Journal of Inequalities and Applications
a SpringerOpen Journal

REVIEW Open Access

Monotonicity inequalities for L_p Blaschke-Minkowski homomorphisms

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Abstract

Schuster introduced the notion of Blaschke-Minkowski homomorphism and considered its Shephard problems. Wang gave the definition of L_p Blaschke-Minkowski homomorphisms and considered its Shephard problems for volume. In this paper, we obtain its Shephard type inequalities for the affine surface area and two monotonicity inequalities for L_p Blaschke-Minkowski homomorphisms are established.

MSC: 52A20; 52A40

Keywords: L_p Blaschke-Minkowski homomorphisms; Shephard problem; monotonicity inequality

1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}_o^n$ denote the set of convex bodies and containing the origin in their interiors, and let $\mathcal{K}_e^n$ denote origin-symmetric convex bodies in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the n-dimensional volume of body K.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty,+\infty)$, is defined by (see [1, 2])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

A function Φ defined on $\mathcal{K}^n$ and taking values in an Ablelian semigroup is called a valuation if

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi K + \Phi L,$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{K}^n$.

The theory of real valued valuations is at the center of convex geometry. A systematic study was initiated by Blaschke in the 1930s, and then Hadwiger [3] focused on classifying valuations on compact convex sets in $\mathbb{R}^n$ and obtained the famous Hadwiger's characterization theorem. Schneider obtained first results on convex body valued valuations with Minkowski addition in 1970s. The survey [4, 5] and the book [6] are an excellent sources for the classical theory of valuations. Some more recent results can see [4, 5, 7–9].

Recently, Schuster in [10] gave the definition of Blaschke-Minkowski homomorphism as follows:

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) Φ is continuous.

(b) Φ is a Blaschke-Minkowski addition, *i.e.*, for all $K, L \in \mathcal{K}^n$

$$\Phi(K\#L) = \Phi K + \Phi L.$$

(c) Φ intertwines rotation, *i.e.*, for all $K \in \mathcal{K}^n$ and $\vartheta \in SO(n)$

$$\Phi(\vartheta K) = \vartheta \Phi K.$$

Here $K\#L$ is the Blaschke sum of the convex bodies K and L, *i.e.*, $S(K\#L,\cdot) = S(K,\cdot) + S(L,\cdot)$. $SO(n)$ is the group of rotation in n dimensions.

The L_p Minkowski valuation was introduced by Ludwig (see [11]). A function $\Psi : \mathcal{K}_o^n \to \mathcal{K}_o^n$ is called an L_p Minkowski valuation if

$$\Psi(K \cup L) +_p \Psi(K \cap L) = \Psi K +_p \Psi L,$$

whenever $K, L, K \cup L \in \mathcal{K}_o^n$, and here '$+_p$' is L_p Minkowski addition (see (2.2)).

Then, Wang in [12] introduced the L_p Blaschke-Minkowski homomorphism and gave Theorem 1.A.

Definition 1.1 Let $p > 1$, a map $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ satisfying the following properties (a), (b) and (c) is called an L_p Blaschke-Minkowski homomorphism.

(a) Φ_p is continuous with respect to Hausdorff metric.

(b) $\Phi_p(K\#_p L) = \Phi_p K +_p \Phi_p L$ for all $K, L \in \mathcal{K}_e^n$.

(c) Φ_p is $SO(n)$ equivariant, *i.e.*, $\Phi_p(\vartheta K) = \vartheta \Phi_p K$ for all $\vartheta \in SO(n)$ and all $K \in \mathcal{K}_e^n$.

Here $K\#_p L$ denotes the L_p Blaschke sum of $K, L \in \mathcal{K}_e^n$, *i.e.*, $S_p(K\#_p L,\cdot) = S_p(K,\cdot) +_p S_p(L,\cdot)$.

Theorem 1.A *Let $p > 1$ and $p \neq n$. If $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ is an L_p Blaschke-Minkowski homomorphism, then there is a nonnegative function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$, such that*

$$h^p(\Phi_p K,\cdot) = S_p(K,\cdot) * g.$$

A map $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ is even because of $\Phi_p(K) = \Phi_p(-K)$ for $K \in \mathcal{K}_e^n$.

A map $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ is an even L_p Blaschke-Minkowski homomorphism, if and only if there is a convex body of revolution $F \in \mathcal{K}_e^n$, unique up to translation, such that

$$h^p(\Phi_p K,\cdot) = S_p(K,\cdot) * h(F,\cdot). \tag{1.1}$$

In [12], together with the L_p Blaschke-Minkowski homomorphisms, Wang studied the Shephard problems of L_p Blaschke-Minkowski homomorphisms.

