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Weighted boundedness of multilinear singular integral operators

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Abstract

In this paper, we establish the weighted sharp maximal function inequalities for the multilinear singular integral operators. As an application, we obtain the boundedness of the multilinear operators on weighted Lebesgue and Morrey spaces.

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Keywords: multilinear operator; singular integral operator; sharp maximal function; weighted BMO; weighted Lipschitz function

1 Introduction

As the development of singular integral operators (see [1–3]), their commutators operators have been well studied. In [4–6], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [7]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [8, 9], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces are obtained. In [10, 11], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces are obtained (also see [12, 13]). In [14–17], the authors studied some multilinear singular integral operators as follows (also see [18, 19]):

$$T^b(f)(x) = \int \frac{R_{m+1}(b; x, y)}{|x - y|^m} K(x, y) f(y) dy,$$

and they obtained some variant sharp function estimates and boundedness of the multilinear operators if $D^\alpha b \in \text{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m$. In this paper, we will study the multilinear operator generated by the singular integral operator and the weighted Lipschitz and BMO functions, that is, $D^\alpha b \in \text{BMO}(w)$ or $D^\alpha b \in \text{Lip}_\beta(w)$ for all α with $|\alpha| = m$.

2 Preliminaries

First, let us introduce some notations. Throughout this paper, Q will denote a cube of $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function f , the sharp maximal function of f is defined by

$$M^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy;$$

here, and in the following, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well known that (see [1, 2])

$$M^\#(f)(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta^\#(f)(x) = M^\#(|f|^\eta)^{1/\eta}(x)$ and $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \leq p < \infty$ and the non-negative weight function w , set

$$M_{\eta,p,w}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{w(Q)^{1-p\eta/n}} \int_Q |f(y)|^p w(y) dy \right)^{1/p}.$$

We write $M_{\eta,p,w}(f) = M_{p,w}(f)$ if $\eta = 0$.

The A_p weight is defined by (see [1]), for $1 < p < \infty$,

$$A_p = \left\{ w \in L^1_{\text{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}$$

and

$$A_1 = \{ w \in L^p_{\text{loc}}(R^n) : M(w)(x) \leq Cw(x), \text{ a.e.} \}.$$

Given a non-negative weight function w . For $1 \leq p < \infty$, the weighted Lebesgue space $L^p(R^n, w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For $0 < \beta < 1$ and the non-negative weight function w , the weighted Lipschitz space $\text{Lip}_\beta(w)$ is the space of functions b such that

$$\|b\|_{\text{Lip}_\beta(w)} = \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p} dy \right)^{1/p} < \infty,$$

and the weighted BMO space $\text{BMO}(w)$ is the space of functions b such that

$$\|b\|_{\text{BMO}(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q |b(y) - b_Q|^p w(x)^{1-p} dy \right)^{1/p} < \infty.$$

Remark

(1) It is well known that (see [10, 20]), for $b \in \text{Lip}_\beta(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \leq Ck \|b\|_{\text{Lip}_\beta(w)} w(x) w(2^k Q)^{\beta/n}.$$

(2) It is well known that (see [11, 20]), for $b \in \text{BMO}(w)$, $w \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^k Q}| \leq Ck \|b\|_{\text{BMO}(w)} w(x).$$

(3) Let $b \in \text{Lip}_\beta(w)$ or $b \in \text{BMO}(w)$ and $w \in A_1$. By [20], we know that spaces $\text{Lip}_\beta(w)$ or $\text{BMO}(w)$ coincide and the norms $\|b\|_{\text{Lip}_\beta(w)}$ or $\|b\|_{\text{BMO}(w)}$ are equivalent with respect to different values $1 \leq p < \infty$.

Definition 1 Let φ be a positive, increasing function on R^+ and let there exist a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for } t \geq 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n . Set, for $1 \leq p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x \in R^n, d > 0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p w(y) dy \right)^{1/p},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\varphi}(R^n, w) = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n, w) = L^{p,\delta}(R^n, w)$, which is the classical Morrey spaces (see [21, 22]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n, w) = L^p(R^n, w)$, which is the weighted Lebesgue spaces (see [1]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [9, 23–28]).

