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# Wave breaking and infinite propagation speed for a modified two-component Camassa-Holm system with $\kappa \neq 0$

Wujun Lv[1], Ahmed Alsaedi[2], Tasawar Hayat[2,3] and Yong Zhou[1,2*]

*Correspondence:
yzhoumath@zjnu.edu.cn
[1]Department of Mathematics, Zhejiang Normal University, Jinhua, China
[2]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

**Abstract**

In this paper, we investigate the modified two-component Camassa-Holm equation with $\kappa \neq 0$ on the real line. Firstly, we establish sufficient conditions on the initial data to guarantee that the corresponding solution blows up in finite time for the modified two-component Camassa-Holm (MCH2) system. Then an infinite propagation speed for MCH2 is proved in the following sense: the corresponding solution $u(x,t)+\kappa$ with compactly supported initial data $(u_0(x)+\kappa, \rho_0(x))$ does not have compact $x$-support in its lifespan.

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**Keywords:** MCH2; blow-up; infinite propagation speed

## 1 Introduction

In this paper, we consider the following modified two-component Camassa-Holm system:

$$\begin{cases} y_t + uy_x + 2yu_x + 2\kappa u_x = -g\rho\bar{\rho}_x, & t>0, x\in\mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t>0, x\in\mathbb{R}, \\ y(0,x) = y_0(x), & x\in\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $y = u - u_{xx}$, $\rho = (1-\partial_x^2)(\bar{\rho}-\bar{\rho}_0)$, $u$ denotes the velocity field, and $\rho$ is related to the free surface density with the boundary assumptions; $\bar{\rho}$ expresses an averaged or filtered density, $\kappa$ is a nonnegative dissipative parameter, $g$ is the downward constant acceleration of gravity in applications to shallow water waves. For convenience we assume $g = 1$ in this paper. Moreover, $u$ and $\gamma$ satisfy the boundary conditions: $u \to -\kappa$ and $\gamma \to 0$ as $|x| \to \infty$.

Let $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$, then the operator $\Lambda^{-2}$ can be expressed by its associated Green's function $G = \frac{1}{2}e^{-|x|}$ as

$$\Lambda^{-2}f(x) = G * f(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}f(y)\,dy.$$

Let $\gamma = \bar{\rho}-\bar{\rho}_0$, then $\gamma = G*\rho$, and let $\widetilde{u} = u+\kappa$. It is convenient to rewrite system (1.1) in the following equivalent integral-differential form:

$$\begin{cases} \widetilde{u}_t + (\widetilde{u} - \kappa)\widetilde{u}_x + \partial_x G * (\widetilde{u}^2 + \frac{1}{2}\widetilde{u}_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2) = 0, & t > 0, x \in \mathbb{R}, \\ \gamma_t + (\widetilde{u} - \kappa)\gamma_x + G * ((\widetilde{u}_x\gamma_x)_x + \widetilde{u}_x\gamma) = 0, & t > 0, x \in \mathbb{R}, \\ \widetilde{u}(0,x) = \widetilde{u}_0(x), & x \in \mathbb{R}, \\ \gamma(0,x) = \gamma_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

In what follows, we will consider system (1.2) for $\widetilde{u}$ instead of system (1.1) for $u$, and we omit the tilde on the $u$ for simplicity. So we consider the following system:

$$\begin{cases} u_t - u_{xxt} + 3uu_x - uu_{xxx} - 2u_xu_{xx} - \kappa u_x + \kappa u_{xxx} = -g\rho\bar{\rho}_x, & t > 0, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x - \kappa\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ y(0,x) = y_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.3}$$

Obviously, under the constraints of $\rho(x,t) \equiv 0$ and $\kappa = 0$, system (1.1) reduces to the Camassa-Holm equation, which was derived physically by Camassa and Holm in [1] (found earlier by Fokas and Fuchssteniner [2] as a bi-Hamiltonian generalization of the KdV equation) by directly approximating the Hamiltonian for Euler's equation in the shallow water region with $u(x,t)$ representing the free surface above a flat bottom. There have been extensive studies on Camassa-Holm equation. Now, we mention some results that are related to our results. Firstly, wave breaking for a large class of initial data has been established in [3–6]. Recently, Zhou and his collaborators [7] give a direct proof for McKean's theorem [5]. In addition, the large time behavior for the support of momentum density of the Camassa-Holm equation was studied in [8]. An interesting phenomenon of the propagation speed for the Camassa-Holm equation with $\kappa = 0$ was presented by Zhou and his collaborators in their work [9] in the sense that a strong solution of the Cauchy problem with compact initial profile cannot be compactly supported at any later time unless it is the zero solution. Meanwhile, for the same problem about the equation $\kappa \neq 0$, we refer to [10] for details.

The Camassa-Holm equation [1] has recently been extended to a two-component Camassa-Holm (CH2) system:

$$\begin{cases} u_t + u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + g\rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \gamma_t + (u - \kappa)\gamma_x + G * ((u_x\gamma_x)_x + u_x\gamma) = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \gamma(0,x) = \gamma_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.4}$$

The CH2 system appeared initially in [11], and recently Constantin and Ivanov in [12] gave a demonstration about its derivation in view of the fluid shallow water theory from the hydrodynamic point of view. This generalization, similar to the Camassa-Holm equation, possessed the peakon, multi-kink solutions and the bi-Hamiltonian structure [13, 14] and is always integrable. The wave breaking mechanism was discussed in [15–17] and the existence of global solutions was analyzed in [12, 16, 18]. A geometric investigation can be found in [15, 19].