Theorem 1.B *Let $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$ is an L_p Blaschke-Minkowski homomorphism, $K \in \mathcal{K}_e^n$, $L \in \Phi_p \mathcal{K}_e^n$ and p is not an even integer. If $1 < p < n$, then*

$$\Phi_p K \subseteq \Phi_p L \quad \Rightarrow \quad V(K) \leq V(L).$$

If $p > n$, then

$$\Phi_p K \subseteq \Phi_p L \quad \Rightarrow \quad V(K) \geq V(L),$$

and $V(K) = V(L)$, if and only if $K = L$.

In this article, we continuously study the L_p Blaschke-Minkowski homomorphisms. Firstly, comparing with Theorem 1.B, we give the L_p-affine surface area of Shephard type inequalities for the L_p Blaschke-Minkowski homomorphisms.

Theorem 1.1 *Let $K \in \mathcal{F}_e^n$, $L \in \omega_p^n$ and $n \neq p > 1$. If $\Phi_p K \subseteq \Phi_p L$, then*

$$\Omega_p(K) \leq \Omega_p(L),$$

with equality if and only if K and L are dilates.

Here $\omega_p^n = \{N \in \mathcal{F}_e^n :$ there exists $Z \in \mathcal{Z}_p^n$ with $f_p(N, \cdot) = h(Z, \cdot)^{-(n+p)}\}$, where $f_p(N, \cdot)$ is the p-curvature function of N, $\mathcal{F}_e^n$ denotes the set of convex bodies in $\mathcal{K}_e^n$ with positive continuous curvature function and $\mathcal{Z}_p^n$ denotes the set of L_p Blaschke-Minkowski homomorphisms. Besides, $\Omega_p(K)$ denotes the L_p-affine surface area of $K \in \mathcal{K}_o^n$.

Actually, we will prove a more general result than Theorem 1.1 in Section 3.

Further, associated with the L_p Blaschke-Minkowski homomorphisms, we establish the following monotonicity inequalities.

Theorem 1.2 *Let $K, L \in K_e^n$, $n \neq p > 1$. If for every $Q \in K_e^n$, $V_p(K, Q) \leq V_p(L, Q)$, then*

$$V(\Phi_p K) \leq V(\Phi_p L),$$

with equality if and only if K and L are dilates.

Theorem 1.3 *Let $K, L \in \mathcal{K}_e^n$, $n \neq p > 1$. If for every $Q \in K_e^n$, $V_p(K, Q) \leq V_p(L, Q)$, then*

$$V\big(\Phi_p^* L\big) \leq V\big(\Phi_p^* K\big),$$

with equality if and only if K and L are dilates.

Here and the following we write $\Phi_p^* K$ for the polar of $\Phi_p K$.

2 Notations and background materials

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0, +\infty)$, is defined by (see [1])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin), and let $\mathcal{S}_e^n$ denote the set of origin-symmetric star bodies.

If $K \in \mathcal{K}^n$, the polar body of K, K^*, is defined by (see [1])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1, y \in K\}.$$

If $K \in \mathcal{K}_o^n$, then the support function and radial function of K^*, the polar body of K, are given (see [1]), respectively, by

$$h_{K^*} = \frac{1}{\rho_K}, \qquad \rho_{K^*} = \frac{1}{h_K}. \tag{2.1}$$

2.1 L_p-mixed volume

For $K, L \in K_o^n$, $p \ge 1$ and $\lambda, \mu \ge 0$ (not both zero), the Firey L_p-combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined by (see [13])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p, \tag{2.2}$$

where '$\cdot$' in $\lambda \cdot K$ denotes the Firey scalar multiplication.

