9)d\bigl(F(x_n,y_n,z_n),gx\bigr) + (a_4 + a_{10} + a_{13})d(gy,gy_n) \\
&\qquad + (a_5 + a_{11} + a_{14})d(gz,gz_n) + (a_6 + a_{12} + a_{15})d(gx,gx_n)
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&(1 - a_1 - a_{10})d\bigl(F(z,x,y),gz\bigr) - (a_2 + a_{11})d\bigl(F(x,y,z),gx\bigr) - (a_3 + a_{12})d\bigl(F(y,z,x),gy\bigr) \\
&\quad \preceq (1 + a_4 + a_7)d\bigl(F(z_n,x_n,y_n),gz\bigr) + (a_5 + a_8)d\bigl(F(x_n,y_n,z_n),gx\bigr) \\
&\qquad + (a_6 + a_9)d\bigl(F(y_n,z_n,x_n),gy\bigr) + (a_4 + a_{10} + a_{13})d(gz,gz_n) \\
&\qquad + (a_5 + a_{11} + a_{14})d(gx,gx_n) + (a_6 + a_{12} + a_{15})d(gy,gy_n)
\end{aligned} \tag{7}$$

for all $n \in \mathbb{N}$. Adding (5), (6) and (7), we get

$$\begin{aligned}
&(1 - a_1 - a_2 - a_3 - a_{10} - a_{11} - a_{12})\bigl[d\bigl(F(x,y,z),gx\bigr) + d\bigl(F(y,z,x),gy\bigr) + d\bigl(F(z,x,y),gz\bigr)\bigr] \\
&\quad \preceq (1 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9)d\bigl(F(x_n,y_n,z_n),gx\bigr) \\
&\qquad + (1 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9)d\bigl(F(y_n,z_n,x_n),gy\bigr) \\
&\qquad + (1 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9)d\bigl(F(z_n,x_n,y_n),gz_n\bigr) \\
&\qquad + (a_4 + a_5 + a_6 + a_{10} + a_{11} + a_{12}a_{13} + a_{14} + a_{15})d(gx,gx_n) \\
&\qquad + (a_4 + a_5 + a_6 + a_{10} + a_{11} + a_{12}a_{13} + a_{14} + a_{15})d(gy,gy_n) \\
&\qquad + (a_4 + a_5 + a_6 + a_{10} + a_{11} + a_{12}a_{13} + a_{14} + a_{15})d(gz,gz_n)
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&d\bigl(F(x,y,z),gx\bigr) + d\bigl(F(y,z,x),gy\bigr) + d\bigl(F(z,x,y),gz\bigr) \\
&\quad \preceq \alpha d\bigl(F(x_n,y_n,z_n),gx\bigr) + \alpha d\bigl(F(y_n,z_n,x_n),gy\bigr) + \alpha d\bigl(F(z_n,x_n,y_n),gz_n\bigr) \\
&\qquad + \beta d(gx,gx_n) + \beta d(gy,gy_n) + \beta d(gz,gz_n),
\end{aligned}$$

where

$$\alpha = \frac{2}{1 - a_1 - a_2 - a_3 - a_{10} - a_{11} - a_{12}}, \qquad \beta = \frac{1}{1 - a_1 - a_2 - a_3 - a_{10} - a_{11} - a_{12}}.$$

From (1), (2) and (3), for any $c \in E$ with $0_E \ll c$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} &d\big(F(x_n, y_n, z_n), gx\big) \preceq \frac{c}{6\max\{\alpha, \beta\}}, \\ &d\big(F(y_n, z_n, x_n), gy\big) \preceq \frac{c}{6\max\{\alpha, \beta\}}, \\ &d\big(F(z_n, x_n, y_n), gz\big) \preceq \frac{c}{6\max\{\alpha, \beta\}}, \\ &d(gx_n, gx) \preceq \frac{c}{6\max\{\alpha, \beta\}}, \\ &d(gy_n, gy) \preceq \frac{c}{6\max\{\alpha, \beta\}}, \\ &d(gz_n, gz) \preceq \frac{c}{6\max\{\alpha, \beta\}} \end{aligned}$$

for all $n \geq N$. Thus, for all $n \geq N$, we have

$$d\big(F(x, y, z), gx\big) + d\big(F(y, z, x), gy\big) + d\big(F(z, x, y), gz\big) \preceq \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} + \frac{c}{6} = c.$$

It follows that $d(F(x, y, z), gx) = d(F(y, z, x), gy) = d(F(z, x, y), gz) = 0_E$, that is, $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$. □