Recently the CH2 system was generalized into the following modified two-component Camassa-Holm (MCH2) system with $\rho \neq 0$ and $\kappa = 0$:

$$\begin{cases} u_t + uu_x + \partial_x G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2) = 0, & t > 0, x \in \mathbb{R}, \\ \gamma_t + u\gamma_x + G * ((u_x\gamma_x)_x + u_x\gamma) = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \gamma(0,x) = \gamma_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.5}$$

Note that the MCH2 system is a modified version of the two-component Camassa-Holm (CH2) system to allow a dependence on the average density $\bar{\rho}$ (or depth, in the shallow water interpretation) as well as the pointwise density $\rho$. This MCH2 system admits peaked solutions in the velocity and average density [20–23]. We find that the MCH2 system is expressed in terms of an averaged or filtered density $\bar{\rho}$ in analogy to the relation between momentum and velocity by setting $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$. Meanwhile, the MCH2 may not be integrable unlike the CH2 system. The characteristic is that it will amount to strengthening the norm for $\bar{\rho}$ from $L^2$ to $H^1$ in the potential energy term [21]. It means we have the following conserved quantity:

$$E(t) = \int_{\mathbb{R}} \left(u^2 + u_x^2 + \gamma^2 + \gamma_x^2\right) dx.$$

We cannot obtain the conservation of $H^1$ norm for the CH2 system.

In what follows, we always assume $\kappa \neq 0$ and $\rho \neq 0$.

This paper is organized as follows. In Section 2, we will present some results, which will be used in this paper. In Section 3, we will establish several sufficient conditions to guarantee that the corresponding strong solution brows up. In Section 4, we will investigate the infinite propagation speed of MCH2 with $\kappa \neq 0$.

## 2 Preliminaries

In this section, for completeness, we recall some elementary results. We list them and skip their proofs for conciseness. Local well-posedness for the MCH2 system (1.3) can be obtained by Kato's semigroup theory [24].

**Theorem 2.1** *Given $X_0 = (u_0, \gamma_0)^T \in H^s \times H^s$, $s > \frac{5}{2}$, there exist a maximal $T = T(\|X_0\|_{H^s \times H^s}) > 0$ and a unique solution $X = (u, \gamma)^T$ to system* (1.3) *such that*

$$X = X(\cdot, X_0) \in C\big([0,T); H^s \times H^s\big) \cap C^1\big([0,T); H^{s-1} \times H^{s-1}\big).$$

*Moreover, the solution depends continuously on the initial data, i.e. the mapping*

$$X_0 \to X(\cdot, X_0) : H^s \times H^s \to C\big([0,T); H^s \times H^s\big) \cap C^1\big([0,T); H^{s-1} \times H^{s-1}\big),$$

*is continuous.*

The proof of the theorem is similar to that in [20] and [25].

**Lemma 2.2** [26] *Suppose that $\Psi(t)$ is a twice continuously differential satisfying*

$$\begin{cases} \Psi''(t) \geq C_0 \Psi'(t)\Psi(t), & t > 0, C_0 > 0, \\ \Psi(t) > 0, & \Psi'(t) > 0. \end{cases} \tag{2.1}$$

*Then $\psi(t)$ blows up in finite time. Moreover, the blow-up time can be estimated in terms of the initial data as*

$$T \leq \max\left\{\frac{2}{C_0\Psi(0)}, \frac{\Psi(0)}{\Psi'(0)}\right\}.$$

**Lemma 2.3** [27] *Assume that a differentiable function $y(t)$ satisfies*

$$y'(t) \leq -Cy^2(t) + K, \tag{2.2}$$

*with constants $C, K > 0$. If the initial data $y(0) = y_0 < -\sqrt{\frac{K}{C}}$, then the solution to (2.2) goes to $-\infty$ before $t$ tends to $\frac{1}{-Cy_0 + \frac{K}{y_0}}$.*

We also need to introduce the standard particle trajectory method for later use. Motivated by McKeans deep observation on the Camassa-Holm equation in [5], we can do a similar particle trajectory as

$$\begin{cases} q_t = u(q,t) - \kappa, & 0 < t < T, x \in \mathbb{R}, \\ q(x,0) = x, & x \in \mathbb{R}. \end{cases} \tag{2.3}$$

Differentiating the first equation in (2.3) with respect to $x$, one has

$$\frac{dq_t}{dx} = q_{xt} = u_x(q,t)q_x, \quad t \in (0,T).$$

Hence

$$q_x(x,t) = \exp\left\{\int_0^t u_x(q,s)\,ds\right\}, \qquad q_x(x,0) = 1, \tag{2.4}$$

which is always positive before the blow-up time. Therefore, the function $q(\cdot,t)$ is an increasing diffeomorphism of a line.

## 3 Blow-up

In this section, we establish sufficient conditions on the initial data to guarantee blow-up for system (1.3). We start this section with the following useful lemma.