Associated with Firey L_p-combination (2.2) of convex bodies, Lutwak (see [14]) introduced the following. For $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$ and $p \ge 1$, the L_p-mixed volume, $V_p(K, L)$, of K and L is defined by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

It was shown in [14] that corresponding to each $K \in \mathcal{K}_o^n$, there exists a positive Borel measure on S^{n-1}, $S_p(K, \cdot)$ of K, such that for each $L \in \mathcal{K}_o^n$,

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v)\, dS_p(K, v). \tag{2.3}$$

The measure $S_p(K, \cdot)$ is just the L_p surface area measure of K, which is absolutely continuous with respect to classical surface area measure $S(K, \cdot)$ and has a Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h(K, \cdot)^{1-p}.$$

Obviously, from (2.3), it follows immediately that, for each $K \in \mathcal{K}_o^n$,

$$V_p(K, K) = V(K). \tag{2.4}$$

The Minkowski inequality for the L_p-mixed volume is called L_p-Minkowski inequality. The L_p-Minkowski inequality can be stated that (see [14]): *If $K, L \in \mathcal{K}_o^n$ and $p \ge 1$, then*

$$V_p(K, L) \ge V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}, \tag{2.5}$$

with equality for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

A convex body $K \in \mathcal{K}_o^n$ is said to have a L_p-curvature function (see [14]) $f_p(K,\cdot): S^{n-1} \to \mathbb{R}$, if its L_p surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S and

$$\frac{dS_p(K,\cdot)}{dS} = f_p(K,\cdot). \tag{2.6}$$

2.2 L_p-dual mixed volume

For $K, L \in S_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in \mathcal{S}_o^n$, of K and L is defined by (see [14])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda\rho(K,\cdot)^{-p} + \mu\rho(L,\cdot)^{-p}. \tag{2.7}$$

Using the L_p-harmonic radial combination (2.7), Lutwak (see [14]) introduced the notion of L_p-dual mixed volume. For $K, L \in S_o^n$ and $p \geq 1$, the L_p-dual mixed volume, $\tilde{V}_{-p}(K,L)$, of K and L is defined by

$$\frac{n}{-p}\tilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the L_p-dual mixed volume:

$$\tilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(\upsilon)\rho_L^{-p}(\upsilon)\, dS(\upsilon), \tag{2.8}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From (2.8), it follows that for each $K \in S_o^n$ and $p \geq 1$,

$$\tilde{V}_{-p}(K,K) = V(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(\upsilon)\, dS(\upsilon). \tag{2.9}$$

Lutwak in [14] established the L_p-dual Minkowski inequality: *If $K, L \in S_o^n$, and $p \geq 1$, then*

$$\tilde{V}_{-p}(K,L) \geq V(K)^{\frac{n+p}{n}} V(L)^{\frac{-p}{n}}, \tag{2.10}$$

with equality if and only if K and L are dilates.

2.3 L_p-mixed affine surface area

Let $\mathcal{F}^n$, $\mathcal{F}_o^n$ denote the set of convex bodies in $\mathcal{K}^n$, $\mathcal{K}_o^n$ with positive continuous curvature function.

Lutwak (see [15]) defined the ith mixed affine surface area as follows: For $K, L \in \mathcal{F}^n$ and $i \in \mathbb{R}$, the ith mixed affine surface area, $\Omega_i(K,L)$, of K and L is defined by

$$\Omega_i(K,L) = \int_{S^{n-1}} f(K,u)^{\frac{n-i}{n+1}} f(L,u)^{\frac{i}{n+1}}\, dS(u).$$

For $K, L \in \mathcal{F}_o^n$, $p \geq 1$ and $i \in \mathbb{R}$, the L_p-mixed affine surface area, $\Omega_{p,i}(K,L)$, of K and L is defined by Wang and Leng (see [16])

$$\Omega_{p,i}(K,L) = \int_{S^{n-1}} f_p(K,u)^{\frac{n-i}{n+p}} f_p(L,u)^{\frac{i}{n+p}}\, dS(u). \tag{2.11}$$

Obviously, from (2.11), we have

$$\Omega_{p,i}(K,K) = \Omega_p(K). \tag{2.12}$$

Specially, for the case $i = -p$, we write $\Omega_{p,-p}(K,L) = \Omega_{-p}(K,L)$. Associated with (2.6), then

$$\begin{aligned}\Omega_{-p}(K,L) &= \int_{S^{n-1}} f_p(K,u) f_p(L,u)^{\frac{-p}{n+p}}\, dS(u)\\ &= \int_{S^{n-1}} f_p(L,u)^{\frac{-p}{n+p}}\, dS_p(K,u).\end{aligned} \tag{2.13}$$

The Minkowski inequality for the L_p-mixed affine surface area was given by Wang and Leng (see [16]): *If $K, L \in \mathcal{F}_o^n$, $p \geq 1$ and $i \in \mathbb{R}$, then for $i < 0$ or $i > n$,*

$$\Omega_{p,i}(K,L)^n \geq \Omega_p(K)^{n-i}\Omega_p(L)^i, \tag{2.14}$$

with equality for $p = 1$ if and only if K and L are homothetic, for $n \neq p > 1$ if and only if K and L are dilates; for $0 < i < n$, (2.14) is reverse; for $i = 0$ or $i = n$, (2.14) is identical.