In this paper, we will study the singular integral operators as follows (see [5]).

Definition 2 Let $T : S \rightarrow S'$ be a linear operator such that T is bounded on $L^p(R^n)$ for $1 < p < \infty$ and weak (L^1, L^1) -bounded and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function f , where K satisfies, for fixed $\varepsilon > 0$,

$$|K(x, y)| \leq C|x - y|^{-n}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leq C|y - z|^\varepsilon |x - z|^{-n-\varepsilon}$$

if $2|y - z| \leq |x - z|$.

Moreover, let m be the positive integer and b be the function on R^n . Set

$$R_{m+1}(b; x, y) = b(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha b(y) (x - y)^\alpha.$$

The multilinear operator related to the operator T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{R_{m+1}(b; x, y)}{|x - y|^m} K(x, y) f(y) dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies the conditions of Definition 2 (see [1, 4]) and that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the multilinear operator T^b if $m = 0$. The multilinear operator T^b are the non-trivial generalizations of the commutator. It is well known that commutators and multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [5, 6, 19]). The main purpose of this paper is to prove the sharp maximal inequalities for the multilinear operator T^b . As the application, we obtain the weighted L^p -norm inequality and Morrey space boundedness for the multilinear operator T^b .

3 Theorems and lemmas

We shall prove the following theorems.

Theorem 1 *Let T be the singular integral operator as Definition 2, $w \in A_1$, $0 < \eta < 1$, $1 < r < \infty$ and $D^\alpha b \in \text{BMO}(w)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M_\eta^\#(T^b(f))(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

Theorem 2 *Let T be the singular integral operator as Definition 2, $w \in A_1$, $0 < \eta < 1$, $1 < r < \infty$, $0 < \beta < 1$ and $D^\alpha b \in \text{Lip}_\beta(w)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M_\eta^\#(T^b(f))(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).$$

Theorem 3 *Let T be the singular integral operator as Definition 2, $w \in A_1$, $1 < p < \infty$ and $D^\alpha b \in \text{BMO}(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^p(R^n, w)$ to $L^p(R^n, w^{1-p})$.*

Theorem 4 *Let T be the singular integral operator as Definition 2, $w \in A_1$, $1 < p < \infty$, $0 < D < 2^n$ and $D^\alpha b \in \text{BMO}(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^{p,\varphi}(R^n, w)$ to $L^{p,\varphi}(R^n, w^{1-p})$.*

Theorem 5 *Let T be the singular integral operator as Definition 2, $w \in A_1$, $0 < \beta < 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $D^\alpha b \in \text{Lip}_\beta(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^p(R^n, w)$ to $L^q(R^n, w^{1-q})$.*

Theorem 6 Let T be the singular integral operator as Definition 2, $w \in A_1$, $0 < \beta < 1$, $0 < D < 2^n$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $D^\alpha b \in \text{Lip}_\beta(w)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^{p,\varphi}(R^n, w)$ to $L^{q,\varphi}(R^n, w^{1-q})$.

To prove the theorems, we need the following lemmas.

Lemma 1 (See [1, p.485]) Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define, for $1/r = 1/p - 1/q$,

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda \left| \left\{ x \in R^n : f(x) > \lambda \right\} \right|^{1/q}, \quad N_{p,q}(f) = \sup_Q \|f \chi_Q\|_{L^p} / \|\chi_Q\|_{L^r},$$

where the sup is taken for all measurable sets Q with $0 < |Q| < \infty$. Then

$$\|f\|_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} \|f\|_{WL^q}.$$

Lemma 2 (See [1, 20]) Let $0 \leq \eta < n$, $1 \leq s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A_1$. Then

$$\|M_{\eta,s,w}(f)\|_{L^q(w)} \leq C \|f\|_{L^p(w)}.$$

Lemma 3 (See [1]) Let $0 < p, \eta < \infty$ and $w \in \bigcup_{1 \leq r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M_\eta(f)(x)^p w(x) dx \leq C \int_{R^n} M_\eta^\#(f)(x)^p w(x) dx.$$