Corollary 3.4 *Let (X, d) be a K-metric space with a cone P having nonempty interior (normal or non-normal), let $F : X^3 \to X$ and $g : X \to X$ be mappings satisfying the E.A. property, and let $g(X)$ be a closed subspace of X. Suppose that for any $x, y, z, u, v, w \in X$, the following condition holds*:

$$\begin{aligned} &d\big(F(x, y, z), F(u, v, w)\big) \\ &\quad \preceq a_1 d\big(F(x, y, z), gx\big) + a_2 d\big(F(y, z, x), gy\big) + a_3 d\big(F(z, x, y), gz\big) \\ &\qquad + a_4 d\big(F(u, v, w), gu\big) + a_5 d\big(F(v, w, u), gv\big) + a_6 d\big(F(w, u, v), gw\big) \\ &\qquad + a_7 d\big(F(u, v, w), gx\big) + a_8 d\big(F(v, w, u), gy\big) + a_9 d\big(F(w, u, v), gz\big) \\ &\qquad + a_{10} d\big(F(x, y, z), gu\big) + a_{11} d\big(F(y, z, x), gv\big) + a_{12} d\big(F(z, x, y), gw\big) \\ &\qquad + a_{13} d(gx, gu) + a_{14} d(gy, gv) + a_{15} d(gz, gw), \end{aligned}$$

where a_i, $i = 1, \ldots, 15$, are nonnegative real numbers such that $\sum_{i=1}^{15} a_i < 1$. Then F and g have a tripled coincidence point.

Proof Since F and g satisfy the *E.A.* property, there exist sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that

$$\begin{aligned} &\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = p, \\ &\lim_{n\to\infty} F(y_n, z_n, x_n) = \lim_{n\to\infty} g(y_n) = q, \\ &\lim_{n\to\infty} F(z_n, x_n, y_n) = \lim_{n\to\infty} g(z_n) = r \end{aligned}$$

for some $p, q, r \in X$. It follows from $g(X)$ is a closed subspace of X that $p = gx$, $q = gy$ and $r = gz$ for some $x, y, z \in X$, and then F and g satisfy the CLR_g property. By Theorem 3.3, we get that F and g have a tripled coincidence point. □

Corollary 3.5 (Aydi *et al.* [20]) *Let (X, d) be a K-metric space with a cone P having nonempty interior (normal or non-normal), and let $F : X^3 \to X$ and $g : X \to X$ be mappings such that $F(X^3) \subseteq g(X)$. Suppose that for any $x, y, z, u, v, w \in X$, the following condition holds*:

$$\begin{aligned} &d\bigl(F(x,y,z), F(u,v,w)\bigr) \\ &\quad \preceq a_1 d\bigl(F(x,y,z), gx\bigr) + a_2 d\bigl(F(y,z,x), gy\bigr) + a_3 d\bigl(F(z,x,y), gz\bigr) \\ &\qquad + a_4 d\bigl(F(u,v,w), gu\bigr) + a_5 d\bigl(F(v,w,u), gv\bigr) + a_6 d\bigl(F(w,u,v), gw\bigr) \\ &\qquad + a_7 d\bigl(F(u,v,w), gx\bigr) + a_8 d\bigl(F(v,w,u), gy\bigr) + a_9 d\bigl(F(w,u,v), gz\bigr) \\ &\qquad + a_{10} d\bigl(F(x,y,z), gu\bigr) + a_{11} d\bigl(F(y,z,x), gv\bigr) + a_{12} d\bigl(F(z,x,y), gw\bigr) \\ &\qquad + a_{13} d(gx, gu) + a_{14} d(gy, gv) + a_{15} d(gz, gw), \end{aligned}$$

where a_i, $i = 1, \ldots, 15$, are nonnegative real numbers such that $\sum_{i=1}^{15} a_i < 1$. Then F and g have a tripled coincidence point provided that $g(X)$ is a complete subspace of X.

Corollary 3.6 *Let (X, d) be a K-metric space with a cone P having nonempty interior (normal or non-normal), and let $F : X^3 \to X$ and $g : X \to X$ be mappings satisfying the CLR_g property. Suppose that for any $x, y, z, u, v, w \in X$, the following holds*:

$$\begin{aligned} &d\bigl(F(x,y,z), F(u,v,w)\bigr) \\ &\quad \preceq \alpha_1 \bigl[d\bigl(F(x,y,z), gx\bigr) + d\bigl(F(y,z,x), gy\bigr) + d\bigl(F(z,x,y), gz\bigr)\bigr] \\ &\qquad + \alpha_2 \bigl[d\bigl(F(u,v,w), gu\bigr) + d\bigl(F(v,w,u), gv\bigr) + d\bigl(F(w,u,v), gw\bigr)\bigr] \\ &\qquad + \alpha_3 \bigl[d\bigl(F(u,v,w), gx\bigr) + d\bigl(F(v,w,u), gy\bigr) + d\bigl(F(w,u,v), gz\bigr)\bigr] \\ &\qquad + \alpha_4 \bigl[d\bigl(F(x,y,z), gu\bigr) + d\bigl(F(y,z,x), gv\bigr) + d\bigl(F(z,x,y), gw\bigr)\bigr] \\ &\qquad + \alpha_5 \bigl[d(gx, gu) + d(gy, gv) + d(gz, gw)\bigr], \end{aligned}$$

where α_i, $i = 1, \ldots, 5$, are nonnegative real numbers such that $\sum_{i=1}^{5} \alpha_i < 1/3$. Then F and g have a tripled coincidence point.