Combining with (2.14), they in [16] obtain the following result. *If $K, L \in \mathcal{F}_o^n$ and $p \geq 1$,*

$$\Omega_{-p}(K,L) = \Omega_{p,-p}(K,L) \geq \Omega_p(K)^{\frac{n+p}{n}}\Omega_p(L)^{\frac{-p}{n}}, \tag{2.15}$$

with equality for $n \neq p > 1$ if and only if K and L are dilates, for $p = 1$ if and only if K and L are homothetic.

2.4 Spherical convolution and spherical harmonics

In the following we state some material on convolution and spherical harmonics, and they can be found in the references (see [17, 18]).

In order to state the material on spherical harmonics, we first introduce further basic notions connected to $SO(n)$ and S^{n-1}. As usual, $SO(n)$ and S^{n-1} will be equipped with invariant probability measures. Let $\mathcal{C}(SO(n))$, $\mathcal{C}(S^{n-1})$ be the spaces of continuous function on $SO(n)$ and S^{n-1} with uniform topology and $\mathcal{M}(SO(n))$, $\mathcal{M}(S^{n-1})$ their dual spaces of signed finite Borel measures with weak topology. If $\mu, \sigma \in \mathcal{M}(SO(n))$, the convolution $\mu * \sigma$ is defined by

$$\int_{SO(n)} f(\vartheta)\, d(\mu * \sigma)(\vartheta) = \int_{SO(n)}\int_{SO(n)} f(\eta\tau)\, d\mu(\eta)\, d\sigma(\tau),$$

for every $f \in \mathcal{C}(SO(n))$ and $\vartheta \in SO(n)$. The sphere S^{n-1} is identical with the honogeneous space $SO(n)/SO(n-1)$, where $SO(n-1)$ denotes the subgroup of rotations leaving the pole $\hat{e}$ of S^{n-1} fixed.

For $\mu \in \mathcal{M}(SO(n))$, the convolutions $\mu * f \in \mathcal{C}(SO(n))$ and $f * \mu \in \mathcal{C}(SO(n))$ with a function $f \in \mathcal{C}(SO(n))$ are defined by

$$\begin{aligned}(f * \mu)(\eta) &= \int_{SO(n)} f\left(\eta\vartheta^{-1}\right) d\mu(\vartheta),\\ (\mu * f)(\eta) &= \int_{SO(n)} \vartheta f(\eta)\, d\mu(\vartheta).\end{aligned} \tag{2.16}$$

The canonical pairing of $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$ is defined by

$$\langle \mu, f\rangle = \langle f, \mu\rangle = \int_{S^{n-1}} f(u)\, d\mu(u). \tag{2.17}$$

From (2.16) and (2.17), it follows that (see [18]) if $\mu, \nu \in \mathcal{M}(S^{n-1})$ and $f \in \mathcal{C}(S^{n-1})$, then

$$\langle \mu * \nu, f\rangle = \langle \mu, f * \nu\rangle. \tag{2.18}$$

3 Proofs of theorems

In this section, firstly, we will prove the general form of Theorem 1.1.

Theorem 3.1 *Let $K \in \mathcal{F}_e^n$, $L \in \omega_p^n$ and $n \neq p > 1$. For every $Q \in \mathcal{K}_e^n$, if $V_p(Q, \Phi_p K) \leq V_p(Q, \Phi_p L)$, then*

$$\Omega_p(K) \leq \Omega_p(L),$$

with equality if and only if K and L are dilates.

Wang in [12] gave the following conclusion; this result is a very useful tool for the following proofs.

Lemma 3.1 *If $\Phi_p : \mathcal{K}_e^n \to \mathcal{K}_e^n$, is an L_p Blaschke-Minkowski homomorphism, then for $K, L \in \mathcal{K}_e^n$,*

$$V_p(K, \Phi_p L) = V_p(L, \Phi_p K).$$

Proof of Theorem 3.1 Since $N \in \omega_p^n$, then there exists $Z \in \mathcal{Z}_p^n$ such that

$$h(Z, \cdot) = f_p(N, \cdot)^{\frac{-1}{n+p}}. \tag{3.1}$$

By (2.3), (2.13), and (3.1), we consider

$$\begin{aligned}\frac{\Omega_{-p}(L,N)}{\Omega_{-p}(K,N)} &= \frac{\int_{S^{n-1}} f_p(N,u)^{\frac{-p}{n+p}}\, dS_p(L,u)}{\int_{S^{n-1}} f_p(N,u)^{\frac{-p}{n+p}}\, dS_p(K,u)}\\ &= \frac{\int_{S^{n-1}} h(Z,\cdot)^p\, dS_p(L,u)}{\int_{S^{n-1}} h(Z,\cdot)^p\, dS_p(K,u)}\\ &= \frac{V_p(L,Z)}{V_p(K,Z)}.\end{aligned}$$