Lemma 4 (See [1]) Let $0 < p < \infty$, $0 < \eta < \infty$, $0 < D < 2^n$ and $w \in A_1$. Then, for any smooth function f for which the left-hand side is finite,

$$\|M_\eta(f)\|_{L^{p,\varphi}(w)} \leq C \|M_\eta^\#(f)\|_{L^{p,\varphi}(w)}.$$

Lemma 5 (See [20]) Let $0 \leq \eta < n$, $0 < D < 2^n$, $1 \leq s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A_1$. Then

$$\|M_{\eta,s,w}(f)\|_{L^{q,\varphi}(w)} \leq C \|f\|_{L^{p,\varphi}(w)}.$$

Lemma 6 (See [16]) Let b be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and any $q > n$. Then

$$|R_m(b; x, y)| \leq C |x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

4 Proofs of theorems

Proof of Theorem 1 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q |T^b(f)(x) - C_0|^\eta dx \right)^{1/\eta} \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{s,w}(f)(\tilde{x}).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}(x) = b(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha b)_{\tilde{Q}} x^\alpha$, then $R_m(b; x, y) = R_m(\tilde{b}; x, y)$ and $D^\alpha \tilde{b} = D^\alpha b - (D^\alpha b)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$\begin{aligned} T^b(f)(x) &= \int_{\mathbb{R}^n} \frac{R_m(\tilde{b}; x, y)}{|x-y|^m} K(x, y) f_1(y) dy - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{(x-y)^\alpha D^\alpha \tilde{b}(y)}{|x-y|^m} K(x, y) f_1(y) dy \\ &\quad + \int_{\mathbb{R}^n} \frac{R_{m+1}(\tilde{b}; x, y)}{|x-y|^{m+1}} K(x, y) f_2(y) dy \\ &= T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right) - T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right) + T^{\tilde{b}}(f_2)(x), \end{aligned}$$

then

$$\begin{aligned} &\left(\frac{1}{|Q|} \int_Q |T^b(f)(x) - T^b(f_2)(x_0)|^\eta dx\right)^{1/\eta} \\ &\leq C \left(\frac{1}{|Q|} \int_Q \left|T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\ &\quad + C \left(\frac{1}{|Q|} \int_Q \left|T\left(\sum_{|\alpha|=m} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\ &\quad + C \left(\frac{1}{|Q|} \int_Q |T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)|^\eta dx\right)^{1/\eta} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

For I_1 , noting that $w \in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|} \int_Q w(x)^{p_0} dx\right)^{1/p_0} \leq \frac{C}{|Q|} \int_Q w(x) dx$$

for all cube Q and some $1 < p_0 < \infty$ (see [1]). We take $q = rp_0/(r + p_0 - 1)$ in Lemma 6 and have $1 < q < r$ and $p_0 = q(r-1)/(r-q)$, then by the Lemma 6 and Hölder's inequality, we obtain

$$\begin{aligned} &|R_m(\tilde{b}; x, y)| \\ &\leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha \tilde{b}(z)|^q dz\right)^{1/q} \\ &\leq C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \left(\int_{\tilde{Q}(x, y)} |D^\alpha \tilde{b}(z)|^q w(z)^{q(1-r)/r} w(z)^{q(r-1)/r} dz\right)^{1/q} \\ &\leq C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \left(\int_{\tilde{Q}(x, y)} |D^\alpha \tilde{b}(z)|^r w(z)^{1-r} dz\right)^{1/r} \\ &\quad \times \left(\int_{\tilde{Q}(x, y)} w(z)^{q(r-1)/(r-q)} dz\right)^{(r-q)/(rq)} \end{aligned}$$

