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Commutators of intrinsic square functions on generalized Morrey spaces

Xiaomei Wu[1*] and Taotao Zheng[2]

*Correspondence: wuxm@zjnu.cn
[1]Department of Mathematics, Xingzhi College, Zhejiang Normal University, Jinhua, 321004, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, we obtain the boundedness of intrinsic square functions and their commutators generated with BMO functions on generalized Morrey spaces. Our theorems extend some well-known results.
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1 Introduction

The intrinsic square functions were first introduced by Wilson in [1, 2]. They are defined as follows. For $0 < \alpha \leq 1$, let $\mathcal{C}_\alpha$ be the family of functions $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ such that ϕ's support is contained in $\{x : |x| \leq 1\}$, $\int \phi\,dx = 0$, and for $x, x' \in \mathbb{R}^n$,

$$\left|\phi(x) - \phi\left(x'\right)\right| \leq \left|x - x'\right|^\alpha.$$

For $(y,t) \in \mathbb{R}^{n+1}_+$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, set

$$A_\alpha f(t,y) \equiv \sup_{\phi \in \mathcal{C}_\alpha} \left|f * \phi_t(y)\right|,$$

where $\phi_t(y) = t^{-n}\phi(\frac{y}{t})$. Then we define the varying-aperture intrinsic square (intrinsic Lusin) function of f by the formula

$$G_{\alpha,\beta}(f)(x) = \left(\int\int_{\Gamma_\beta(x)} \left(A_\alpha f(t,y)\right)^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma_\beta(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\}$. Denote $G_{\alpha,1}(f) = G_\alpha(f)$.

This function is independent of any particular kernel, such as Poisson kernel. It dominates pointwise the classical square function (Lusin area integral) and its real-variable generalizations. Although the function $G_{\alpha,\beta}(f)$ depends on the kernels with uniform compact support, there is a pointwise relation between $G_{\alpha,\beta}(f)$ with different β $(\beta \geq 1)$:

$$G_{\alpha,\beta}(f)(x) \leq \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x).$$

We refer for details to [1].

The intrinsic Littlewood-Paley g-function and the intrinsic g_λ^*-function are defined, respectively, by

$$g_\alpha f(x) = \left(\int_0^\infty \left(A_\alpha f(t,y)\right)^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

$$g_{\lambda,\alpha}^* f(x) = \left(\int\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left(A_\alpha f(t,y)\right)^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

In [1], Wilson proved the following result.

Theorem A *Let* $1 < p < \infty$, $0 < \alpha \leq 1$, *then* G_α *is bounded from* $L^p(\mathbb{R}^n)$ *to itself.*

After that, Huang and Liu [3] studied the boundedness of intrinsic square functions on weighted Hardy spaces. Moreover, they characterized the weighted Hardy spaces by intrinsic square functions. In [4] and [5], Wang and Liu obtained some weak type estimates on weighted Hardy spaces. In [6] and [7], Wang considered intrinsic functions and the commutators generated with BMO functions on weighted Morrey spaces. Let b be a locally integrable function on $\mathbb{R}^n$. Setting

$$A_{\alpha,b} f(t,y) \equiv \sup_{\phi\in\mathcal{C}_\alpha} \left|\int_{\mathbb{R}^n} \left[b(x)-b(z)\right]\phi_t(y-z)f(z)\,dz\right|,$$

the commutators are defined by

$$[b,G_\alpha]f(x) = \left(\int\int_{\Gamma(x)} \left(A_{\alpha,b} f(t,y)\right)^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

$$[b,g_\alpha]f(x) = \left(\int_0^\infty \left(A_{\alpha,b} f(t,y)\right)^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

and

$$\left[b,g_{\lambda,\alpha}^*\right]f(x) = \left(\int\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left(A_{\alpha,b} f(t,y)\right)^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right| dy < \infty,$$

where $f_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$.

In this paper, we will consider G_α, g_α, $g_{\lambda,\alpha}^*$ and their commutators on generalized Morrey spaces. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times \mathbb{R}^+$. For any $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, we denote by $L^{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey spaces, if

$$\|f\|_{L^{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \left(\int_{B(x,r)} \left|f(x)\right|^p dx\right)^{\frac{1}{p}} < \infty.$$