**Lemma 3.1** *Given $X_0 = (u_0, \gamma_0)^T \in H^s \times H^s$, $s > \frac{5}{2}$, $T$ is assumed to be the maximal existence time of the solution $X = (u, \gamma)^T$ to system (1.3) corresponding to the initial data $X_0$. Then for all $t \in [0, T)$, we have the following conservation law*:

$$E(t) = \int_{\mathbb{R}} \left(u^2 + u_x^2 + \gamma^2 + \gamma_x^2\right) dx. \tag{3.1}$$

*Proof* We will prove that $E(t)$ is a conserved quantity with respect to the time variable. Here we use the classical energy method. Multiplying the first equation in (1.3) by $u(x,t)$ and integrating by parts, we obtain

$$\int_{\mathbb{R}} uu_t\,dx + \int_{\mathbb{R}} u_x u_{xt}\,dx = -\int_{\mathbb{R}} u\gamma\gamma_x\,dx + \int_{\mathbb{R}} u\gamma_x\gamma_{xx}\,dx. \tag{3.2}$$

Similarly, we have the following identity for the second equation in (1.3):

$$\int_{\mathbb{R}} \gamma\gamma_t\,dx + \int_{\mathbb{R}} \gamma_x\gamma_{xt}\,dx = \int_{\mathbb{R}} u\gamma\gamma_x\,dx - \int_{\mathbb{R}} u\gamma_x\gamma_{xx}\,dx + \kappa\int_{\mathbb{R}} \gamma(\gamma_x - \gamma_{xxx})\,dx. \tag{3.3}$$

Combining the above equalities, we get

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2 + u_x^2 + \gamma^2 + \gamma_x^2\right)dx = 2\int_{\mathbb{R}}(uu_t + u_x u_{xt} + \gamma\gamma_t + \gamma_x\gamma_{xt}) = 0.$$

Therefore, $E(t)$ is conserved. Using the conservation law, we obtain

$$\begin{aligned}\left\|u(\cdot,t)^2\right\|_{L^\infty(\mathbb{R})} + \left\|\gamma^2(\cdot,t)\right\|_{L^\infty(\mathbb{R})} &\le \frac{1}{2}\|u\|_{H^1(\mathbb{R})}^2 + \frac{1}{2}\|\gamma\|_{H^1(\mathbb{R})}^2\\ &= \frac{1}{2}\|u_0\|_{H^1(\mathbb{R})}^2 + \frac{1}{2}\|\gamma_0\|_{H^1(\mathbb{R})}^2\\ &= \frac{1}{2}E_0,\end{aligned}$$

for all $t \in [0,T)$, where $E_0$ is the initial value of $E(t)$. □

The next result describes the precise blow-up scenarios for sufficiently regular solutions to system (1.3).

**Theorem 3.2** *Given $X_0 = (u_0,\gamma_0)^T \in H^s \times H^s$, $s > \frac{5}{2}$, there exist a maximal $T = T(\|X_0\|_{H^s\times H^s}) > 0$ and a unique solution $X = (u,\gamma)^T$ to system* (1.3). *Then the corresponding solutions blows up in finite time if and only if*

$$\liminf_{t\to T}\inf_{x\in\mathbb{R}}\left\{u_x(x,t)\right\} = -\infty \quad \text{or} \quad \liminf_{t\to T}\inf_{x\in\mathbb{R}}\left\{\gamma_x(x,t)\right\} = -\infty.$$

*Proof* By a simple density argument, one needs only to show that the desired results are valid when $s \ge 3$, so in the following section, $s = 3$ is taken for simplicity of notation. Firstly, multiplying the first and second equations in (1.3) by $u_{xx}$ and $\gamma_{xx}$, respectively, and integrating by parts, we have

$$\begin{aligned}&2\left(\int_{\mathbb{R}} u_x u_{xt}\,dx + \int_{\mathbb{R}} u_{xx}u_{xxt}\,dx\right)\\ &\quad = -3\int_{\mathbb{R}} u_x\left(u_x^2 + u_{xx}^2\right)dx + 2\int u_{xx}(\gamma\gamma_x - \gamma_x\gamma_{xx})\,dx,\end{aligned} \tag{3.4}$$

$$2\left(\int_{\mathbb{R}} \gamma_x\gamma_{xt}\,dx + \int_{\mathbb{R}} \gamma_{xx}\gamma_{xxt}\,dx\right) = -\int_{\mathbb{R}} u_x\left(3\gamma_x^2 + \gamma_{xx}^2\right)dx - 2\int u_{xx}\gamma\gamma_x\,dx, \tag{3.5}$$

and then combining (3.2) and (3.3), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} u^2 + 2u_x^2 + u_{xx}^2 + \gamma^2 + 2\gamma_x^2 + \gamma_{xx}^2\,dx\\&\quad = -\int_{\mathbb{R}} u_x\big(3u_x^2 + 3u_{xx}^2 + 3\gamma_x^2 + \gamma_{xx}^2\big)\,dx - 2\int_{\mathbb{R}} \gamma_{xx}\gamma_x u_{xx}\,dx.\end{aligned} \tag{3.6}$$

Assume that $T < \infty$ and there exists $M > 0$, such that

$$u_x(t,x) \geq -M \quad \text{and} \quad \gamma_x(t,x) \geq -M \quad \text{for } \forall (t,x) \in [0,T) \times \mathbb{R}. \tag{3.7}$$

Then, applying Gronwall's inequality to (3.6) and (3.7), we get

$$\|u_{xx}(t)\|_{L^2}^2 + \|\gamma_{xx}(t)\|_{L^2}^2 \leq \|u(t)\|_{H^2}^2 + \|\gamma(t)\|_{H^2}^2 \leq \big(\|u_0\|_{H^2}^2 + \|\gamma_0\|_{H^2}^2\big)e^{4MT}. \tag{3.8}$$