$$\begin{aligned}
&\leq C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{Q})^{1/r} |\tilde{Q}|^{(r-q)/rq} \\
&\quad \times \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z)^{p_0} dz \right)^{(r-q)/rq} \\
&\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{1/r} |\tilde{Q}|^{1/q-1/r} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z) dz \right)^{(r-1)/r} \\
&\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{1/r} |\tilde{Q}|^{1/q-1/r} w(\tilde{Q})^{1-1/r} |\tilde{Q}|^{1/r-1} \\
&\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} \\
&\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}),
\end{aligned}$$

thus, by the L^s -boundedness of T (see Lemma 2) for $1 < s < r$ and $w \in A_1 \subseteq A_{r/s}$, we obtain

$$\begin{aligned}
I_1 &\leq \frac{C}{|Q|} \int_Q \left| T \left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \right) \right| dx \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \left(\frac{1}{|Q|} \int_{R^n} |T(f_1)(x)|^s dx \right)^{1/s} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{R^n} |f_1(x)|^s dx \right)^{1/s} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} |f(x)|^s w(x)^{s/r} w(x)^{-s/r} dx \right)^{1/s} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \left(\int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) |Q|^{-1/s} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{1/r} |\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

For I_2 , by the weak (L^1, L^1) boundedness of T (see Lemma 2) and Kolmogoro's inequality (see Lemma 1), we obtain

$$\begin{aligned}
I_2 &\leq C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_Q |T(D^\alpha \tilde{b} f_1)(x)|^\eta dx \right)^{1/\eta} \\
&\leq C \sum_{|\alpha|=m} \frac{|Q|^{1/\eta-1} \|T(D^\alpha \tilde{b} f_1) \chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}} \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|T(D^\alpha \tilde{b} f_1)\|_{WL^1}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{R^n} |D^\alpha \tilde{b}(x) f_1(x)| dx \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}| w(x)^{-1/r} |f(x)| w(x)^{1/r} dx \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\tilde{Q}} |(D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}})|^{r'} w(x)^{1-r'} dx \right)^{1/r'} \left(\int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{Q})^{1/r'} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} M_{r,w}(f)(\tilde{x}) \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

For I_3 , note that $|x - y| \approx |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus Q$, we write

$$\begin{aligned}
&T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0) \\
&= \int_{R^n} (R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)) \frac{K(x, y)}{|x - y|^m} |f_2(y)| dy \\
&\quad + \int_{R^n} \left(\frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right) R_m(\tilde{b}; x_0, y) f_2(y) dy \\
&\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left(\frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right) (x - y)^\alpha D^\alpha \tilde{b}(y) f_2(y) dy \\
&\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left(\frac{(x - y)^\alpha}{|x - y|^m} - \frac{(x_0 - y)^\alpha}{|x_0 - y|^m} \right) K(x_0, y) D^\alpha \tilde{b}(y) f_2(y) dy \\
&= I_3^{(1)}(x) + I_3^{(2)}(x) + I_3^{(3)}(x) + I_3^{(4)}(x).
\end{aligned}$$

For $I_3^{(1)}(x)$, by the formula (see [16]):

$$R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y) = \sum_{|\gamma| < m} \frac{1}{\gamma!} R_{m-|\gamma|}(D^\gamma \tilde{b}; x, x_0) (x - y)^\gamma$$

and Lemma 6, we have, similar to the proof of I_1 ,

$$|R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \leq C \sum_{|\gamma| < m} \sum_{|\alpha|=m} |x - x_0|^{m-|\gamma|} |x - y|^{|\gamma|} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}),$$

thus, by $w \in A_1 \subseteq A_r$,

$$\begin{aligned}
&|I_3^{(1)}(x)| \\
&\leq \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \frac{|K(x, y)|}{|x - y|^m} |f(y)| dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n+1}} |f(y)| dy
\end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \int_{2^k \tilde{Q}} |f(y)| w(y)^{1/r} w(y)^{-1/r} dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \\
&\quad \times \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \left(\int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n+1}} w(2^k \tilde{Q})^{1/r} \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
&\quad \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} |2^k \tilde{Q}| w(2^k \tilde{Q})^{-1/r} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}) \sum_{k=1}^{\infty} 2^{-k} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

