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An hybrid mean value of quadratic Gauss sums and a sum analogous to Kloosterman sums

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Abstract
The main purpose of this paper is, using the analytic methods and the properties of character sums, to study the computational problem of one kind of hybrid mean value involving the quadratic Gauss sums and a new sum analogous to Kloosterman sums, and to give an interesting hybrid mean value formula for it.

MSC: 11L03; 11L40

Keywords: quadratic Gauss sums; a sum analogous to Kloosterman sums; hybrid mean value; identity

1 Introduction

Let $q \geq 3$ be an integer, and let χ be a Dirichlet character mod q. Then for any integer n, the famous quadratic Gauss sums $G(\chi,n;q)$ is defined as follows:

$$G(\chi,n;q)=\sum_{a=1}^{q}\chi(a)\cdot e\left(\frac{na^2}{q}\right),$$

where $e(y)=e^{2\pi iy}$.

This sum plays a very important role in the study of analytic number theory, many famous number theoretic problems are closely related to it. For example, the distribution of primes, the Goldbach problem, the properties of Dirichlet L-functions are some good examples. About the arithmetic properties of $G(\chi,n;q)$, some authors had studied it and obtained many interesting results. For example, if $q=p$ is a prime and $(p,n)=1$, then one can get the estimate $|G(\chi,n;p)| \leq 2\sqrt{p}$. Some other results can be found in references [1–6].

On the other hand, the classical Kloosterman sums $K(m,n;q)$ is defined as

$$K(m,n;q)=\sum_{a=1}^{q-1}{}' e\left(\frac{ma+n\overline{a}}{q}\right),$$

where $\sum_{a=1}^{\prime q-1}$ denotes the summation over all $1 \leq a \leq q$ such that $(a,q)=1$, and $\overline{a}$ denotes the solution of the congruence equation $ax \equiv 1 \bmod q$.

Now we define another sum analogous to Kloosterman sums as follows:

$$S(\chi,q)=\sum_{a=1}^{q-1}{}'\chi(a+\overline{a}).$$

In fact, this sum is a special case of the general character of polynomials, some related results can be found in [7, 8] and [9, 10].

The main purpose of this paper is using the analytic method and the properties of the character sums to study the hybrid mean value properties of $G(\chi,n;p)$ and $S(\chi,p)$, and to give an interesting mean value formula. That is, we shall prove the following two conclusions.

Theorem 1 *Let p be an odd prime, χ be any non-principal even character* (*i.e.* $\chi(-1)=1$) mod p. *Then for any integer n with* $(n,p)=1$, *we have the identity*

$$\left|\sum_{a=1}^{p-1}\chi(a)\cdot e\left(\frac{na^2}{p}\right)\right|^2=2p+\overline{\chi}(2)\cdot\left(\frac{n}{p}\right)\cdot\tau(\chi_2)\cdot\sum_{a=1}^{p-1}\chi(a+\overline{a}),$$

where $(\frac{*}{p})=\chi_2$ denotes the Legendre symbol, and $\tau(\chi_2)=\sum_{a=1}^{p-1}\chi_2(a)\cdot e(\frac{a}{p})$ denotes the classical Gauss sums with $\tau^2(\chi_2)=(\frac{-1}{p})\cdot p$.

Theorem 2 *Let p be an odd prime with $p\equiv 3$* mod 4. *Then for any integer n with* $(n,p)=1$, *we have the identity*

$$\sum_{\chi \bmod p}{}'\left|\sum_{a=1}^{p-1}\chi(a)\cdot e\left(\frac{na^2}{p}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\right|^2=(p-1)\cdot\left(3p^2-6p-1\right),$$

where $\sum'_{\chi \bmod p}$ denotes the summation over all even character mod p, *i.e.* $\chi(-1)=1$.

Some notes: Theorem 1 tells us that there exists a close relationship between $G(\chi,n;p)$ and $S(\chi,p)$. That is, $|G(\chi,n;p)|^2$ can be represented by $S(\chi,p)$.