On the other hand, differentiating the two equations in (1.3) with respect to $x$, then multiplying by $u_{xxx}$ and $\gamma_{xxx}$, respectively, and combining with (3.1) and (3.6), we obtain

$$\begin{aligned}\frac{d}{dt}\big(\|u(t)\|_{H^3}^2 + \|\gamma(t)\|_{H^3}^2\big) &\leq CM\big(\|u(t)\|_{H^3}^2 + \|\gamma(t)\|_{H^3}^2\big)\\&\quad - 2\int_{\mathbb{R}} \gamma_{xx}\gamma_{xxx}u_{xx}\,dx.\end{aligned} \tag{3.9}$$

Then applying the Sobolev embedding result $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ (with $s > \frac{1}{2}$), we get

$$-\int_{\mathbb{R}} \gamma_{xx}\gamma_{xxx}u_{xx}\,dx \leq C\big(\|u(t)\|_{H^3}^2 + \|\gamma(t)\|_{H^3}^2\big)\big(\|u_{xx}(t)\|_{L^2}^2 + \|\gamma_{xx}(t)\|_{L^2}^2\big)^{\frac{1}{2}}, \tag{3.10}$$

because of (3.8), (3.9), (3.10), and the Gronwall inequality, we obtain

$$\|u(t)\|_{H^3}^2 + \|\gamma(t)\|_{H^3}^2 \leq \big(\|u_0\|_{H^3}^2 + \|\gamma_0\|_{H^3}^2\big)e^{A}, \quad \text{for } \forall t \in [0,T), \tag{3.11}$$

where $A = C[M + (\|u_0\|_{H^2}^2 + \|\gamma_0\|_{H^2}^2)^{\frac{1}{2}}e^{2MT}]T$.

This contradicts the assumption. Conversely, the Sobelev embedding result $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ (with $s > \frac{1}{2}$) implies that if Theorem 3.2 holds, the solution blows up in finite time, which completes the proof of Theorem 3.2. □

We state our first criterion via the associated initial potential as follows.

**Theorem 3.3** *Suppose* $X_0 = (u_0, \gamma_0)^T \in H^s \times H^s$, $s > \frac{5}{2}$, $\rho_0(x_0) = 0$, $y_0(x_0) = 0$, *and the initial data satisfies the following conditions*:

(i) $\rho_0 \geq 0$ *on* $(-\infty, x_0)$ *and* $\rho_0 \leq 0$ *on* $(x_0, \infty)$,

(ii) $\int_{-\infty}^{x_0} e^{\xi} y_0(\xi)\,d\xi \geq 0$ *and* $\int_{x_0}^{\infty} e^{-\xi} y_0(\xi)\,d\xi \leq 0$,

*for some point* $x_0 \in \mathbb{R}$. *Then the solution to our system* (1.3) *with initial value* $X_0$ *blows up in finite time*.

**Remark 3.1** This theorem is similar to the result proved by Zhou in [6].

*Proof* Differentiating the first equation (1.2) with respect to variable $x$, we obtain

$$u_{tx} + (u-\kappa)u_{xx} + u_x^2 + \partial_x^2\left(G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right)\right) = 0. \tag{3.12}$$

Applying $\partial_x^2(G*f) = G*f - f$ to (3.12) yields

$$u_{tx} + (u-\kappa)u_{xx} = u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2 - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right). \tag{3.13}$$

This equation gives

$$\begin{aligned}\frac{d}{dt}u_x\big(q(x_0,t),t\big) &= u_{xt}\big(q(x_0,t),t\big) + \big(u\big(q(x_0,t),t\big) - \kappa\big)u_{xx}\big(q(x_0,t),t\big) \\ &\leq \frac{1}{2}u^2\big(q(x_0,t),t\big) - \frac{1}{2}u_x^2\big(q(x_0,t),t\big) \\ &\quad + \frac{1}{2}\gamma^2\big(q(x_0,t),t\big) - \frac{1}{2}\gamma_x^2\big(q(x_0,t),t\big) - G * \left(\frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right),\end{aligned} \tag{3.14}$$

where we used the fact

$$G * \left(u^2 + \frac{1}{2}u_x^2\right) \geq \frac{1}{2}u^2.$$

In order to arrive at our result, we need the following three claims.

Claim 1. $y(q(x_0,t),t) + \kappa = 0$ for all $t$ in its lifespan.

It is worth to notice the first two equations in (1.3) as follows:

$$y_t + 2yu_x + y_x u - \kappa y_x + \rho\gamma_x = 0, \quad t > 0, x \in \mathbb{R}, \tag{3.15}$$

$$\rho_t + (\rho u)_x - \kappa\rho_x = 0, \quad t > 0, x \in \mathbb{R}. \tag{3.16}$$

By applying the particle trajectory method to the above two equations, we obtain

$$\begin{aligned}\frac{d}{dt}y\big(q(x,t),t\big)q_x^2(x,t) &= y_t q_x^2 + y_x q_t q_x^2 + 2yq_x q_{xt} \\ &= \big(y_t + 2yu_x + y_x(u-\kappa)\big)q_x^2(x,t) \\ &= -\rho\big(q(x,t),t\big)\gamma_x\big(q(x,t),t\big)q_x^2(x,t)\end{aligned} \tag{3.17}$$

and

$$\frac{d}{dt}\big(\rho\big(q(x,t),t\big)q_x(x,t)\big) = (\rho_t + \rho_x u + \rho u_x)q_x = 0,$$

which implies that

$$\rho\big(q(x,t),t\big)q_x(x,t) = \rho_0(x). \tag{3.18}$$

Obviously, we can obtain $\rho(q(x_0,t),t) = 0$ since $\rho_0(x_0) = 0$; then we deduce that

$$\frac{d}{dt} y\big(q(x_0,t),t\big) q_x^2(x_0,t) = 0.$$

Thus it is easier to see that $y(q(x_0,t),t)q_x^2(x_0,t)$ is independent on time $t$. Without loss of generality, we take $t = 0$, and we have

$$y\big(q(x_0,t),t\big) q_x^2(x_0,t) = y_0(x_0) = 0.$$

Therefore, thanks to (2.4) we obtain $y(q(x_0,t),t) = 0$, for all $t$ in its lifespan.