For $I_3^{(2)}(x)$, by the conditions of K , we get

$$\begin{aligned}
|I_3^{(2)}(x)| &\leq C \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} \left| \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right| |R_m(\tilde{b}; x_0, y)| |f(y)| dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} \left(\frac{|x - x_0|}{|x_0 - y|^{n+1}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon}} \right) |f(y)| dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) \int_{2^k \tilde{Q}} |f(y)| dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}) \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\varepsilon}) \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&|I_3^{(3)}(x)| + |I_3^{(4)}(x)| \\
&\leq C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q}} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) |f(y)| |D^\alpha \tilde{b}(y)| dy \\
&\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) \\
&\quad \times \int_{2^k \tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^k \tilde{Q}}| w(y)^{-1/r} |f(y)| w(y)^{1/r} dy \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \int_{2^k \tilde{Q}} |(D^\alpha b)_{2^k \tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| |f(y)| w(y)^{1/r} w(y)^{-1/r} dy \\
& \leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) \left(\int_{2^k \tilde{Q}} |(D^\alpha b)(y) - (D^\alpha b)_{2^k \tilde{Q}}|^{r'} w(y)^{1-r'} dy \right)^{1/r'} \\
& \quad \times \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
& \quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
& \quad \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} |2^k \tilde{Q}| w(2^k \tilde{Q})^{-1/r} \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\varepsilon}) \frac{w(2^k \tilde{Q})}{|2^k \tilde{Q}|} \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
& \quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k (2^{-k} + 2^{-k\varepsilon}) \left(\frac{1}{w(2^k \tilde{Q})} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).
\end{aligned}$$

Thus

$$I_3 \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} w(\tilde{x}) M_{r,w}(f)(\tilde{x}).$$

These complete the proof of Theorem 1. $\square$

Proof of Theorem 2 It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0 , the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q |T^b(f)(x) - C_0|^\eta dx \right)^{1/\eta} \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$\begin{aligned}
& \left(\frac{1}{|Q|} \int_Q |T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0)|^\eta dx \right)^{1/\eta} \\
& \leq C \left(\frac{1}{|Q|} \int_Q \left| T \left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \right) \right|^\eta dx \right)^{1/\eta} \\
& \quad + C \left(\frac{1}{|Q|} \int_Q \left| T \left(\sum_{|\alpha|=m} \frac{(x - \cdot)^\alpha D^\alpha \tilde{b}}{|x - \cdot|^m} f_1 \right) \right|^\eta dx \right)^{1/\eta} \\
& \quad + C \left(\frac{1}{|Q|} \int_Q |T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)|^\eta dx \right)^{1/\eta} \\
& = J_1 + J_2 + J_3.
\end{aligned}$$

For J_1 and J_2 , by using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned}
 & |R_m(\tilde{b}; x, y)| \\
 & \leq C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \left(\int_{\tilde{Q}(x,y)} |D^\alpha \tilde{b}(z)|^q w(z)^{q(1-r)/r} w(z)^{q(r-1)/r} dz \right)^{1/q} \\
 & \leq C|x-y|^m \\
 & \quad \times \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \left(\int_{\tilde{Q}(x,y)} |D^\alpha \tilde{b}(z)|^r w(z)^{1-r} dz \right)^{1/r} \left(\int_{\tilde{Q}(x,y)} w(z)^{q(r-1)/(r-q)} dz \right)^{(r-q)/rq} \\
 & \leq C|x-y|^m \\
 & \quad \times \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{Q})^{\beta/n+1/r} |\tilde{Q}|^{(r-q)/rq} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z)^{p_0} dz \right)^{(r-q)/rq} \\
 & \leq C|x-y|^m \\
 & \quad \times \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{\beta/n+1/r} |\tilde{Q}|^{1/q-1/r} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} w(z) dz \right)^{(r-1)/r} \\
 & \leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} |\tilde{Q}|^{-1/q} w(\tilde{Q})^{\beta/n+1/r} |\tilde{Q}|^{1/q-1/r} w(\tilde{Q})^{1-1/r} |\tilde{Q}|^{1/r-1} \\
 & \leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \frac{w(\tilde{Q})^{\beta/n+1}}{|\tilde{Q}|} \\
 & \leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x}),
 \end{aligned}$$