Since for any odd character χ mod p, we have $G(\chi,n;p)=S(\chi,p)=0$, we only discussed the summation for all even characters χ mod p in Theorem 2.

If $p\equiv 1$ mod 4, then we cannot give a computational formula for the hybrid mean value in Theorem 2. In this case, the difficulty is that we cannot obtain an exact value for the behind formula (13). We hope that the interested reader will stay with us as we turn to further study.

For general integer $q\geq 3$, whether there exists a computational formula for the hybrid mean value

$$\sum_{\chi \bmod q}{}'\left|\sum_{a=1}^{q}\chi(a)\cdot e\left(\frac{na^2}{q}\right)\right|^2\cdot\left|\sum_{a=1}^{q-1}{}'\chi(a+\overline{a})\right|^2$$

is an interesting open problem, where n is any integer with $(n,q)=1$.

2 Several lemmas

In this section, we shall give two simple lemmas, which are necessary in the proofs of our theorems. Hereinafter, we shall use many properties of character sums and Gauss sums, all of these can be found in references [1, 2] and [11]. First we have the following.

Lemma 1 *Let p be an odd prime, χ be any non-principal even character* mod p. *Then we have the identity*

$$\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\right|^2 = 2p + \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{b(b-1)(a^2b-1)}{p}\right),$$

where $(\frac{*}{p})$ denotes the Legendre symbol.

Proof Let $a+\overline{a}=u$, then we have

$$\begin{aligned}\sum_{a=1}^{p-1}\chi(a+\overline{a}) &= \sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=1\\ a+\overline{a}\equiv u \bmod p}}^{p-1}1 = \sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=1\\ a^2-au+1\equiv 0 \bmod p}}^{p-1}1\\ &= \sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=0\\ (2a-u)^2\equiv u^2-4 \bmod p}}^{p-1}1 = \sum_{u=1}^{p-1}\chi(u)\sum_{\substack{a=0\\ a^2\equiv u^2-4 \bmod p}}^{p-1}1. \qquad (1)\end{aligned}$$

Note that for any fixed integer u^2-4, the number of the solutions of the congruence equation $x^2\equiv u^2-4 \bmod p$ are $1+(\frac{u^2-4}{p})$, so from (1) we have

$$\begin{aligned}\sum_{a=1}^{p-1}\chi(a+\overline{a}) &= \sum_{u=1}^{p-1}\chi(u)\left(1+\left(\frac{u^2-4}{p}\right)\right)\\ &= \sum_{u=1}^{p-1}\chi(u)\left(\frac{u^2-4}{p}\right) = \chi(2)\sum_{u=1}^{p-1}\chi(u)\left(\frac{u^2-1}{p}\right). \qquad (2)\end{aligned}$$

Now from (2) and the properties of reduced residue system mod p we have

$$\begin{aligned}\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\right|^2 &= \left|\sum_{u=1}^{p-1}\chi(u)\left(\frac{u^2-1}{p}\right)\right|^2 = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\left(\frac{a^2-1}{p}\right)\left(\frac{b^2-1}{p}\right)\\ &= \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{a^2b^2-1}{p}\right)\left(\frac{b^2-1}{p}\right)\\ &= \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(1+\left(\frac{b}{p}\right)\right)\left(\frac{a^2b-1}{p}\right)\left(\frac{b-1}{p}\right)\\ &= \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{(a^2b-1)(b-1)}{p}\right)\\ &\quad + \sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{(a^2b-1)b(b-1)}{p}\right). \qquad (3)\end{aligned}$$

Note that $\chi(-1) = 1$, from the properties of the complete residue system mod p we also have

$$\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{(a^2b-1)(b-1)}{p}\right) = \sum_{a=1}^{p-1}\chi(a)\sum_{b=0}^{p-1}\left(\frac{(2a^2b-a^2-1)^2-(a^2-1)^2}{p}\right) = \sum_{a=1}^{p-1}\chi(a)\sum_{b=0}^{p-1}\left(\frac{b^2-(a^2-1)^2}{p}\right) \tag{4}$$

and

$$\sum_{a=1}^{p}\left(\frac{a^2+n}{p}\right) = \begin{cases} -1, & \text{if } (n,p)=1; \\ p-1, & \text{if } (n,p)=p. \end{cases} \tag{5}$$

(This formula can be found in Hua's book [11], Section 7.8, Theorem 8.2.)