Claim 2. For any fixed $t$, $\gamma_x^2(x,t) - \gamma^2(x,t) \le (\gamma_x^2 - \gamma^2)(q(x_0,t),t)$ for all $x \in \mathbb{R}$.

If $x \le q(x_0,t)$, then

$$\begin{aligned} \gamma_x^2(x,t) - \gamma^2(x,t) &= -\left(\int_{-\infty}^{q(x_0,t)} e^{\xi} \rho(\xi,t)\,d\xi - \int_x^{q(x_0,t)} e^{\xi} \rho(\xi,t)\,d\xi\right) \\ &\quad \times \left(\int_{q(x_0,t)}^{\infty} e^{-\xi} \rho(\xi,t)\,d\xi + \int_x^{q(x_0,t)} e^{-\xi} \rho(\xi,t)\,d\xi\right) \\ &= \gamma_x^2\big(q(x_0,t),t\big) - \gamma^2\big(q(x_0,t),t\big) \\ &\quad - \int_{-\infty}^{x} e^{\xi} \rho(\xi,t)\,d\xi \int_x^{q(x_0,t)} e^{-\xi} \rho(\xi,t)\,d\xi \\ &\quad + \int_x^{q(x_0,t)} e^{\xi} \rho(\xi,t)\,d\xi \int_{q(x_0,t)}^{\infty} e^{-\xi} \rho(\xi,t)\,d\xi \\ &\le \gamma_x^2\big(q(x_0,t),t\big) - \gamma^2\big(q(x_0,t),t\big), \end{aligned} \tag{3.19}$$

where the condition (i) is used. Similarly, for $x \ge q(x_0,t)$, we also have

$$\gamma_x^2(x,t) - \gamma^2(x,t) \le \gamma_x^2\big(q(x_0,t),t\big) - \gamma^2\big(q(x_0,t),t\big), \tag{3.20}$$

for any fixed $t$. Combining (3.19) and (3.20), we get

$$\gamma_x^2(x,t) - \gamma^2(x,t) \le \gamma_x^2\big(q(x_0,t),t\big) - \gamma^2\big(q(x_0,t),t\big), \quad \text{for all } x \in \mathbb{R}.$$

Combining Claim 2 with (3.14), we get

$$\frac{d}{dt} u_x\big(q(x_0,t),t\big) \le \frac{1}{2} u^2\big(q(x_0,t),t\big) - \frac{1}{2} u_x^2\big(q(x_0,t),t\big). \tag{3.21}$$

Claim 3. $u_x(q(x_0,t),t) < 0$ is decreasing, $u^2(q(x_0,t),t) < u_x^2(q(x_0,t),t)$ for all $t \ge 0$.

Suppose not, *i.e.* there exists a $t_0$ such that $u^2(q(x_0,t),t) < u_x^2(q(x_0,t),t)$ on $[0,t_0)$ and $u^2(q(x_0,t_0),t_0) = u_x^2(q(x_0,t_0),t_0)$. Now, let

$$I(t) := \frac{1}{2} e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^{\xi} y(\xi,t)\,d\xi$$

and

$$II(t) := \frac{1}{2} e^{q(x_0,t)} \int_{q(x_0,t)}^{\infty} e^{-\xi} y(\xi,t)\,d\xi.$$

Firstly, differentiating $I(t)$, we have

$$\begin{aligned}
\frac{dI(t)}{dt} &= -\frac{1}{2}u\bigl(q(x_0,t),t\bigr)e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi \\
&\quad + \frac{1}{2}\kappa e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y_t(\xi,t)\,d\xi \\
&= \frac{1}{2}u(u_x-u)\bigl(q(x_0,t),t\bigr) + \frac{1}{2}\kappa e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi \\
&\quad - \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}(uy_x + 2u_xy + \rho\gamma_x - \kappa y_x)\,d\xi \\
&\geq \frac{1}{2}u(u_x-u)\bigl(q(x_0,t),t\bigr) + \frac{1}{2}\kappa e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi \\
&\quad + \frac{1}{4}\bigl(u^2+u_x^2-2uu_x\bigr)\bigl(q(x_0,t),t\bigr) + \frac{1}{2}\kappa e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y_x(\xi,t)\,d\xi \\
&\quad - \frac{1}{4}\gamma^2\bigl(q(x_0,t),t\bigr) + \frac{1}{4}\gamma_x^2\bigl(q(x_0,t),t\bigr) + G*\left(\frac{1}{4}\gamma^2 - \frac{1}{4}\gamma_x^2\right) \\
&\geq \frac{1}{4}u_x^2\bigl(q(x_0,t),t\bigr) - \frac{1}{4}u^2\bigl(q(x_0,t),t\bigr) > 0, \quad \text{on } [0,t_0),
\end{aligned} \tag{3.22}$$

where we used the Claim 2.