thus

$$\begin{aligned}
 J_1 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x}) |Q|^{-1/s} \left(\int_{R^n} |f_1(x)|^s dx \right)^{1/s} \\
 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{Q})^{\beta/n} w(\tilde{x}) |Q|^{-1/s} \left(\int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
 & \quad \times \left(\int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \\
 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) |\tilde{Q}|^{-1/s} w(\tilde{Q})^{1/r} \left(\frac{1}{w(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
 & \quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x)^{-s/(r-s)} dx \right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} w(x) dx \right)^{1/r} |\tilde{Q}|^{1/s} w(\tilde{Q})^{-1/r} \\
 & \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}), \\
 J_2 & \leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}| w(x)^{-1/r} |f(x)| w(x)^{1/r} dx \\
 & \leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\tilde{Q}} |(D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}})|^{r'} w(x)^{1-r'} dx \right)^{1/r'} \left(\int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r}
 \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|\alpha|=m} \frac{1}{|\tilde{Q}|} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{Q})^{\beta/n+1/r'} w(\tilde{Q})^{1/r-\beta/n} \left(\frac{1}{w(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} |f(x)|^r w(x) dx \right)^{1/r} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \frac{w(\tilde{Q})}{|\tilde{Q}|} M_{\beta,r,w}(f)(\tilde{x}) \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).
\end{aligned}$$

For J_3 , we have

$$\begin{aligned}
&|R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \\
&\leq C \sum_{|\gamma| < m} \sum_{|\alpha|=m} |x - x_0|^{m-|\gamma|} |x - y|^{|\gamma|} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) w(2^k \tilde{Q})^{\beta/n},
\end{aligned}$$

thus

$$\begin{aligned}
&|T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)| \\
&\leq \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \frac{|K(x, y)|}{|x - y|^m} |f(y)| dy \\
&\quad + \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \left| \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right| |R_m(\tilde{b}; x_0, y)| |f(y)| dy \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \left| \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right| |(x - y)^\alpha| |D^\alpha \tilde{b}(y)| |f(y)| dy \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \left| \frac{(x - y)^\alpha}{|x - y|^m} - \frac{(x_0 - y)^\alpha}{|x_0 - y|^m} \right| |K(x_0, y)| |D^\alpha \tilde{b}(y)| |f(y)| dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) \sum_{k=0}^{\infty} w(2^{k+1}\tilde{Q})^{\beta/n} \\
&\quad \times \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) |f(y)| dy \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) \\
&\quad \times \int_{2^k\tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^k\tilde{Q}}| w(y)^{-1/r} |f(y)| w(y)^{1/r} dy \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) \\
&\quad \times \int_{2^k\tilde{Q}} |(D^\alpha b)_{2^k\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| |f(y)| w(y)^{1/r} w(y)^{-1/r} dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) \\
&\quad \times \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) w(2^k \tilde{Q})^{\beta/n} \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} |2^k \tilde{Q}| w(2^k \tilde{Q})^{-1/r} \\
& + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) \left(\int_{2^k \tilde{Q}} |(D^\alpha b(y) - (D^\alpha b)_{2^k \tilde{Q}})|^{r'} w(y)^{1-r'} dy \right)^{1/r'} \\
& \times \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dy \right)^{1/r} \\
& + C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k w(2^k \tilde{Q})^{\beta/n} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\varepsilon}{(2^k d)^{n+\varepsilon}} \right) \\
& \times \left(\int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y)^{-1/(r-1)} dy \right)^{(r-1)/r} \\
& \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} w(y) dy \right)^{1/r} |2^k \tilde{Q}| w(2^k \tilde{Q})^{-1/r} \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k (2^{-k} + 2^{-k\varepsilon}) \left(\frac{1}{w(2^k \tilde{Q})^{1-r\beta/n}} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
& + C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \\
& \times \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\varepsilon}) \frac{w(2^k \tilde{Q})}{|2^k \tilde{Q}|} \left(\frac{1}{w(2^k \tilde{Q})^{1-r\beta/n}} \int_{2^k \tilde{Q}} |f(y)|^r w(y) dx \right)^{1/r} \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} w(\tilde{x}) M_{\beta,r,w}(f)(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 2. $\square$