In [8], Mizuhara introduced these generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ and discussed the boundedness of the Calderón-Zygmund singular integral operators. Note that the generalized Morrey spaces $L^{p,\omega}(\mathbb{R}^n)$ with normalized norm

$$\|f\|_{L^{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \omega(x,r)^{-1}\big|B(x,r)\big|^{-\frac{1}{p}}\left(\int_{B(x,r)}\big|f(x)\big|^p\,dx\right)^{\frac{1}{p}},$$

were first defined by Guliyev in [9]. When $\omega(x,r) = r^{\frac{\lambda-n}{p}}$, $L^{p,\omega}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$. It is the classical Morrey space which was first introduced by Morrey in [10]. There are many papers discussed the conditions on $\omega(x,r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [8], the function φ is supposed to be a positively growth function and satisfy the double condition: for all $r > 0$, $\varphi(2r) \le D\varphi(r)$, where $D \ge 1$ is a constant independent of r. This type of conditions on φ is studied by many authors; see, for example, [11, 12]. In [13], the following statement was proved by Nakai for the Calderón-Zygmund singular integral operators T.

Theorem B *Let $1 \le p < \infty$ and let $\omega(x,r)$ satisfy the conditions*

$$c^{-1}\omega(x,r) \le \omega(x,t) \le c\omega(x,r),$$

whenever $r \le t \le 2r$, where c (≥ 1) does not depend on $t, r, x \in \mathbb{R}^n$ and

$$\int_r^\infty \omega(x,t)^p\frac{dt}{t} \le c\omega(x,r)^p,$$

where c does not depend on x and r. Then the operator T *is bounded on $L^{p,\omega}(\mathbb{R}^n)$ for $p > 1$ and from $L^{1,\omega}(\mathbb{R}^n)$ to $\mathrm{W}L^{1,\omega}(\mathbb{R}^n)$ for $p = 1$.*

The following statement, containing some results which were obtained in [8] and [13], was proved by Guliyev in [14, 15] (also see [16]).

Theorem C *Let $1 \le p < \infty$ and let the pair (ω_1, ω_2) satisfy the condition*

$$\int_t^\infty \omega_1(x,r)\frac{dr}{r} \le c\omega_2(x,t), \tag{1}$$

where c does not depend on x and t. Then the operator T *is bounded from $L^{p,\omega_1}(\mathbb{R}^n)$ to $L^{p,\omega_2}(\mathbb{R}^n)$ for $p > 1$ and from $L^{1,\omega_1}(\mathbb{R}^n)$ to $\mathrm{W}L^{1,\omega_2}(\mathbb{R}^n)$ for $p = 1$.*

Recently, in [17] and [9], Guliyev *et al.* introduced a weaker condition for the boundedness of Calderón-Zygmund singular integral operators from $L^{p,\omega_1}(\mathbb{R}^n)$ to $L^{p,\omega_2}(\mathbb{R}^n)$: If $1 \le p < +\infty$, for any $x \in \mathbb{R}^n$ and $t > 0$, there exists a constant $c > 0$, such that

$$\int_t^\infty \frac{\operatorname{ess\,inf}_{r<s<\infty}\omega_1(x,s)s^{\frac{n}{p}}}{r^{\frac{n}{p}+1}}\,dr \le c\omega_2(x,t). \tag{2}$$

By an easy computation, we can check that if the pair (ω_1, ω_2) satisfies double condition, then it will satisfy condition (1). Moreover, if (ω_1, ω_2) satisfies condition (1), it will also satisfy condition (2). But the opposite is not true. We refer to [13] and Remark 4.7 in [9] for details.

In this paper, we will obtain the boundedness of the intrinsic function, the intrinsic Littlewood-Paley g function, the intrinsic g_λ^* function and their commutators on generalized Morrey spaces when the pair (ω_1, ω_2) satisfies condition (2) or the following inequality:

$$\int_t^\infty \left(1 + \ln \frac{r}{t}\right) \frac{\operatorname{ess\,inf}_{r<s<\infty} \omega_1(x,s) s^{\frac{n}{p}}}{r^{\frac{n}{p}+1}}\, dr \leq c\omega_2(x,t). \tag{3}$$

Our main results in this paper are stated as follows.

Theorem 1.1 *Let* $1 < p < \infty$, $0 < \alpha \leq 1$, *let* (ω_1, ω_2) *satisfy condition* (2), *then* G_α *is bounded from* $L^{p,\omega_1}(\mathbb{R}^n)$ *to* $L^{p,\omega_2}(\mathbb{R}^n)$.

Theorem 1.2 *Let* $1 < p < \infty$, $0 < \alpha \leq 1$, *let* (ω_1, ω_2) *satisfy condition* (2), *then for* $\lambda > 3 + \frac{2\alpha}{n}$, *we have* $g_{\lambda,\alpha}^*$ *is bounded from* $L^{p,\omega_1}(\mathbb{R}^n)$ *to* $L^{p,\omega_2}(\mathbb{R}^n)$.