Proof It suffices to take $a_1 = a_2 = a_3 = \alpha_1$, $a_4 = a_5 = a_6 = \alpha_2$, $a_7 = a_8 = a_9 = \alpha_3$, $a_{10} = a_{11} = a_{12} = \alpha_4$ and $a_{13} = a_{14} = a_{15} = \alpha_5$ in Theorem 3.3 with $\sum_{i=1}^{5} \alpha_i < 1/3$. □

Corollary 3.7 *Let (X, d) be a K-metric space with a cone P having nonempty interior (normal or non-normal), let $F : X^3 \to X$ and $g : X \to X$ be mappings satisfying the E.A. property, and let $g(X)$ be a closed subspace of X. Suppose that for any $x, y, z, u, v, w \in X$, the following holds*:

$$\begin{aligned} &d\bigl(F(x,y,z), F(u,v,w)\bigr) \\ &\quad \preceq \alpha_1 \bigl[d\bigl(F(x,y,z), gx\bigr) + d\bigl(F(y,z,x), gy\bigr) + d\bigl(F(z,x,y), gz\bigr)\bigr] \end{aligned}$$

$$
\begin{aligned}
&+\alpha_2\big[d\big(F(u,v,w),gu\big)+d\big(F(v,w,u),gv\big)+d\big(F(w,u,v),gw\big)\big]\\
&+\alpha_3\big[d\big(F(u,v,w),gx\big)+d\big(F(v,w,u),gy\big)+d\big(F(w,u,v),gz\big)\big]\\
&+\alpha_4\big[d\big(F(x,y,z),gu\big)+d\big(F(y,z,x),gv\big)+d\big(F(z,x,y),gw\big)\big]\\
&+\alpha_5\big[d(gx,gu)+d(gy,gv)+d(gz,gw)\big],
\end{aligned}
$$

where α_i, $i=1,\dots,5$, *are nonnegative real numbers such that* $\sum_{i=1}^{5}\alpha_i<1/3$. *Then* F *and* g *have a tripled coincidence point.*

Proof It follows immediately from Corollary 3.6. □

Corollary 3.8 (Aydi *et al.* [20]) *Let* (X,d) *be a* K*-metric space with a cone* P *having nonempty interior* (*normal or non-normal*), *and let* $F:X^3\to X$ *and* $g:X\to X$ *be mappings such that* $F(X^3)\subseteq g(X)$ *and for any* $x,y,z,u,v,w\in X$, *the following holds*:

$$
\begin{aligned}
&d\big(F(x,y,z),F(u,v,w)\big)\\
&\quad\preceq\alpha_1\big[d\big(F(x,y,z),gx\big)+d\big(F(y,z,x),gy\big)+d\big(F(z,x,y),gz\big)\big]\\
&\qquad+\alpha_2\big[d\big(F(u,v,w),gu\big)+d\big(F(v,w,u),gv\big)+d\big(F(w,u,v),gw\big)\big]\\
&\qquad+\alpha_3\big[d\big(F(u,v,w),gx\big)+d\big(F(v,w,u),gy\big)+d\big(F(w,u,v),gz\big)\big]\\
&\qquad+\alpha_4\big[d\big(F(x,y,z),gu\big)+d\big(F(y,z,x),gv\big)+d\big(F(z,x,y),gw\big)\big]\\
&\qquad+\alpha_5\big[d(gx,gu)+d(gy,gv)+d(gz,gw)\big],
\end{aligned}
$$

where α_i, $i=1,\dots,5$, *are nonnegative real numbers such that* $\sum_{i=1}^{5}\alpha_i<1/3$. *Then* F *and* g *have a tripled coincidence point provided that* $g(X)$ *is a complete subspace of* X.

Next, we prove the existence of a common tripled fixed point theorem for a W-compatible mapping.

Theorem 3.9 *Let* $F:X^3\to X$ *and* $g:X\to X$ *be two mappings which satisfy all the conditions of Theorem* 3.3. *If* F *and* g *are* W*-compatible, then* F *and* g *have a unique common tripled fixed point. Moreover, a common tripled fixed point of* F *and* g *is of the form* (u,u,u) *for some* $u\in X$.