Proof of Theorem 3 By putting $1 < r < p$ in Theorem 1 and noticing that $w^{1-p} \in A_1$, by Lemma 3 and 4, we obtain

$$\begin{aligned}
\|T^b(f)\|_{L^p(w^{1-p})} & \leq \|M_\eta(T^b(f))\|_{L^p(w^{1-p})} \leq C \|M_\eta^\#(T^b(f))\|_{L^p(w^{1-p})} \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \|w M_{r,w}(f)\|_{L^p(w^{1-p})} \\
& = C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \|M_{r,w}(f)\|_{L^p(w)} \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 3. $\square$

Proof of Theorem 4 By choosing $1 < r < p$ in Theorem 1 and noticing that $w^{1-p} \in A_1$, by Lemmas 5 and 6, we get

$$\begin{aligned}
\|T^b(f)\|_{L^{p,\varphi}(w^{1-p})} & \leq \|M_\eta(T^b(f))\|_{L^{p,\varphi}(w^{1-p})} \leq C \|M_\eta^\#(T^b(f))\|_{L^{p,\varphi}(w^{1-p})} \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \|w M_{r,w}(f)\|_{L^{p,\varphi}(w^{1-p})}
\end{aligned}$$

$$\begin{aligned} &= C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \|M_{r,w}(f)\|_{L^{p,\varphi}(w)} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{BMO}(w)} \|f\|_{L^{p,\varphi}(w)}. \end{aligned}$$

This completes the proof of Theorem 4. $\square$

Proof of Theorem 5 By setting $1 < r < p$ in Theorem 2 and noting that $w^{1-q} \in A_1$, by Lemmas 3 and 4, we have

$$\begin{aligned} \|T^b(f)\|_{L^q(w^{1-q})} &\leq \|M_\eta(T^b(f))\|_{L^q(w^{1-q})} \leq C \|M_\eta^\#(T^b(f))\|_{L^q(w^{1-q})} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \|wM_{\beta,r,w}(f)\|_{L^q(w^{1-q})} \\ &= C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^q(w)} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \|f\|_{L^p(w)}. \end{aligned}$$

This completes the proof of Theorem 5. $\square$

Proof of Theorem 6 By choosing $1 < r < p$ in Theorem 2, and noticing that $w^{1-q} \in A_1$, by Lemmas 5 and 6, we get

$$\begin{aligned} \|T^b(f)\|_{L^{q,\varphi}(w^{1-q})} &\leq \|M_\eta(T^b(f))\|_{L^{q,\varphi}(w^{1-q})} \leq C \|M_\eta^\#(T^b(f))\|_{L^{q,\varphi}(w^{1-q})} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \|wM_{\beta,r,w}(f)\|_{L^{q,\varphi}(w^{1-q})} \\ &= C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \|M_{\beta,r,w}(f)\|_{L^{q,\varphi}(w)} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\text{Lip}_\beta(w)} \|f\|_{L^{p,\varphi}(w)}. \end{aligned}$$

This completes the proof of Theorem 6. $\square$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors completed the paper together. They also read and approved the final manuscript.

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