Secondly, by the same argument, we get

$$\begin{aligned}
\frac{dII(t)}{dt} &= \frac{1}{2}u\bigl(q(x_0,t),t\bigr)e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\,d\xi \\
&\quad - \frac{1}{2}\kappa e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y_t(\xi,t)\,d\xi \\
&= \frac{1}{2}u(u_x+u)\bigl(q(x_0,t),t\bigr) - \frac{1}{2}\kappa e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\,d\xi \\
&\quad - \frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}(uy_x + 2u_xy + \rho\gamma_x - \kappa y_x)\,d\xi \\
&\leq \frac{1}{2}u(u_x+u)\bigl(q(x_0,t),t\bigr) - \frac{1}{2}\kappa e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\,d\xi \\
&\quad - \frac{1}{4}\bigl(u^2+u_x^2+2uu_x\bigr)\bigl(q(x_0,t),t\bigr) + \frac{1}{2}\kappa e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y_x(\xi,t)\,d\xi \\
&\quad + \frac{1}{4}\gamma^2\bigl(q(x_0,t),t\bigr) - \frac{1}{4}\gamma_x^2\bigl(q(x_0,t),t\bigr) - G*\left(\frac{1}{4}\gamma^2 - \frac{1}{4}\gamma_x^2\right) \\
&\leq -\frac{1}{4}u_x^2\bigl(q(x_0,t),t\bigr) + \frac{1}{4}u^2\bigl(q(x_0,t),t\bigr) < 0, \quad \text{on } [0,t_0).
\end{aligned} \tag{3.23}$$

Hence, based on (3.22), (3.23), and the continuity property of ODEs, we have

$$u_x^2\bigl(q(x_0,t),t\bigr) - u^2\bigl(q(x_0,t),t\bigr) = -4I(t)II(t) > -4I(0)II(0) \geq 0,$$

for all $t > 0$, which implies $t_0$ can be extended to infinity.

Using (3.22) and (3.23) again, we have the following equation for $u_x^2(q(x_0,t),t) - u^2(q(x_0,t),t)$:

$$\begin{aligned}
&\frac{d}{dt}\big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big)\big) \\
&\quad = -4\frac{d}{dt}I(t)II(t) \\
&\quad = -4II(t)\frac{d}{dt}I(t) - 4I(t)\frac{d}{dt}II(t) \\
&\quad \geq -II(t)\big(u_x^2 - u^2\big)\big(q(x_0,t),t\big) + I(t)\big(u_x^2 - u^2\big)\big(q(x_0,t),t\big) \\
&\quad = -u_x\big(q(x_0,t),t\big)\big(u_x^2 - u^2\big)\big(q(x_0,t),t\big),
\end{aligned} \tag{3.24}$$

where we used $u_x(q(x_0,t),t) = -I(t) + II(t)$.

Now, substituting (3.21) into (3.24), it yields

$$\begin{aligned}
&\frac{d}{dt}\big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big)\big) \\
&\quad \geq \frac{1}{2}\big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big)\big) \times (-2u_x)\big(q(x_0,t),t\big) \\
&\quad \geq \frac{1}{2}\big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big)\big) \\
&\qquad \times \left(\int_0^t \big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big)\big)\,d\tau - 2u_{0x}(x_0)\right).
\end{aligned} \tag{3.25}$$

Let $\Psi(t) = \int_0^t (u_x^2(q(x_0,t),t) - u^2(q(x_0,t),t))\,d\tau - 2u_{0x}(x_0)$, then (3.22) is an equation of type (3.1) with $C_0 = \frac{1}{2}$. The proof is completed by applying Lemma 2.3. □

**Remark 3.2** Scrutinizing the proof, we discover that the condition (i) guarantees that inequality (3.19), (3.20) hold. If it is replaced by

$$\rho_0 \leq 0 \text{ on } (-\infty, x_0) \quad \text{and} \quad \rho_0 \geq 0 \text{ on } (x_0, \infty),$$

we find that the inequalities (3.19), (3.20) still hold. As is well known, McKean [5] states that only the sign of the initial potential $y_0(x)$, not the size of it, affects the wave breaking phenomenon. Similar to his theorem, we apply a similar initial potential $y_0(x) + \kappa$ to the two-component case, and it reveals that the sign of the initial density $\rho_0(x)$ also plays an important role.

Then we give the second criterion in this paper.

**Theorem 3.4** *Suppose* $X_0 = (u_0, \gamma_0)^T \in H^s \times H^s$, $s > \frac{5}{2}$, *and that we have the following inequality*:

$$\left(\int_{\mathbb{R}} u_{0x}^3\,dx\right)^2 > 3E_0^3 \quad \textit{and} \quad \int_{\mathbb{R}} u_{0x}^3\,dx < 0.$$

*Then the corresponding solution to system* (1.3) *blows up in finite time.*

*Proof* Let

$$m(t) = \int_{\mathbb{R}} u_x^3 \, dx, \quad t \geq 0.$$

Multiplying (3.13) with $u_x^2$ and integrating by parts subsequently, we obtain the equation for $m(t)$ as

$$\begin{aligned}
\frac{1}{3}\frac{dm(t)}{dt} &= -\frac{1}{2}\int_{\mathbb{R}} u_x^4 \, dx + \frac{1}{3}\int_{\mathbb{R}} u_x^4 \, dx - \int_{\mathbb{R}} u_x^2 G * \left[u^2 + \frac{u_x^2}{2} + \frac{\gamma^2}{2} - \frac{\gamma_x^2}{2}\right] dx \\
&\quad + \int_{\mathbb{R}} u_x^2 \left[u^2 + \frac{\gamma^2}{2} - \frac{\gamma_x^2}{2}\right] dx \\
&\leq -\frac{1}{6}\int_{\mathbb{R}} u_x^4 \, dx + \frac{1}{2}\int_{\mathbb{R}} u^2 u_x^2 \, dx + \frac{1}{2}\int_{\mathbb{R}} u_x^2 G * \gamma_x^2 \, dx + \frac{1}{2}\int_{\mathbb{R}} u_x^2 \gamma^2 \, dx \\
&\leq -\frac{1}{6}\int_{\mathbb{R}} u_x^4 \, dx + \frac{1}{2}\int_{\mathbb{R}} u_x^2 \left[u^2 + \gamma^2\right] dx + \frac{1}{4}\left\|\gamma_x^2\right\|_{L^1} \int_{\mathbb{R}} u_x^2 \, dx,
\end{aligned}$$