Since $Z \in \mathcal{Z}_p^n$, letting $Z = \Phi_p Q$ for $Q \in \mathcal{K}_e^n$, combining with Lemma 3.1, we obtain

$$\frac{V_p(L,Z)}{V_p(K,Z)} = \frac{V_p(L,\Phi_p Q)}{V_p(K,\Phi_p Q)} = \frac{V_p(Q,\Phi_p L)}{V_p(Q,\Phi_p K)}.$$

Therefore, if $V_p(Q,\Phi_p K) \leq V_p(Q,\Phi_p L)$, then we have

$$\Omega_{-p}(L,N) \geq \Omega_{-p}(K,N). \tag{3.2}$$

Due to $L \in \omega_p^n$, taking $N = L$ in (3.2), and together with (2.12) and inequality (2.15), we get

$$\Omega_p(L) \geq \Omega_{-p}(K,L) \geq \Omega_p(K)^{\frac{n+p}{n}} \Omega_p(L)^{\frac{-p}{n}},$$

i.e.,

$$\Omega_p(K) \leq \Omega_p(L). \tag{3.3}$$

According to the equality conditions of (2.15) and (3.2), we see that equality holds in (3.3) for $n \neq p > 1$ if and only if K and L are dilates. □

Proof of Theorem 1.2 Since $Q \in K_e^n$, taking $Q = \Phi_p M$ for $M \in K_e^n$, then

$$V_p(K,Q) \leq V_p(L,Q)$$

can be written as

$$V_p(K,\Phi_p M) \leq V_p(L,\Phi_p M),$$

then from Lemma 3.1, it follows that

$$V_p(M,\Phi_p K) \leq V_p(M,\Phi_p L). \tag{3.4}$$

Since $\Phi_p L \in K_e^n$, let $M = \Phi_p L$ in (3.4), together with (2.4) and (2.5), we can get

$$V(\Phi_p L) \geq V_p(\Phi_p L,\Phi_p K) \geq V(\Phi_p L)^{\frac{n-p}{n}} V(\Phi_p K)^{\frac{p}{n}}, \tag{3.5}$$

such that

$$V(\Phi_p K) \leq V(\Phi_p L). \tag{3.6}$$

According to the equality conditions of (2.5) and (3.5), we see that equality holds in (3.6) for $n \neq p > 1$ if and only if K and L are dilates. □

We turn now to proof of Theorem 1.3. To this end, associate with the L_p Blaschke-Minkowski homomorphism Φ_p, we define a new operator $M_{\Phi_p} : \mathcal{S}_e^n \to \mathcal{K}_e^n$ by

$$h^p(M_{\Phi_p}L,\cdot) = \rho^{n+p}(L,\cdot) * h(F,\cdot). \tag{3.7}$$

By (2.16), the operator M_{Φ_p} is well defined.

Lemma 3.2 *If* $K \in \mathcal{K}_e^n$, $L \in S_e^n$, $n \neq p > 1$, *then*

$$\tilde{V}_{-p}(L, \Phi_p^* K) = V_p(K, M_{\Phi_p L}). \tag{3.8}$$

Proof By (1.1), (2.1), (2.3), (2.8), (2.18), and (3.7), we have

$$\begin{aligned}
\tilde{V}_{-p}(L, \Phi_p^* K) &= \frac{1}{n}\langle \rho_L^{n+p}(u), \rho_{\Phi_p^* K}^{-p}(u)\rangle \\
&= \frac{1}{n}\langle \rho_L^{n+p}(u), h_{\Phi_p K}^{p}(u)\rangle \\
&= \frac{1}{n}\langle \rho_L^{n+p}(u), (S_p(K,u) * h(F,u))\rangle \\
&= \frac{1}{n}\langle \rho_L^{n+p}(u) * h(F,u), S_p(K,u)\rangle \\
&= \frac{1}{n}\langle h^p(M_{\Phi_p} L, u), S_p(K,u)\rangle \\
&= \frac{1}{n}\int_{S^{n-1}} h^p(M_{\Phi_p} L, u)\, dS_p(K,u) \\
&= V_p(K, M_{\Phi_p L}).
\end{aligned}$$