Theorem 1.3 *Let* $1 < p < \infty$, $0 < \alpha \leq 1$, $b \in \mathrm{BMO}$, *let* (ω_1, ω_2) *satisfy condition* (3), *then* $[b, G_\alpha]$ *is bounded from* $L^{p,\omega_1}(\mathbb{R}^n)$ *to* $L^{p,\omega_2}(\mathbb{R}^n)$.

Theorem 1.4 *Let* $1 < p < \infty$, $0 < \alpha \leq 1$, $b \in \mathrm{BMO}$, *let* (ω_1, ω_2) *satisfy condition* (3), *then for* $\lambda > 3 + \frac{2\alpha}{n}$, $[b, g_{\lambda,\alpha}^*]$ *is bounded from* $L^{p,\omega_1}(\mathbb{R}^n)$ *to* $L^{p,\omega_2}(\mathbb{R}^n)$.

In [1], the author proved that the functions G_α and g_α are pointwise comparable. Thus, as a consequence of Theorem 1.1 and Theorem 1.3, we have the following results.

Corollary 1.5 *Let* $1 < p < \infty$, $0 < \alpha \leq 1$, *let* (ω_1, ω_2) *satisfy condition* (2), *then* g_α *is bounded from* $L^{p,\omega_1}(\mathbb{R}^n)$ *to* $L^{p,\omega_2}(\mathbb{R}^n)$.

Corollary 1.6 *Let* $1 < p < \infty$, $0 < \alpha \leq 1$, $b \in \mathrm{BMO}$, *and let* (ω_1, ω_2) *satisfy condition* (3), *then* $[b, g_\alpha]$ *is bounded from* $L^{p,\omega_1}(\mathbb{R}^n)$ *to* $L^{p,\omega_2}(\mathbb{R}^n)$.

Throughout this paper, we use the notation $A \preceq B$ to mean that there is a positive constant C (≥ 1) independent of all essential variables such that $A \leq CB$. Moreover, C maybe different from place to place.

2 Proofs of main theorems

Before proving the main theorems, we need the following lemmas.

Lemma 2.1 ([18]) *The inequality* $\operatorname{ess\,sup}_{t>0} \omega(t) Hg(t) \preceq \operatorname{ess\,sup}_{t>0} v(t) g(t)$ *holds for all non-negative and non-increasing* g *on* $(0, \infty)$ *if and only if*

$$A := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \frac{dr}{\operatorname{ess\,sup}_{0<s<r} v(s)} < \infty, \tag{4}$$

where $Hg(t)$ *is the Hardy operator* $Hg(t) := \frac{1}{t} \int_0^t g(r)\, dr$, $0 < t < \infty$.

Lemma 2.2 ([19]) (1) *For* $1 < p < \infty$,

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \left| f(y) - f_{B(x,r)} \right|^p dy \right)^{\frac{1}{p}}.$$

(2) *Let* $f \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < 2r < t$, *then*

$$|f_{B(x,r)} - f_{B(x,t)}| \preceq \|f\|_* \ln \frac{t}{r}.$$

Lemma 2.3 *For* $j \in \mathbb{Z}^+$, *denote*

$$G_{\alpha,2^j}(f)(x) = \left(\int_0^\infty \int_{|x-y|\leq 2^j t} \left(A_\alpha f(y,t) \right)^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

Let $1 < p < \infty$, $0 < \alpha \leq 1$, *then we have*

$$\left\| G_{\alpha,2^j}(f) \right\|_{L^p(\mathbb{R}^n)} \preceq 2^{j(\frac{3n}{2}+\alpha)} \left\| G_\alpha(f) \right\|_{L^p(\mathbb{R}^n)}.$$

From [1], we know that

$$G_{\alpha,\beta}(f)(x) \leq \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x).$$

Then, by an easy computation, we get Lemma 2.3.

By a similar argument as in [20], we can easily get the following lemma.

Lemma 2.4 *Let* $1 < p < \infty$, $0 < \alpha \leq 1$, *then the commutators* $[b, G_\alpha]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to itself whenever* $b \in \mathrm{BMO}$.

Now we are in a position to prove the theorems.