where we used

$$\left\|G * \gamma_x^2\right\|_{L^\infty} \leq \|G\|_{L^\infty} \left\|\gamma_x^2\right\|_{L^1} \leq \frac{1}{2}\left\|\gamma_x^2\right\|_{L^1}.$$

According to the invariant property of $E(t)$, we get

$$\frac{1}{2}\int_{\mathbb{R}} u_x^2 \left[u^2 + \gamma^2\right] dx + \frac{1}{4}\left\|\gamma_x^2\right\|_{L^1} \int_{\mathbb{R}} u_x^2 \, dx \leq \frac{1}{4}E_0^2 + \frac{1}{4}E_0^2 = \frac{1}{2}E_0^2.$$

On the other hand, the Cauchy-Schwarz inequality implies that

$$\left|\int_{\mathbb{R}} u_x^3 \, dx\right| \leq \left(\int_{\mathbb{R}} u_x^4 \, dx\right)^{1/2} \left(\int_{\mathbb{R}} u_x^2 \, dx\right)^{1/2},$$

therefore

$$\int_{\mathbb{R}} u_x^4 \, dx \geq \frac{1}{E_0}\left(\int_{\mathbb{R}} u_x^3 \, dx\right)^2.$$

Thus we obtain from the above that

$$\frac{dm(t)}{dt} \leq -\frac{m^2}{2E_0} + \frac{3}{2}E_0^2.$$

The hypothesis of this theorem and the standard argument on the Riccati type equation ensure that there exists a time $T$ such that

$$\lim_{t \to T} \int_{\mathbb{R}} u_x^3 \, dx = -\infty.$$

Since

$$\int_{\mathbb{R}} u_x^3 \, dx \geq \inf u_x(x,t) \int_{\mathbb{R}} u_x^2 \, dx \geq \inf u_x(x,t) E_0,$$

we have

$$\liminf_{t\to T}\inf_{x\in\mathbb{R}} u_x(x,t) = -\infty.$$

This completes the proof. □

Finally, we give the third criterion.

**Theorem 3.5** *Suppose* $X_0 = (u_0,\gamma_0)^T \in H^s \times H^s$, $s > \frac{5}{2}$, $\rho_0$ *and* $u_0$ *satisfy the following conditions*:

(i) $\rho_0 \geq 0$ *on* $(-\infty, x_0)$ *and* $\rho_0 \leq 0$ *on* $(x_0,\infty)$
$\big($*or* $\rho_0 \leq 0$ *on* $(-\infty, x_0)$ *and* $\rho_0 \geq 0$ *on* $(x_0,\infty)\big)$,

(ii) $u_0'(x_0) \leq -\frac{\sqrt{2}}{2}E_0^{\frac{1}{2}}$.

*Proof* As mentioned in Claim 2 of Theorem 3.3, condition (i) means that for any fixed $t$, $\gamma_x^2(x,t) - \gamma^2(x,t) \leq \gamma_x^2(q(x_0,t),t) - \gamma^2(q(x_0,t),t)$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned}
\frac{d}{dt}u_x\big(q(x_0,t),t\big) &\leq \frac{1}{2}u^2\big(q(x_0,t),t\big) - \frac{1}{2}u_x^2\big(q(x_0,t),t\big)\\
&\leq \frac{1}{4}\|u\|_{H^1}^2 - \frac{1}{2}u_x^2\big(q(x_0,t),t\big)\\
&\leq \frac{1}{4}E_0 - \frac{1}{2}u_x^2\big(q(x_0,t),t\big),
\end{aligned}$$

let $\varphi(t) = u_x(q(x_0,t),t)$, we obtain

$$\frac{d\varphi}{dt} \leq -\frac{1}{2}\varphi^2 + K^2,$$

where $K = \frac{1}{2}E_0^{\frac{1}{2}}$. By applying Lemma 2.3, we have

$$\lim_{t\to T}\varphi(t) = -\infty, \quad \text{with } T = \frac{1}{-\frac{1}{2}\varphi_0 + \frac{K^2}{\varphi_0}},$$

from the condition that

$$\varphi_0 < -\sqrt{2}K = -\frac{\sqrt{2}}{2}E_0^{\frac{1}{2}}.$$

This completes the proof. □

## 4 Infinite propagation speed

In this section, we consider the infinite propagation speed for system (1.1). It can be shown as follows.