Proof First, we will show that the tripled point of coincidence is unique. Suppose that (x,y,z) and $(x^*,y^*,z^*)\in X^3$ with

$$
\begin{cases} gx=F(x,y,z),\\ gy=F(y,z,x),\\ gz=F(z,x,y)\end{cases}\quad\text{and}\quad\begin{cases} gx^*=F(x^*,y^*,z^*),\\ gy^*=F(y^*,z^*,x^*),\\ gz^*=F(z^*,x^*,y^*).\end{cases}
$$

Using the contractive condition in Theorem 3.3, we obtain

$$
\begin{aligned}
&d\big(gx,gx^*\big)\\
&\quad=d\big(F(x,y,z),F\big(x^*,y^*,z^*\big)\big)\\
&\quad\preceq a_1d\big(F(x,y,z),gx\big)+a_2d\big(F(y,z,x),gy\big)+a_3d\big(F(z,x,y),gz\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + a_4 d\big(F\big(x^*, y^*, z^*\big), gx^*\big) + a_5 d\big(F\big(y^*, z^*, x^*\big), gy^*\big) + a_6 d\big(F\big(z^*, x^*, y^*\big), gz^*\big) \\
&\quad + a_7 d\big(F\big(x^*, y^*, z^*\big), gx\big) + a_8 d\big(F\big(y^*, z^*, x^*\big), gy\big) + a_9 d\big(F\big(z^*, x^*, y^*\big), gz\big) \\
&\quad + a_{10} d\big(F(x, y, z), gx^*\big) + a_{11} d\big(F(y, z, x), gy^*\big) + a_{12} d\big(F(z, x, y), gz^*\big) \\
&\quad + a_{13} d\big(gx, gx^*\big) + a_{14} d\big(gy, gy^*\big) + a_{15} d\big(gz, gz^*\big) \\
&= (a_7 + a_{10} + a_{13}) d\big(gx^*, gx\big) + (a_8 + a_{11} + a_{14}) d\big(gy^*, gy\big) + (a_9 + a_{12} + a_{15}) d\big(gz^*, gz\big).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
d\big(gy, gy^*\big) &= d\big(F(y, z, x), F\big(y^*, z^*, x^*\big)\big) \\
&\preceq (a_7 + a_{10} + a_{13}) d\big(gy^*, gy\big) + (a_8 + a_{11} + a_{14}) d\big(gz^*, gz\big) \\
&\quad + (a_9 + a_{12} + a_{15}) d\big(gx^*, gx\big)
\end{aligned}
$$

and

$$
\begin{aligned}
d\big(gz, gz^*\big) &= d\big(F(z, x, y), F\big(z^*, x^*, y^*\big)\big) \\
&\preceq (a_7 + a_{10} + a_{13}) d\big(gz^*, gz\big) + (a_8 + a_{11} + a_{14}) d\big(gx^*, gx\big) \\
&\quad + (a_9 + a_{12} + a_{15}) d\big(gy^*, gy\big).
\end{aligned}
$$

Adding the above three inequalities, we get

$$
d\big(gx, gx^*\big) + d\big(gy, gy^*\big) + d\big(gz, gz^*\big) \preceq \left(\sum_{i=7}^{15} a_i\right)\big[d\big(gx, gx^*\big) + d\big(gy, gy^*\big) + d\big(gz, gz^*\big)\big].
$$

Since $\sum_{i=7}^{15} a_i < 1$, we obtain that

$$
d\big(gx, gx^*\big) + d\big(gy, gy^*\big) + d\big(gz, gz^*\big) = 0_E,
$$

which implies that

$$
gx = gx^*, \qquad gy = gy^* \quad \text{and} \quad gz = gz^*. \tag{8}
$$

This shows the uniqueness of the tripled point of coincidence of F and g, that is, (gx, gy, gz).

From the contractive condition in Theorem 3.3, we have

$$
\begin{aligned}
&d\big(gx, gy^*\big) \\
&\quad = d\big(F(x, y, z), F\big(y^*, z^*, x^*\big)\big) \\
&\quad \preceq a_1 d\big(F(x, y, z), gx\big) + a_2 d\big(F(y, z, x), gy\big) + a_3 d\big(F(z, x, y), gz\big) \\
&\qquad + a_4 d\big(F\big(y^*, z^*, x^*\big), gy^*\big) + a_5 d\big(F\big(z^*, x^*, y^*\big), gz^*\big) + a_6 d\big(F\big(x^*, y^*, z^*\big), gx^*\big) \\
&\qquad + a_7 d\big(F\big(y^*, z^*, x^*\big), gx\big) + a_8 d\big(F\big(z^*, x^*, y^*\big), gy\big) + a_9 d\big(F\big(x^*, y^*, z^*\big), gz\big) \\
&\qquad + a_{10} d\big(F(x, y, z), gy^*\big) + a_{11} d\big(F(y, z, x), gz^*\big) + a_{12} d\big(F(z, x, y), gx^*\big)
\end{aligned}
$$

$$\begin{aligned}&+ a_{13}d\big(gx,gy^*\big) + a_{14}d\big(gy,gz^*\big) + a_{15}d\big(gz,gx^*\big)\\ &= (a_7 + a_{10} + a_{13})d\big(gy^*,gx\big) + (a_8 + a_{11} + a_{14})d\big(gz^*,gy\big) + (a_9 + a_{12} + a_{15})d\big(gx^*,gz\big).\end{aligned}$$