□

Proof of Theorem 1.3 Since $Q \in \mathcal{K}_e^n$, taking $Q = M_{\Phi_p} N$ for any $N \in S_e^n$, then

$$V_p(K,Q) \leq V_p(L,Q)$$

can be written as

$$V_p(K, M_{\Phi_p} N) \leq V_p(L, M_{\Phi_p} N). \tag{3.9}$$

Combining with (3.8), (3.9) can be written as

$$\tilde{V}_{-p}(N, \Phi_p^* K) \leq \tilde{V}_{-p}(N, \Phi_p^* L).$$

But $N \in S_e^n$, taking $N = \Phi_p^* L$, together with (2.9) and inequality (2.10), we get

$$\begin{aligned}
V(\Phi_p^* L) &\geq \tilde{V}_{-p}(\Phi_p^* L, \Phi_p^* K) \\
&\geq V(\Phi_p^* L)^{\frac{n+p}{n}} V(\Phi_p^* K)^{\frac{-p}{n}},
\end{aligned} \tag{3.10}$$

such that

$$V(\Phi_p^* L) \leq V(\Phi_p^* K). \tag{3.11}$$

According to the equality conditions of (2.9) and (3.10), we see that equality holds in (3.11) for $n \neq p > 1$ if and only if K and L are dilates. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The authors would like to deeply thank the referees for very valuable and helpful comments and suggestions, which made the paper more accurate and readable. Research is supported in part by the Natural Science Foundation of China (Grant No. 11371224).

Received: 30 August 2013 Accepted: 26 February 2014 Published: 31 Mar 2014

References

1. Gardner, RJ: Geometric Tomography, 2nd edn. Cambridge University Press, Cambridge, (2006)
2. Schneider, R: Convex Bodies: The Brunn-Minkowski Theory. Cambridge University Press, Cambridge (1993)
3. Hadwiger, H: Vorlesungen uber Inhalt, Oberflache und Isoperimetrie. Springer, Berlin (1957)
4. McMullen, P: Valuations and dissections. In: Gruber, PM, Wills, JM (eds.) Handbook of Convex Geometry, vol. B, pp. 933-990. North-Holland, Amsterdam (1993)
5. McMullen, P, Schneider, R: Valuations on convex bodies. In: Gruber, PM, Wills, JM (eds.) Convexity and Its Applications, pp. 170-247. Birkhäuser, Basel (1983)
6. Klain, DA, Rota, G: Introduction to Geometric Probability. Cambridge University Press, Cambridge, (1997)
7. Alesker, S: Continuous rotation invariant valuations on convex sets. Ann. Math. **149**, 977-1005 (1999)
8. Alesker, S: Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. Geom. Funct. Anal. **11**, 244-272 (2001)
9. Ludwig, M: Ellipsoids and matrix valued valuations. Duke Math. J. **119**, 159-188 (2003)
10. Schuster, FE: Volume inequalities and additive maps of convex bodies. Mathematica **53**, 211-234 (2006)
11. Ludwig, E: Minkowski valuations. Trans. Am. Math. Soc. **357**, 4191-4213 (2005)
12. Wang, W: L_p Blaschke-Minkowski homomorphisms. J. Inequal. Appl. **2013**, 140 (2013)
13. Firey, WJ: p-means of convex bodies. Math. Scand. **10**, 17-24 (1962)
14. Lutwak, E: The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas. Adv. Math. **118**, 244-294 (1996)
15. Lutwak, E: Mixed affine surface area. J. Math. Anal. Appl. **125**, 351-360 (1987)
16. Wang, WD, Leng, GS: L_p-mixed affine surface area. J. Math. Anal. Appl. **335**, 341-354 (2007)
17. Grinberg, E, Zhang, GY: Convolutions, transforms and convex bodies. Proc. Lond. Math. Soc. **78**, 77-115 (1999)
18. Schuster, FE: Convolutions and multiplier transformations of convex bodies. Trans. Am. Math. Soc. **359**, 5567-5591 (2007)

10.1186/1029-242X-2014-131
Cite this article as: Li and Wang: Monotonicity inequalities for L_p Blaschke-Minkowski homomorphisms. *Journal of Inequalities and Applications* 2014, 2014:131