Proof of Theorem 1.1 The main ideas of these proofs come from [9]. We decompose $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{2B}(y)$, $f_2(y) = f(y) - f_1(y)$, $B := B(x_0, r)$. Then

$$\|G_\alpha f\|_{L^p(B(x_0,r))} \leq \|G_\alpha f_1\|_{L^p(B(x_0,r))} + \|G_\alpha f_2\|_{L^p(B(x_0,r))} := I + II.$$

First, let us estimate I. By Theorem A, we obtain

$$I \leq \|G_\alpha f_1\|_{L^p(\mathbb{R}^n)} \preceq \|f_1\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(2B)} \preceq r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1}\,dt. \tag{5}$$

Then let us estimate II. Recalling the properties of function ϕ, we know that

$$\left| f_2 * \phi_t(y) \right| = \left| t^{-n} \int_{|y-z|\leq t} \phi\left(\frac{y-z}{t} \right) f_2(z)\,dz \right| \preceq t^{-n} \int_{|y-z|\leq t} \left| f_2(z) \right| dz.$$

Since $x \in B(x_0, r)$, $(y,t) \in \Gamma(x)$ and $|z - x_0| \geq 2r$, we have

$$r \leq |z - x_0| - |x_0 - x| \leq |x - z| \leq |x - y| + |y - z| \leq 2t.$$

So, we obtain

$$\begin{aligned}
G_\alpha f_2(x) &\preceq \left(\int\int_{\Gamma(x)} \left| t^{-n}\int_{|y-z|\le t} \left|f_2(z)\right| dz\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\preceq \left(\int_{t>r/2}\int_{|x-y|<t}\left(\int_{|z-x|\le 2t}\left|f_2(z)\right| dz\right)^2 \frac{dy\,dt}{t^{3n+1}}\right)^{\frac{1}{2}} \\
&\preceq \left(\int_{t>r/2}\left(\int_{|z-x|\le 2t}\left|f_2(z)\right| dz\right)^2 \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}.
\end{aligned}$$

By Minkowski's inequality and $|z-x| \ge |z-x_0| - |x_0-x| \ge \frac{1}{2}|z-x_0|$, we have

$$\begin{aligned}
G_\alpha f_2(x) &\preceq \int_{\mathbb{R}^n}\left(\int_{t>\frac{|z-x|}{2}} \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} \left|f_2(z)\right| dz \\
&\preceq \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x|^n}\,dz \preceq \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x_0|^n}\,dz \\
&\preceq \int_{|z-x_0|>2r} \left|f(z)\right| \int_{|z-x_0|}^{+\infty} \frac{1}{t^{n+1}}\,dt\,dz \\
&= \int_{2r}^{+\infty}\int_{2r<|z-x_0|<t} \left|f(z)\right| dz \frac{1}{t^{n+1}}\,dt \preceq \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1}\,dt.
\end{aligned}$$

The last inequality is due to Hölder's inequality. Thus,

$$\|G_\alpha f_2\|_{L^p(B(x_0,r))} \preceq r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1}\,dt. \tag{6}$$

By combining (5) and (6), we have

$$\|G_\alpha f\|_{L^p(B(x_0,r))} \preceq r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} t^{-\frac{n}{p}-1}\,dt.$$

So, let $t = s^{-\frac{p}{n}}$; we have

$$\begin{aligned}
\|G_\alpha f\|_{L^{p,\omega_2}(\mathbb{R}^n)} &\preceq \sup_{x_0\in\mathbb{R}^n, r>0} \omega_2(x_0,r)^{-1}\left|B(x_0,r)\right|^{-\frac{1}{p}} r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \frac{1}{t^{\frac{n}{p}+1}}\,dt \\
&\preceq \sup_{x_0\in\mathbb{R}^n, r>0} \omega_2(x_0,r)^{-1} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L^p(B(x_0,s^{-\frac{p}{n}}))}\,ds \\
&= \sup_{x_0\in\mathbb{R}^n, r>0} \omega_2\bigl(x_0,r^{-\frac{p}{n}}\bigr)^{-1} \int_0^{r} \|f\|_{L^p(B(x_0,s^{-\frac{p}{n}}))}\,ds.
\end{aligned}$$

Take $w(t) = \omega_2(x_0,t^{-\frac{p}{n}})^{-1}t$, $v(t) = \omega_1(x_0,t^{-\frac{p}{n}})^{-1}t$. Since (ω_1,ω_2) satisfies condition (2), we can verify that $w(t)$, $v(t)$ satisfy condition (4). Let $g(s) = \|f\|_{L^p(B(x_0,s^{-\frac{p}{n}}))}$. Obviously, it is decreasing on variable s. So, by Lemma 2.1, we can conclude the following estimates:

$$\|G_\alpha f\|_{L^{p,\omega_2}(\mathbb{R}^n)} \preceq \sup_{x_0\in\mathbb{R}^n, r>0} \omega_1\bigl(x_0,r^{-\frac{p}{n}}\bigr)^{-1} r \|f\|_{L^p(B(x_0,r^{-\frac{p}{n}}))} = \|f\|_{L^{p,\omega_1}(\mathbb{R}^n)}.$$