Similarly, we get

$$\begin{aligned}d\big(gy,gz^*\big) &\preceq (a_7 + a_{10} + a_{13})d\big(gz^*,gy\big) + (a_8 + a_{11} + a_{14})d\big(gx^*,gz\big)\\ &\quad + (a_9 + a_{12} + a_{15})d\big(gy^*,gx\big)\end{aligned}$$

and

$$\begin{aligned}d\big(gz,gx^*\big) &\preceq (a_7 + a_{10} + a_{13})d\big(gx^*,gz\big) + (a_8 + a_{11} + a_{14})d\big(gy^*,gx\big)\\ &\quad + (a_9 + a_{12} + a_{15})d\big(gz^*,gy\big).\end{aligned}$$

Adding the above inequalities, we obtain

$$d\big(gx,gy^*\big) + d\big(gy,gz^*\big) + d\big(gz,gx^*\big) \preceq \left(\sum_{i=7}^{15} a_i\right)\big(d\big(gx,gy^*\big) + d\big(gy,gz^*\big) + d\big(gz,gx^*\big)\big).$$

It follows from $\sum_{i=7}^{15} a_i < 1$ that

$$gx = gy^*, \qquad gy = gz^* \quad \text{and} \quad gz = gx^*. \tag{9}$$

From (8) and (9), we can conclude that

$$gx = gy = gz. \tag{10}$$

This implies that (gx,gx,gx) is the unique tripled point of coincidence of F and g.

Now, let $u = gx$, then we have $u = gx = F(x,y,z) = gy = F(y,z,x) = gz = F(z,x,y)$. Since F and g are W-compatible, we have

$$F(gx,gy,gz) = g\big(F(x,y,z)\big),$$

which due to (10) gives that

$$F(u,u,u) = gu.$$

Consequently, (u,u,u) is a tripled coincidence point of F and g, and so (gu,gu,gu) is a tripled point of coincidence of F and g, and by its uniqueness, we get $gu = gx$. Thus, we obtain

$$u = gx = gu = F(u,u,u).$$

Hence, (u,u,u) is the unique common tripled fixed point of F and g. This completes the proof. □

Corollary 3.10 *Let $F : X^3 \to X$ and $g : X \to X$ be two mappings which satisfy all the conditions of* Corollary 3.4. *If F and g are W-compatible, then F and g have a unique common* *tripled fixed point. Moreover, a common tripled fixed point of F and g is of the form (u,u,u) for some $u \in X$.*

Proof It is similar to the proof of Theorem 3.9. □

Corollary 3.11 (Aydi *et al.* [20]) *Let $F : X^3 \to X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Corollary* 3.5. *If F and g are W-compatible, then F and g have a unique common tripled fixed point. Moreover, the common tripled fixed point of F and g is of the form (u,u,u) for some $u \in X$.*

Here, we give some illustrative examples which demonstrate the validity of the hypotheses and the degree of utility of our results. These examples cannot conclude the existence of a tripled coincidence point and a common tripled fixed point by using the main results of Aydi *et al.* [20].

Example 3.12 Let $X = [0, \frac{1}{2}]$ and $E = \mathbb{R}^2$ with the usual norm. Define the cone $P = \{(x,y) \in E : x, y \geq 0\}$ (*this cone is normal*) and $d : X^2 \to E$ by $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. It is easy to see that (X,d) is a K-metric space over a normal solid cone P.

Consider the mappings $F : X^3 \to X$ and $g : X \to X$ defined as

$$F(x,y,z) = \begin{cases} \frac{1}{20}; & (x,y,z) = (\frac{1}{2},\frac{1}{2},\frac{1}{2}), \\ \frac{x^2+y^2+z^2}{60}; & (x,y,z) \neq (\frac{1}{2},\frac{1}{2},\frac{1}{2}) \end{cases} \quad \text{and} \quad gx = \begin{cases} \frac{1}{2}; & x = \frac{1}{2}, \\ \frac{x}{10}; & x \neq \frac{1}{2}. \end{cases}$$

Since $F(X^3) = [0, \frac{1}{80}) \cup \{\frac{1}{20}\} \nsubseteq g(X) = [0, \frac{1}{20}) \cup \{\frac{1}{2}\}$, the main results of Aydi *et al.* [20] cannot be applied in this case.

Next, we show that our results can be used for this case.

- Let us prove that f and g satisfy the CLR_g property.