Combining (4) and (5) we can deduce the identity

$$\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{(a^2b-1)(b-1)}{p}\right) = 2(p-1) - \sum_{a=2}^{p-2}\chi(a) = 2p. \tag{6}$$

Now Lemma 1 follows from (3) and (6). □

Lemma 2 *Let p be an odd prime, χ be any non-principal even character* mod p. *Then for any integer m with $(m,p) = 1$, we have the identity*

$$\left|\sum_{a=1}^{p-1}\chi(a)\cdot e\left(\frac{ma^2}{p}\right)\right|^2 = 2p + \left(\frac{m}{p}\right)\cdot\tau(\chi_2)\cdot\sum_{a=1}^{p-1}\chi(a)\left(\frac{a^2-1}{p}\right),$$

where $\chi_2 = (\frac{*}{p})$ denotes the Legendre symbol with $\tau^2(\chi_2) = (\frac{-1}{p})\cdot p$.

Proof If $(m,p) = 1$, then from the properties of Gauss sums and quadratic residue mod p we have

$$\begin{aligned}\sum_{a=0}^{p-1}e\left(\frac{ma^2}{p}\right) &= 1 + \sum_{a=1}^{p-1}e\left(\frac{ma^2}{p}\right) = 1 + \sum_{a=1}^{p-1}\left(1+\left(\frac{a}{p}\right)\right)\cdot e\left(\frac{ma}{p}\right) \\ &= \sum_{a=0}^{p-1}e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\cdot e\left(\frac{ma}{p}\right) \\ &= \left(\frac{m}{p}\right)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\cdot e\left(\frac{a}{p}\right) = \left(\frac{m}{p}\right)\cdot\tau(\chi_2). \end{aligned} \tag{7}$$

Since χ is a non-principal even character mod p, so from identity (7) and the definition of $G(\chi,m;p)$ we have

$$\begin{aligned}\left|G(\chi,m;p)\right|^2 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a\overline{b})\cdot e\left(\frac{ma^2-mb^2}{p}\right) \\ &= \sum_{a=1}^{p-1}\chi(a)\cdot\sum_{b=1}^{p-1}e\left(\frac{mb^2(a^2-1)}{p}\right) = \sum_{a=1}^{p-1}\chi(a)\cdot\left(\sum_{b=0}^{p-1}e\left(\frac{mb^2(a^2-1)}{p}\right)-1\right)\end{aligned}$$

$$\begin{aligned}
&= 2(p-1) + \sum_{a=2}^{p-2} \chi(a) \cdot \left(\sum_{b=0}^{p-1} e\left(\frac{mb^2(a^2-1)}{p} \right) - 1 \right) \\
&= 2(p-1) - \sum_{a=2}^{p-2} \chi(a) + \tau(\chi_2) \cdot \sum_{a=2}^{p-2} \chi(a) \left(\frac{m(a^2-1)}{p} \right) \\
&= 2p - \sum_{a=1}^{p-1} \chi(a) + \tau(\chi_2) \cdot \sum_{a=1}^{p-1} \chi(a) \left(\frac{m(a^2-1)}{p} \right) \\
&= 2p + \left(\frac{m}{p} \right) \cdot \tau(\chi_2) \cdot \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right).
\end{aligned}$$