□

Proof of Theorem 1.2

$$\begin{aligned}\left[g^*_{\lambda,\alpha}(f)(x)\right]^2 &= \int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left(A_\alpha f(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}} \\ &\quad + \int_0^\infty \int_{|x-y|\geq t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left(A_\alpha f(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}} \\ &:= III + IV.\end{aligned}$$

First, let us estimate III:

$$III \leq \int_0^{+\infty} \int_{|x-y|<t} \left(A_\alpha f(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}} = \left(G_\alpha f(x)\right)^2.$$

Then let us estimate IV:

$$\begin{aligned}IV &\leq \sum_{j=1}^\infty \int_0^\infty \int_{2^{j-1}t\leq|x-y|\leq 2^j t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left(A_\alpha f(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}} \\ &\preceq \sum_{j=1}^\infty \int_0^\infty \int_{2^{j-1}t\leq|x-y|\leq 2^j t} 2^{-jn\lambda} \left(A_\alpha f(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}} \\ &\preceq \sum_{j=1}^\infty 2^{-jn\lambda} \int_0^\infty \int_{|x-y|\leq 2^j t} \left(A_\alpha f(y,t)\right)^2 \frac{dy\,dt}{t^{n+1}} \\ &:= \sum_{j=1}^\infty 2^{-jn\lambda} \left(G_{\alpha,2^j}(f)(x)\right)^2.\end{aligned}$$

Thus,

$$\left\|g^*_{\lambda,\alpha}(f)\right\|_{L^{p,\omega_2}(\mathbb{R}^n)} \leq \|G_\alpha f\|_{L^{p,\omega_2}(\mathbb{R}^n)} + \sum_{j=1}^\infty 2^{-\frac{jn\lambda}{2}} \left\|G_{\alpha,2^j}(f)\right\|_{L^{p,\omega_2}(\mathbb{R}^n)}. \tag{7}$$

By Theorem 1.1, we have

$$\|G_\alpha f\|_{L^{p,\omega_2}(\mathbb{R}^n)} \preceq \|f\|_{L^{p,\omega_1}(\mathbb{R}^n)}. \tag{8}$$

To complete the proof, it suffices to estimate $\|G_{\alpha,2^j}(f)\|_{L^{p,\omega_2}(\mathbb{R}^n)}$. Take $f_1(y) = f(y)\chi_{2B}(y)$, $f_2(y) = f(y) - f_1(y)$, $2B = B(x_0, 2r)$. Then

$$\left\|G_{\alpha,2^j}(f)\right\|_{L^p(B(x_0,r))} \leq \left\|G_{\alpha,2^j}(f_1)\right\|_{L^p(B(x_0,r))} + \left\|G_{\alpha,2^j}(f_2)\right\|_{L^p(B(x_0,r))}. \tag{9}$$

For the first part, by Lemma 2.3, we obtain

$$\begin{aligned}\left\|G_{\alpha,2^j}(f_1)\right\|_{L^p(B(x_0,r))} &\preceq 2^{j(\frac{3n}{2}+\alpha)} \left\|G_\alpha(f_1)\right\|_{L^p(\mathbb{R}^n)} \preceq 2^{j(\frac{3n}{2}+\alpha)} \|f\|_{L^p(2B)} \\ &\preceq 2^{j(\frac{3n}{2}+\alpha)} r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} \frac{1}{t^{\frac{n}{p}+1}}\,dt.\end{aligned} \tag{10}$$

For the other part, we know

$$\begin{aligned}
G_{\alpha,2^j}(f_2)(x) &= \left(\int_0^\infty \int_{|x-y|\le 2^j t} \big(A_\alpha f_2(y,t)\big)^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty \int_{|x-y|\le 2^j t} \Big(\sup_{\phi\in\mathcal{C}_\alpha} \big|f_2 * \phi_t(y)\big|\Big)^2 \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\preceq \left(\int_0^\infty \int_{|x-y|\le 2^j t} \left(\int_{|z-y|\le t} \big|f_2(z)\big|\,dz\right)^2 \frac{dy\,dt}{t^{3n+1}}\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $|z-x| \le |z-y| + |y-x| \le 2^{j+1}t$, by Minkowski's inequality, we get

