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A geometric property for a class of meromorphic analytic functions

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Abstract

In this paper, we investigate a geometric property of a class of meromorphic functions. This property implies concavity. A sufficient condition, for a function in this class, is considered utilizing Jack's lemma. We show that, for a meromorphic function $f(z)$, the sufficient condition for concavity is $\Re\{\frac{zf'''(z)}{f''(z)}\} < 0$, $z \in U$.

1 Introduction

A conformal, meromorphic function f on the punctured unit disk $\widehat{U} := \{z \in \mathbb{C} : 0 < |z| < 1\}$ is said to be a concave mapping if $f(\widehat{U})$ is the complement of a convex, compact set. Recently, Chuaqui *et al.* [1] studied the normalized conformal mappings of the disk onto the exterior of a convex polygon via an exemplification formula furnished by the Schwarz lemma. Let Σ be the family of functions analytic in the punctured unit disk $\widehat{U}$ of the form

$$f(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \cdots, \tag{1.1}$$

then the necessary and sufficient condition for f to be concave mapping is

$$1 + \Re\left\{z\frac{f''(z)}{f'(z)}\right\} < 0, \quad z \in \widehat{U},$$

where

$$z\frac{f''(z)}{f'(z)} = -2 - 2b_1 z^2 - 6b_2 z^3 - \left(12b_3 + 2b_1^2\right)z^4 - (20b_4 + 10b_1 b_2)z^5 - \cdots.$$

Furthermore, an analytic function $f \in \widehat{U}$ is called a concave function of order $\alpha \geq 0$ if it satisfies

$$1 + \Re\left\{z\frac{f''(z)}{f'(z)}\right\} < -\alpha, \quad z \in \widehat{U}.$$

Denote this class by Σ_α.

In this work, we investigate a geometric property of a class of meromorphic functions. This property implies concavity. A sufficient condition, for a function in this class, is considered utilizing Jack's lemma. We show that, for a meromorphic function $f(z) \in \Sigma$, a sufficient condition for concavity is

$$\mathfrak{Re}\left\{\frac{zf'''(z)}{f''(z)}\right\} < 0, \quad z \in U.$$

2 Main result

We have the following result.

Theorem 2.1 *If $f \in \Sigma$ satisfies the following inequality*:

$$\mathfrak{Re}\left\{\frac{zf'''(z)}{f''(z)}\right\} < 0, \quad z \in U, \tag{2.1}$$

such that

$$\mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \neq 0, \quad z \in U,$$

then f is concave in $\widehat{U}$.

Proof To show that f is concave, we need

$$\mathfrak{Re}\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in U.$$

Let $\omega(z)$ be a function defined by

$$-1 - \frac{zf''(z)}{f'(z)} = \frac{1 + w(z)}{1 - w(z)}. \tag{2.2}$$

Then $w(z)$ is analytic in U with $w(0) = w'(0) = 0$ and

$$\frac{zf''(z)}{f'(z)} = \frac{-2}{1 - w(z)}. \tag{2.3}$$

Therefore, we need to show that $|w(z)| < 1$ in U. If not, then there exists a $z_0 \in U$ such that $|w(z_0)| = 1$. By Jack's lemma $z_0 w'(z_0) = kw(z_0)$, where $k \geq 2$, because $w'(0) = 0$. By (2.3) we have

$$-z^3 f''(z)\big(1 - w(z)\big) = 2z^2 f'(z). \tag{2.4}$$

Differentiating logarithmically (2.4) with respect to z, we conclude

$$\frac{3z^2 f''(z) + z^3 f'''(z)}{z^3 f''(z)} - \frac{w'(z)}{1 - w(z)} = \frac{2zf'(z) + z^2 f''(z)}{z^2 f'(z)},$$

hence

$$\frac{3z^3 f''(z) + z^4 f'''(z)}{z^3 f''(z)} - \frac{zw'(z)}{1 - w(z)} = \frac{2z^2 f'(z) + z^3 f''(z)}{z^2 f'(z)}$$

and

$$3+\frac{z^4f'''(z)}{z^3f''(z)}-\frac{zw'(z)}{1-w(z)}=2+\frac{z^3f''(z)}{z^2f'(z)}.$$

It gives for $z=z_0$

$$3+\frac{z_0f'''(z_0)}{f''(z_0)}-\frac{z_0w'(z_0)}{1-w(z_0)}=2+\frac{z_0f''(z_0)}{f'(z_0)}.$$

By (2.3) and by $z_0w'(z_0)=kw(z_0)$, where $k\geq 2$, we have

$$\begin{aligned}\frac{z_0f'''(z_0)}{f''(z_0)}&=\frac{z_0w'(z_0)}{1-w(z_0)}-1+\frac{z_0f''(z_0)}{f'(z_0)}\\&=\frac{z_0w'(z_0)}{1-w(z_0)}-1-\frac{2}{1-w(z)}\\&=\frac{kw(z_0)}{1-w(z_0)}-1-\frac{2}{1-w(z)}\\&=\frac{(k+1)w(z_0)-3}{1-w(z_0)}.\end{aligned}$$

Because $k+1\geq 3$, a simple geometric observation yields

$$\mathfrak{Re}\left\{\frac{(k+1)w(z_0)-3}{1-w(z_0)}\right\}\geq 0,$$

hence

$$\mathfrak{Re}\left\{\frac{z_0f'''(z_0)}{f''(z_0)}\right\}\geq 0.$$

This contradicts the assumption (2.1). Therefore, $|w(z)|<1$ in U and (2.2) means that f is concave. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors jointly worked on deriving the results and approved the final manuscript.

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