This completes the proof of Lemma 2. □

3 Proof of the theorems

In this section, we shall complete the proof of our theorems. First we prove Theorem 1. In fact from (2) and Lemma 2 we may immediately deduce the identity

$$\begin{aligned}
\left| \sum_{a=1}^{p-1} \chi(a) \cdot e\left(\frac{ma^2}{p} \right) \right|^2 &= 2p + \left(\frac{m}{p} \right) \cdot \tau(\chi_2) \cdot \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right) \\
&= 2p + \overline{\chi}(2) \cdot \left(\frac{m}{p} \right) \cdot \tau(\chi_2) \cdot \sum_{a=1}^{p-1} \chi(a + \overline{a}).
\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2; from Lemma 1 and Lemma 2 we have

$$\begin{aligned}
&\sideset{}{'}\sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) \cdot e\left(\frac{na^2}{p} \right) \right|^2 \cdot \left| \sum_{a=1}^{p-1} \chi(a + \overline{a}) \right|^2 \\
&\quad = \left| \sum_{a=1}^{p-1} \chi_0(a) \cdot e\left(\frac{na^2}{p} \right) \right|^2 \cdot \left| \sum_{a=1}^{p-1} \chi_0(a + \overline{a}) \right|^2 \\
&\qquad + \sideset{}{'}\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}} \left(2p + \left(\frac{n}{p} \right) \cdot \tau(\chi_2) \cdot \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right) \right) \\
&\qquad \times \left(2p + \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \left(\frac{b(b-1)(a^2b-1)}{p} \right) \right). \qquad (8)
\end{aligned}$$

If $p \equiv 3 \bmod 4$, then we have $(\frac{-1}{p}) = -1$ and

$$\left| \sum_{a=1}^{p-1} \chi_0(a) \cdot e\left(\frac{na^2}{p} \right) \right|^2 = \left| \sum_{a=0}^{p-1} e\left(\frac{na^2}{p} \right) - 1 \right|^2 = \left| \chi_2(n)\tau(\chi_2) - 1 \right|^2 = p + 1, \qquad (9)$$

$$\left| \sum_{a=1}^{p-1} \chi_0(a + \overline{a}) \right|^2 = \left(\sum_{a=1}^{p-1} 1 \right)^2 = (p-1)^2. \qquad (10)$$

Note that the identity

$$\sum_{\chi \bmod p}' \chi(a) = \begin{cases} \frac{p-1}{2}, & \text{if } a \equiv \pm 1 \bmod p, \\ 0, & \text{otherwise,} \end{cases}$$

from (5) we have

$$\begin{aligned} \sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}}' \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right) &= \sum_{\chi \bmod p}' \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p} \right) - \sum_{a=1}^{p-1} \left(\frac{a^2-1}{p} \right) \\ &= -\sum_{a=0}^{p-1} \left(\frac{a^2-1}{p} \right) + \left(\frac{-1}{p} \right) = 1 + \left(\frac{-1}{p} \right) = 0; \end{aligned} \tag{11}$$

$$\begin{aligned} &\sum_{\substack{\chi \bmod p \\ \chi \neq \chi_0}}' \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \left(\frac{b(b-1)(a^2b-1)}{p} \right) \\ &= \sum_{\chi \bmod p}' \sum_{a=1}^{p-1} \chi(a) \sum_{b=1}^{p-1} \left(\frac{b(b-1)(a^2b-1)}{p} \right) - \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{b(b-1)(a^2b-1)}{p} \right) \\ &= (p-1) \cdot \sum_{b=1}^{p-1} \left(\frac{b(b-1)(b-1)}{p} \right) - \sum_{b=1}^{p-1} \left(\frac{b-1}{p} \right) \sum_{a=1}^{p-1} \left(\frac{a^2-\overline{b}}{p} \right) \\ &= (p-1) \cdot \sum_{b=2}^{p-1} \left(\frac{b}{p} \right) - \sum_{b=1}^{p-1} \left(\frac{b-1}{p} \right) \left(-1 - \left(\frac{-\overline{b}}{p} \right) \right) \\ &= 1 - p - 2 \left(\frac{-1}{p} \right) = -(p-3). \end{aligned} \tag{12}$$