$$\begin{aligned}
G_{\alpha,2^j}(f_2)(x) &\preceq \left(\int_0^\infty \int_{|x-y|\le 2^j t} \left(\int_{|z-x|\le 2^{j+1}t} \big|f_2(z)\big|\,dz\right)^2 \frac{dy\,dt}{t^{3n+1}}\right)^{\frac{1}{2}} \\
&\preceq \left(\int_0^\infty \left(\int_{|z-x|\le 2^{j+1}t} \big|f_2(z)\big|\,dz\right)^2 \frac{2^{jn}\,dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\
&\preceq 2^{\frac{jn}{2}} \int_{\mathbb{R}^n} \left(\int_{t\ge \frac{|z-x|}{2^{j+1}}} \big|f_2(z)\big|^2 \frac{1}{t^{2n+1}}\,dt\right)^{\frac{1}{2}} dz \\
&\preceq 2^{\frac{3jn}{2}} \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x|^n}\,dz.
\end{aligned}$$

For $x \in B(x_0,r)$, we have $|z-x| \ge |z-x_0| - |x_0-x| \ge |z-x_0| - \frac{1}{2}|z-x_0| = \frac{1}{2}|z-x_0|$. So by Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned}
G_{\alpha,2^j}(f_2)(x) &\preceq 2^{\frac{3jn}{2}} \int_{|z-x_0|>2r} \frac{|f(z)|}{|z-x_0|^n}\,dz \\
&\preceq 2^{\frac{3jn}{2}} \int_{|z-x_0|>2r} \big|f(z)\big| \int_{|z-x_0|}^\infty \frac{1}{t^{n+1}}\,dt\,dz \\
&= 2^{\frac{3jn}{2}} \int_{2r}^\infty \int_{|z-x_0|<t} \big|f(z)\big| \frac{1}{t^{n+1}}\,dz\,dt \\
&\preceq 2^{\frac{3jn}{2}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} \frac{1}{t^{\frac{n}{p}+1}}\,dt.
\end{aligned}$$

Thus,

$$\big\|G_{\alpha,2^j}(f_2)\big\|_{L^p(B(x_0,r))} \preceq 2^{\frac{3jn}{2}} r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} \frac{1}{t^{\frac{n}{p}+1}}\,dt. \tag{11}$$

Combining by (9), (10), and (11), we have

$$\big\|G_{\alpha,2^j}(f)\big\|_{L^p(B(x_0,r))} \preceq 2^{j(\frac{3n}{2}+\alpha)} r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))} \frac{1}{t^{\frac{n}{p}+1}}\,dt.$$

Thus, by substitution of variables and Lemma 2.1, we get

$$\begin{aligned}
\|G_{\alpha,2^j}(f)\|_{L^{p,\omega_2}(\mathbb{R}^n)} &\preceq 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n,r>0} \omega_2\bigl(B(x_0,r)\bigr)^{-1}\bigl|B(x_0,r)\bigr|^{-\frac{1}{p}} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L^p(B(x_0,s^{-\frac{p}{n}}))}\,ds \\
&= 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n,r>0} \omega_2\bigl(x_0,r^{-\frac{p}{n}}\bigr)^{-1} \int_0^{r} \|f\|_{L^p(B(x_0,s^{-\frac{p}{n}}))}\,ds \\
&\preceq 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in\mathbb{R}^n,r>0} \omega_1\bigl(x_0,r^{-\frac{p}{n}}\bigr)^{-1} r\|f\|_{L^p(B(x_0,r^{-\frac{p}{n}}))} \\
&= 2^{j(\frac{3n}{2}+\alpha)} \|f\|_{L^{p,\omega_1}(\mathbb{R}^n)}. \qquad (12)
\end{aligned}$$

Since $\lambda > 3 + \frac{2\alpha}{n}$, by (7), (8) and (12), we have the desired theorem. □

Proof of Theorem 1.3 We decompose $f = f_1 + f_2$ as in the proof of Theorem 1.2, where $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. Then

$$\bigl\|[b,G_\alpha]f\bigr\|_{L^p(B(x_0,r))} \le \bigl\|[b,G_\alpha]f_1\bigr\|_{L^p(B(x_0,r))} + \bigl\|[b,G_\alpha]f_2\bigr\|_{L^p(B(x_0,r))}.$$

By Lemma 2.4, we have

$$\bigl\|[b,G_\alpha]f_1\bigr\|_{L^p(B(x_0,r))} \preceq \|f_1\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(2B)} \preceq r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t))} \frac{1}{t^{\frac{n}{p}+1}}\,dt.$$