  Consider the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X which are defined by

$$x_n = \frac{1}{3n}, \qquad y_n = \frac{1}{4n} \quad \text{and} \quad z_n = \frac{1}{5n}; \quad n = 1, 2, 3 \ldots.$$

  Since

$$\begin{aligned} \lim_{n\to\infty} F(x_n,y_n,z_n) &= \lim_{n\to\infty} g(x_n) = g0, \\ \lim_{n\to\infty} F(y_n,z_n,x_n) &= \lim_{n\to\infty} g(y_n) = g0, \\ \lim_{n\to\infty} F(z_n,x_n,y_n) &= \lim_{n\to\infty} g(z_n) = g0, \end{aligned}$$

  thus F and g satisfy the CLR_g property with these sequences.

- Next, we will show that F and g are W-compatible.

  It easy to see that $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,x,y) = gz$ if and only if $x = y = z = 0$. Since

$$F(g0, g0, g0) = g\bigl(F(0,0,0)\bigr),$$

  mappings F and g are W-compatible.

- Finally, we prove that for $x, y, z, u, v, w \in X$,

$$\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr) \\
&\quad\preceq a_1 d\bigl(F(x,y,z),gx\bigr) + a_2 d\bigl(F(y,z,x),gy\bigr) + a_3 d\bigl(F(z,x,y),gz\bigr) \\
&\qquad + a_4 d\bigl(F(u,v,w),gu\bigr) + a_5 d\bigl(F(v,w,u),gv\bigr) + a_6 d\bigl(F(w,u,v),gw\bigr) \\
&\qquad + a_7 d\bigl(F(u,v,w),gx\bigr) + a_8 d\bigl(F(v,w,u),gy\bigr) + a_9 d\bigl(F(w,u,v),gz\bigr) \\
&\qquad + a_{10} d\bigl(F(x,y,z),gu\bigr) + a_{11} d\bigl(F(y,z,x),gv\bigr) + a_{12} d\bigl(F(z,x,y),gw\bigr) \\
&\qquad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw),
\end{aligned}$$

where $a_1 = a_4 = \frac{2}{9}$, $a_2 = a_3 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{11} = a_{12} = 0$ and $a_{13} = a_{14} = a_{15} = \frac{1}{6}$ such that $\sum_{i=1}^{15} a_i < 1$.

For $x, y, z, u, v, w \in X$, we distinguish the following cases.

Case 1: $(x,y,z) \neq (\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(u,v,w) \neq (\frac{1}{2},\frac{1}{2},\frac{1}{2})$. In this case, we have

$$\begin{aligned}
d\bigl(F(x,y,z),F(u,v,w)\bigr) &= \left(\left|\frac{x^2+y^2+z^2}{60} - \frac{u^2+v^2+w^2}{60}\right|,\right. \\
&\qquad \left.\alpha\left|\frac{x^2+y^2+z^2}{60} - \frac{u^2+v^2+w^2}{60}\right|\right) \\
&\preceq \left(\frac{|x^2-u^2|}{60} + \frac{|y^2-v^2|}{60} + \frac{|z^2-w^2|}{60},\right. \\
&\qquad \left.\alpha\frac{|x^2-u^2|}{60} + \alpha\frac{|y^2-v^2|}{60} + \alpha\frac{|z^2-w^2|}{60}\right) \\
&\preceq \left(\frac{|x-u|}{60} + \frac{|y-v|}{60} + \frac{|z-w|}{60}, \alpha\frac{|x-u|}{60} + \alpha\frac{|y-v|}{60} + \alpha\frac{|z-w|}{60}\right) \\
&= \frac{1}{6}\left(\frac{|x-u|}{10}, \alpha\frac{|x-u|}{10}\right) + \frac{1}{6}\left(\frac{|y-v|}{10}, \alpha\frac{|y-v|}{10}\right) \\
&\qquad + \frac{1}{6}\left(\frac{|z-w|}{10}, \alpha\frac{|z-w|}{10}\right) \\
&= a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw) \\
&\preceq a_1 d\bigl(F(x,y,z),gx\bigr) + a_4 d\bigl(F(u,v,w),gu\bigr) \\
&\qquad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}$$

Case 2: $(x,y,z) \neq (\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(u,v,w) = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$. In this case, we have

$$\begin{aligned}
d\bigl(F(x,y,z),F(u,v,w)\bigr) &= \left(\left|\frac{x^2+y^2+z^2}{60} - \frac{1}{20}\right|, \alpha\left|\frac{x^2+y^2+z^2}{60} - \frac{1}{20}\right|\right) \\
&\preceq \left(\frac{1}{20}, \frac{\alpha}{20}\right) \\
&\preceq \frac{2}{9}\left(\frac{9}{20}, \frac{9\alpha}{20}\right) \\
&= a_4 d\bigl(F(u,v,w),gu\bigr)
\end{aligned}$$

$$\begin{aligned}&\preceq a_1 d\big(F(x,y,z),gx\big)+a_4 d\big(F(u,v,w),gu\big)\\&\quad+a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw).\end{aligned}$$