Note that $(\frac{-1}{p}) = -1$ and

$$\begin{aligned} &\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{(a^2-1)(b^2-1)(a^2b^2-1)}{p} \right) \\ &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{(a^2\overline{b}^2-1)(b^2-1)(a^2-1)}{p} \right) \\ &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{(a^2-b^2)(b^2-1)(a^2-1)}{p} \right) \\ &= -\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{(b^2-a^2)(b^2-1)(a^2-1)}{p} \right) \\ &= -\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{(b^2a^2-a^2)(b^2a^2-1)(a^2-1)}{p} \right) \\ &= -\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \left(\frac{(a^2-1)(b^2-1)(a^2b^2-1)}{p} \right), \end{aligned}$$

so that we have the identities

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a^2-1)(b^2-1)(a^2b^2-1)}{p}\right)=0 \tag{13}$$

and

$$\begin{aligned}
&\sideset{}{'}\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{b(b-1)(a^2b-1)}{p}\right)\right)\left(\sum_{a=1}^{p-1}\chi(a)\left(\frac{a^2-1}{p}\right)\right)\\
&\quad=\sideset{}{'}\sum_{\chi \bmod p}\left(\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{b(b-1)(a^2b-1)}{p}\right)\right)\left(\sum_{c=1}^{p-1}\chi(c)\left(\frac{c^2-1}{p}\right)\right)\\
&\qquad-\left(\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{b(b-1)(a^2b-1)}{p}\right)\right)\left(\sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right)\right)\\
&\quad=(p-1)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{b(b-1)(a^2b-1)}{p}\right)\left(\frac{\overline{a}^2-1}{p}\right)-2-2\left(\frac{-1}{p}\right)\\
&\quad=-(p-1)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(1+\left(\frac{b}{p}\right)\right)\left(\frac{(b-1)(a^2b-1)}{p}\right)\left(\frac{a^2-1}{p}\right)\\
&\qquad+(p-1)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(b-1)(a^2b-1)}{p}\right)\left(\frac{a^2-1}{p}\right)\\
&\quad=-(p-1)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\left(\frac{(a^2-1)(b^2-1)(a^2b^2-1)}{p}\right)\\
&\qquad+(p-1)\cdot\sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right)\sum_{b=0}^{p-1}\left(\frac{(2a^2b-a^2-1)^2-(a^2-1)^2}{p}\right)\\
&\quad=(p-1)\cdot\sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right)\sum_{b=0}^{p-1}\left(\frac{b^2-(a^2-1)^2}{p}\right)=0.
\end{aligned}\tag{14}$$

Combining (8)-(12) and (14) we may immediately deduce

$$\begin{aligned}
&\sideset{}{'}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)\cdot e\left(\frac{na^2}{p}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}\chi(a+\overline{a})\right|^2\\
&\quad=\left|\sum_{a=1}^{p-1}\chi_0(a)\cdot e\left(\frac{na^2}{p}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}\chi_0(a+\overline{a})\right|^2\\
&\qquad+\sideset{}{'}\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\left(2p+\left(\frac{n}{p}\right)\cdot\tau(\chi_2)\cdot\sum_{a=1}^{p-1}\chi(a)\left(\frac{a^2-1}{p}\right)\right)\\
&\qquad\times\left(2p+\sum_{a=1}^{p-1}\chi(a)\sum_{b=1}^{p-1}\left(\frac{b(b-1)(a^2b-1)}{p}\right)\right)
\end{aligned}$$

$$= (p+1)(p-1)^2 + 4p^2\left(\frac{p-1}{2} - 1\right) - 2p(p-3)$$
$$= 3p^3 - 9p^2 + 5p + 1 = (p-1)\cdot\left(3p^2 - 6p - 1\right).$$

This completes the proof of our theorems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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