Next, we estimate the second part. We divide it into two parts. We have

$$\begin{aligned}
[b,G_\alpha]f_2(x) &= \biggl(\int\int_{\Gamma(x)} \sup_{\phi\in\mathcal{C}_\alpha} \biggl|\int_{\mathbb{R}^n} \bigl[b(x)-b(z)\bigr]\phi_t(y-z)f_2(z)\,dz\biggr|^2 \frac{dy\,dt}{t^{n+1}}\biggr)^{\frac{1}{2}} \\
&\le \biggl(\int\int_{\Gamma(x)} \sup_{\phi\in\mathcal{C}_\alpha} \biggl|\int_{\mathbb{R}^n} \bigl[b(x)-b_B\bigr]\phi_t(y-z)f_2(z)\,dz\biggr|^2 \frac{dy\,dt}{t^{n+1}}\biggr)^{\frac{1}{2}} \\
&\quad + \biggl(\int\int_{\Gamma(x)} \sup_{\phi\in\mathcal{C}_\alpha} \biggl|\int_{\mathbb{R}^n} \bigl[b_B-b(z)\bigr]\phi_t(y-z)f_2(z)\,dz\biggr|^2 \frac{dy\,dt}{t^{n+1}}\biggr)^{\frac{1}{2}} \\
&:= V + VI.
\end{aligned}$$

First, for V, we find that

$$V = \bigl|b(x)-b_B\bigr|\biggl(\int\int_{\Gamma(x)} \sup_{\phi\in\mathcal{C}_\alpha} \biggl|\int_{\mathbb{R}^n} \phi_t(y-z)f_2(z)\,dz\biggr|^2 \frac{dy\,dt}{t^{n+1}}\biggr)^{\frac{1}{2}} = \bigl|b(x)-b_B\bigr|G_\alpha f_2(x).$$

Following the proof in Theorem 1.1, we get

$$\begin{aligned}
&\biggl(\int_{B(x_0,r)} \bigl|b(x)-b_B\bigr|^p \bigl|G_\alpha f_2(x)\bigr|^p\,dx\biggr)^{\frac{1}{p}} \\
&\quad\preceq \biggl(\int_{B(x_0,r)} \bigl|b(x)-b_B\bigr|^p\,dx\biggr)^{\frac{1}{p}} \int_{2r}^{+\infty} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\
&\quad\preceq \|b\|_* r^{\frac{n}{p}} \int_{2r}^{+\infty} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.
\end{aligned}$$

For VI, since $|y-x|<t$, we get $|x-z|<2t$. Thus, by Minkowski's inequality, we obtain

$$\begin{aligned} VI &\preceq \left(\int\int_{\Gamma(x)}\left|\int_{|x-z|<2t}\left|b_B-b(z)\right|\left|f_2(z)\right|dz\right|^2\frac{dy\,dt}{t^{3n+1}}\right)^{\frac{1}{2}} \\ &\preceq \left(\int_0^{\infty}\left|\int_{|x-z|<2t}\left|b_B-b(z)\right|\left|f_2(z)\right|dz\right|^2\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\ &\preceq \int_{|x_0-z|>2r}\left|b_B-b(z)\right|\left|f(z)\right|\frac{1}{|x-z|^n}\,dz. \end{aligned}$$

Since $|z-x|\geq\frac{1}{2}|z-x_0|$, by Fubini's theorem, we get

$$\begin{aligned} \left(\int_{B(x_0,r)}|VI|^p\,dx\right)^{\frac{1}{p}} &\preceq \left(\int_{B(x_0,r)}\left|\int_{|x_0-z|>2r}\left|b_B-b(z)\right|\left|f(z)\right|\frac{1}{|x-z|^n}\,dz\right|^p dx\right)^{\frac{1}{p}} \\ &\preceq r^{\frac{n}{p}}\int_{|x_0-z|>2r}\left|b_B-b(z)\right|\left|f(z)\right|\frac{1}{|x_0-z|^n}\,dz \\ &\preceq r^{\frac{n}{p}}\int_{|x_0-z|>2r}\left|b_B-b(z)\right|\left|f(z)\right|\int_{|x_0-z|}^{+\infty}\frac{1}{t^{n+1}}\,dt\,dz \\ &\preceq r^{\frac{n}{p}}\int_{2r}^{+\infty}\int_{B(x_0,t)}\left|b_B-b(z)\right|\left|f(z)\right|dz\frac{1}{t^{n+1}}\,dt \\ &\preceq r^{\frac{n}{p}}\int_{2r}^{+\infty}\int_{B(x_0,t)}|b_B-b_{B(x_0,t)}|\left|f(z)\right|dz\frac{1}{t^{n+1}}\,dt \\ &\quad + r^{\frac{n}{p}}\int_{2r}^{+\infty}\int_{B(x_0,t)}\left|b(z)-b_{B(x_0,t)}\right|\left|f(z)\right|dz\frac{1}{t^{n+1}}\,dt \\ &:= A+B. \end{aligned}$$