**Theorem 4.1** *Assume* $T = T(u_0,\rho_0) > 0$ *is the maximal existence time of the unique classical solutions* $(u,\rho)$ *to system* (1.3). *If* $u_0(x) = u(x,0)$ *has compact support* $[\alpha,\beta]$, *and* $\rho_0$ *is* *also compactly supported on the interval* $[\alpha,\beta]$, *moreover,* $\rho_0 > 0$ (*or* $\rho_0 < 0$ ) *on* $[\alpha,\beta]$, *then for* $t \in (0,T]$, *we have*

$$u(x,t) = \begin{cases} \phi_-(t)e^{-x}, & x > q(\alpha,t), \\ \phi_+(t)e^{x}, & x < q(\beta,t), \end{cases}$$

*where* $\phi_-(t)$ *and* $\phi_+(t)$ *denote continuous nonvanishing functions with* $\phi_-(t) < 0$ *and* $\phi_+(t) > 0$ *for* $t \in (0,T]$. *Furthermore,* $\phi_-(t)$ *is a strictly decreasing function, while* $\phi_+(t)$ *is an increasing function.*

*Proof* Since $\rho_0$ has initially compact support $[\alpha,\beta]$, and thanks to (3.17) and (3.18), we obtain $y(q(x,t),t)q_x^2(x,t) = y_0(x)$, for any $x \in \mathbb{R} - [\alpha,\beta]$; simultaneously, $u_0(x)$ has compact support $[\alpha,\beta]$, which implies that $y_0(x)$ has compact support $[\alpha,\beta]$. It follows that $y(q(x,t),t)q_x^2(x,t) = y_0(x) = 0$, for any $x \in \mathbb{R} - [\alpha,\beta]$. So $y$ is compactly supported with its support contained in the interval $[q(\alpha,t), q(\beta,t)]$. Therefore the following functions are well defined:

$$E_+(t) = \int_{\mathbb{R}} e^x y(x,t)\,dx \quad \text{and} \quad E_-(t) = \int_{\mathbb{R}} e^{-x} y(x,t)\,dx,$$

with

$$E_+(t) = \int_{\mathbb{R}} e^x y_0\,dx = 0 \quad \text{and} \quad E_-(t) = \int_{\mathbb{R}} e^{-x} y_0\,dx = 0. \tag{4.1}$$

Then, for $x > q(\beta,t)$,

$$u(x,t) = \frac{1}{2}e^{-|x|} * y(x,t) = \frac{1}{2}e^{-x}\int_{q(\alpha,t)}^{q(\beta,t)} e^x y(x,t)\,dt = \frac{1}{2}e^{-x}E_+(t), \tag{4.2}$$

similarly, when $x < q(\alpha,t)$

$$u(x,t) = \frac{1}{2}e^{-|x|} * y(x,t) = \frac{1}{2}e^{x}\int_{q(\alpha,t)}^{q(\beta,t)} e^{-x} y(x,t)\,dt = \frac{1}{2}e^{x}E_-(t). \tag{4.3}$$

Hence, as consequences of (4.2) and (4.3), we have

$$u(x,t) = -u_x(x,t) = u_{xx}(x,t) = \frac{1}{2}e^{-x}E_+(t), \quad \text{as } x > q(\beta,t), \tag{4.4}$$

and

$$u(x,t) = u_x(x,t) = u_{xx}(x,t) = \frac{1}{2}e^{x}E_-(t), \quad \text{as } x < q(\alpha,t). \tag{4.5}$$

On the other hand,

$$\begin{aligned} \frac{dE_+(t)}{dt} &= \int_{-\infty}^{\infty} e^x y_t(x,t)\,dx \\ &= \int_{-\infty}^{\infty} e^x \bigl\{-(yu_x) - yu_x + \kappa y_x - \rho\gamma_x\bigr\}\,dx \end{aligned}$$

$$= \int_{-\infty}^{\infty} e^x \left[u^2 + \frac{1}{2}u_x^2 - \rho\gamma_x\right] dx$$
$$= \int_{-\infty}^{\infty} e^x \left[u^2 + \frac{1}{2}u_x^2 + \gamma^2 - \gamma_x^2\right] dx. \tag{4.6}$$

For any fixed $t$, $\gamma^2 - \gamma_x^2 = e^x \int_x^{\infty} e^{\tau} \rho(\tau, t)\, d\tau \times e^{-x} \int_{-\infty}^{x} e^{-\tau} \rho(\tau, t)\, d\tau$, for all $x \in \mathbb{R}$. Then for $x > q(\beta, t)$, we have

$$\gamma^2 - \gamma_x^2 = 0, \tag{4.7}$$

similarly, when $x < q(\alpha, t)$, we get

$$\gamma^2 - \gamma_x^2 = 0, \tag{4.8}$$

when $q(\alpha, t) \le x \le q(\beta, t)$, we obtain

$$\gamma^2 - \gamma_x^2 = e^x \int_x^{q(\beta,t)} e^{\tau} \rho(\tau, t)\, d\tau \times e^{-x} \int_{q(\alpha,t)}^{x} e^{-\tau} \rho(\tau, t)\, d\tau > 0. \tag{4.9}$$

By using (4.7), (4.8), and (4.9), we have

$$\frac{dE_+(t)}{dt} > 0. \tag{4.10}$$

Therefore, $E_+(t)$ is an increasing function in the lifespan. From (4.1), it follows that $E_+(t) > 0$ for $t \in (0, T]$.

Similarly, it is easy to see that $E_-(t)$ is decreasing with $E_-(0) = 0$. Therefore, $E_-(t) < 0$ for $t \in (0, T]$.

Taking $\varphi_{\pm}(t) = \frac{1}{2}E_{\pm}(t)$, we obtain what we want. Then the theorem is proved. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Author details**
[1]Department of Mathematics, Zhejiang Normal University, Jinhua, China. [2]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, 21589, Saudi Arabia. [3]Department of Mathematics, Quaid-I-Azam University 45320, Islamabad, 44000, Pakistan.

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