Case 3: $(x,y,z)=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(u,v,w)\neq(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. In this case, we have

$$\begin{aligned}d\big(F(x,y,z),F(u,v,w)\big)&=\left(\left|\frac{1}{20}-\frac{u^2+v^2+w^2}{60}\right|,\alpha\left|\frac{1}{20}-\frac{u^2+v^2+w^2}{60}\right|\right)\\&\preceq\left(\frac{1}{20},\frac{\alpha}{20}\right)\\&\preceq\frac{2}{9}\left(\frac{9}{20},\frac{9\alpha}{20}\right)\\&=a_1 d\big(F(x,y,z),gx\big)\\&\preceq a_1 d\big(F(x,y,z),gx\big)+a_4 d\big(F(u,v,w),gu\big)\\&\quad+a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw).\end{aligned}$$

Case 4: $(x,y,z)=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(u,v,w)=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Clearly,

$$\begin{aligned}&d\big(F(x,y,z),F(u,v,w)\big)\\&\quad\preceq a_1 d\big(F(x,y,z),gx\big)+a_2 d\big(F(y,z,x),gy\big)+a_3 d\big(F(z,x,y),gz\big)\\&\qquad+a_4 d\big(F(u,v,w),gu\big)+a_5 d\big(F(v,w,u),gv\big)+a_6 d\big(F(w,u,v),gw\big)\\&\qquad+a_7 d\big(F(u,v,w),gx\big)+a_8 d\big(F(v,w,u),gy\big)+a_9 d\big(F(w,u,v),gz\big)\\&\qquad+a_{10} d\big(F(x,y,z),gu\big)+a_{11} d\big(F(y,z,x),gv\big)+a_{12} d\big(F(z,x,y),gw\big)\\&\qquad+a_{13}d(gx,gu)+a_{14}d(gy,gv)+a_{15}d(gz,gw).\end{aligned}$$

Hence, all the hypotheses of Theorem 3.3 and Theorem 3.9 hold. Clearly, $(0,0,0)$ is the unique common tripled fixed point of F and g.

Example 3.13 Let $X=[0,1]$ and $E=C^1_{\mathbb{R}}[0,1]$ with the norm $\|f\|=\|f\|_\infty+\|f'\|_\infty$ for all $f\in E$. Define the cone $P=\{f\in E: f(t)\geq 0 \text{ for } t\in[0,1]\}$ (*this cone is not normal*) and $d: X^2\to E$ by $d(x,y)=|x-y|\varphi$ for a fixed $\varphi\in P$ (*e.g.*, $\varphi(t)=e^t$ for $t\in[0,1]$). It is easy to see that (X,d) is a K-metric space over a non-normal solid cone.

Consider the mappings $F: X^3\to X$ and $g: X\to X$ defined as

$$F(x,y,z)=\begin{cases}\frac{1}{9}; & (x,y,z)=(1,1,1),\\ \frac{x+y+z}{90}; & (x,y,z)\neq(1,1,1)\end{cases}\quad\text{and}\quad gx=\begin{cases}1; & x=1,\\ \frac{x}{9}; & x\neq 1.\end{cases}$$

Since $F(X^3)=[0,\frac{1}{30})\cup\{\frac{1}{9}\}\nsubseteq g(X)=[0,\frac{1}{9})\cup\{1\}$, the main results of Aydi *et al.* [20] cannot be applied in this case.

Next, we show that our results can be used for this case.

- Let us prove that f and g satisfy the CLR_g property.

  Consider the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X which are defined by

$$x_n=\frac{1}{2n},\qquad y_n=\frac{1}{3n}\quad\text{and}\quad z_n=\frac{1}{4n};\quad n=1,2,3\ldots.$$

Since

$$\begin{aligned}
&\lim_{n\to\infty} F(x_n,y_n,z_n) = \lim_{n\to\infty} g(x_n) = g0,\\
&\lim_{n\to\infty} F(y_n,z_n,x_n) = \lim_{n\to\infty} g(y_n) = g0,\\
&\lim_{n\to\infty} F(z_n,x_n,y_n) = \lim_{n\to\infty} g(z_n) = g0,
\end{aligned}$$

thus F and g satisfy the CLR_g property with these sequences.

- Next, we will show that F and g are W-compatible.

  It is obtained that $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,x,y) = gz$ if and only if $x = y = z = 0$. Since

  $$F(g0,g0,g0) = g\bigl(F(0,0,0)\bigr),$$

  mappings F and g are W-compatible.
- Finally, we prove that for $x,y,z,u,v,w \in X$,