For A, using Lemma 2.2 and Hölder's inequality, we have

$$\begin{aligned} A &\preceq \|b\|_* r^{\frac{n}{p}}\int_{2r}^{+\infty}\int_{B(x_0,t)}\left|f(z)\right|dz\frac{1}{t^{n+1}}\ln\frac{t}{r}\,dt \\ &\preceq r^{\frac{n}{p}}\int_{2r}^{+\infty}\ln\frac{t}{r}\|f\|_{L^p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned}$$

For B, we denote $D=\int_{B(x_0,t)}|f(z)||b_{B(x_0,t)}-b(z)|\,dz$. Then, by Hölder's inequality and Lemma 2.2, we get

$$\begin{aligned} D &\leq \left(\int_{B(x_0,t)}\left|f(z)\right|^p dz\right)^{\frac{1}{p}}\left(\int_{B(x_0,t)}\left|b_{B(x_0,t)}-b(z)\right|^{p'}dz\right)^{\frac{1}{p'}} \\ &\preceq t^{\frac{n}{p'}}\|b\|_*\|f\|_{L^p(B(x_0,t))}. \end{aligned}$$

This yields $B\preceq r^{\frac{n}{p}}\int_{2r}^{+\infty}\|f\|_{L^p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}}$. Thus,

$$\left\|[b,G_\alpha]f\right\|_{L^p(B(x_0,r))}\preceq r^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,t))}\frac{1}{t^{\frac{n}{p}+1}}\left(1+\ln\frac{t}{r}\right)dt.$$

By a change of variables, we obtain

$$\begin{aligned}
&\big\|[b,G_\alpha]f\big\|_{L^{p,\omega_2}(\mathbb{R}^n)}\\
&\quad\preceq \sup_{x_0\in\mathbb{R}^n,r>0}\omega_2(x_0,r)^{-1}\big|B(x_0,r)\big|^{-\frac{1}{p}}r^{\frac{n}{p}}\int_{2r}^{\infty}\|f\|_{L^p(B(x_0,t))}\frac{1}{t^{\frac{n}{p}+1}}\left(1+\ln\frac{t}{r}\right)dt\\
&\quad\preceq \sup_{x_0\in\mathbb{R}^n,r>0}\omega_2(x_0,r)^{-1}\int_0^{r^{-\frac{n}{p}}}\|f\|_{L^p(B(x_0,s^{-\frac{p}{n}}))}\left(1+\ln\frac{s^{-\frac{p}{n}}}{r}\right)ds\\
&\quad= \sup_{x_0\in\mathbb{R}^n,r>0}\omega_2\big(x_0,r^{-\frac{p}{n}}\big)^{-1}\int_0^{r}\|f\|_{L^p(B(x_0,s^{-\frac{p}{n}}))}\left(1+\frac{p}{n}\ln\frac{r}{s}\right)ds.
\end{aligned}$$

Let $w(t)=\omega_2(x_0,t^{-\frac{p}{n}})^{-1}t$, $v(t)=\omega_1(x_0,t^{-\frac{p}{n}})^{-1}t$. Since (ω_1,ω_2) satisfies condition (3), by a similarly argument with Theorem 1.1, we conclude the following estimates:

$$\big\|[b,G_\alpha]f\big\|_{L^{p,\omega_2}(\mathbb{R}^n)}\preceq \sup_{x_0\in\mathbb{R}^n,r>0}\omega_1\big(x_0,r^{-\frac{p}{n}}\big)^{-1}r\|f\|_{L^p(B(x_0,r^{-\frac{p}{n}}))}=\|f\|_{L^{p,\omega_1}(\mathbb{R}^n)}.$$

Using an argument similar to the above proofs and that of Theorem 1.2, we can also show the boundedness of $[b,g^*_{\lambda,\alpha}]$. □

Competing interests
The author declares that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Xingzhi College, Zhejiang Normal University, Jinhua, 321004, P.R. China. [2]Department of Mathematics, Zhejiang University, Hangzhou, 310027, P.R. China.

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