  $$\begin{aligned}
  &d\bigl(F(x,y,z),F(u,v,w)\bigr)\\
  &\quad\preceq a_1 d\bigl(F(x,y,z),gx\bigr) + a_2 d\bigl(F(y,z,x),gy\bigr) + a_3 d\bigl(F(z,x,y),gz\bigr)\\
  &\qquad + a_4 d\bigl(F(u,v,w),gu\bigr) + a_5 d\bigl(F(v,w,u),gv\bigr) + a_6 d\bigl(F(w,u,v),gw\bigr)\\
  &\qquad + a_7 d\bigl(F(u,v,w),gx\bigr) + a_8 d\bigl(F(v,w,u),gy\bigr) + a_9 d\bigl(F(w,u,v),gz\bigr)\\
  &\qquad + a_{10} d\bigl(F(x,y,z),gu\bigr) + a_{11} d\bigl(F(y,z,x),gv\bigr) + a_{12} d\bigl(F(z,x,y),gw\bigr)\\
  &\qquad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw),
  \end{aligned}$$

  where $a_1 = a_4 = \frac{1}{4}$, $a_2 = a_3 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{11} = a_{12} = 0$ and $a_{13} = a_{14} = a_{15} = \frac{1}{10}$ such that $\sum_{i=1}^{15} a_i < 1$.

For $x,y,z,u,v,w \in X$, we distinguish the following cases.

Case 1: $(x,y,z) \neq (1,1,1)$ and $(u,v,w) \neq (1,1,1)$. In this case, we have

$$\begin{aligned}
d\bigl(F(x,y,z),F(u,v,w)\bigr) &= \left|\frac{x+y+z}{90} - \frac{u+v+w}{90}\right|\varphi\\
&\preceq \frac{1}{10}\frac{|x-u|}{9}\varphi + \frac{1}{10}\frac{|y-v|}{9}\varphi + \frac{1}{10}\frac{|z-w|}{9}\varphi\\
&= a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw)\\
&\preceq a_1 d\bigl(F(x,y,z),gx\bigr) + a_4 d\bigl(F(u,v,w),gu\bigr)\\
&\quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}$$

Case 2: $(x,y,z) \neq (1,1,1)$ and $(u,v,w) = (1,1,1)$. In this case, we have

$$\begin{aligned}
d\bigl(F(x,y,z),F(u,v,w)\bigr) &= \left|\frac{x+y+z}{90} - \frac{1}{9}\right|\varphi\\
&\preceq \frac{1}{9}\varphi
\end{aligned}$$

$$
\begin{aligned}
&\preceq \left(\frac{1}{4}\right)\left(\frac{8}{9}\right)\varphi \\
&= a_4 d\bigl(F(u,v,w),gu\bigr) \\
&\preceq a_1 d\bigl(F(x,y,z),gx\bigr) + a_4 d\bigl(F(u,v,w),gu\bigr) \\
&\quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}
$$

Case 3: $(x,y,z) = (1,1,1)$ and $(u,v,w) \neq (1,1,1)$. In this case, we have

$$
\begin{aligned}
d\bigl(F(x,y,z),F(u,v,w)\bigr) &= \left|\frac{1}{9} - \frac{u+v+w}{90}\right|\varphi \\
&\preceq \frac{1}{9}\varphi \\
&\preceq \left(\frac{1}{4}\right)\left(\frac{8}{9}\right)\varphi \\
&= a_1 d\bigl(F(x,y,z),gx\bigr) \\
&\preceq a_1 d\bigl(F(x,y,z),gx\bigr) + a_4 d\bigl(F(u,v,w),gu\bigr) \\
&\quad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}
$$

Case 4: $(x,y,z) = (1,1,1)$ and $(u,v,w) = (1,1,1)$. Clearly,

$$
\begin{aligned}
&d\bigl(F(x,y,z),F(u,v,w)\bigr) \\
&\quad\preceq a_1 d\bigl(F(x,y,z),gx\bigr) + a_2 d\bigl(F(y,z,x),gy\bigr) + a_3 d\bigl(F(z,x,y),gz\bigr) \\
&\qquad + a_4 d\bigl(F(u,v,w),gu\bigr) + a_5 d\bigl(F(v,w,u),gv\bigr) + a_6 d\bigl(F(w,u,v),gw\bigr) \\
&\qquad + a_7 d\bigl(F(u,v,w),gx\bigr) + a_8 d\bigl(F(v,w,u),gy\bigr) + a_9 d\bigl(F(w,u,v),gz\bigr) \\
&\qquad + a_{10} d\bigl(F(x,y,z),gu\bigr) + a_{11} d\bigl(F(y,z,x),gv\bigr) + a_{12} d\bigl(F(z,x,y),gw\bigr) \\
&\qquad + a_{13} d(gx,gu) + a_{14} d(gy,gv) + a_{15} d(gz,gw).
\end{aligned}
$$

Therefore, all the hypotheses of Theorem 3.3 and Theorem 3.9 hold. It is easy to see that a point $(0,0,0)$ is the unique common tripled fixed point of F and g.

4 Conclusions

Our results never require the conditions on completeness (or closedness) of the underlying space (or subspaces) together with the conditions on continuity in respect of any one of the involved mappings. Therefore, our results are a generalization of the results of Aydi *et al.* [20], Abbas *et al.* [3], Olaleru [12] and many results in this field.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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