He and Huang *Journal of Inequalities and Applications* 2014, 2014:114
http://www.journalofinequalitiesandapplications.com/content/2014/1/114

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

# Inequalities for $M$-tensors

Jun He* and Ting-Zhu Huang

*Correspondence:
hejunfan1@163.com
School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, P.R. China

**Abstract**
In this paper, we establish some important properties of *M*-tensors. We derive upper and lower bounds for the minimum eigenvalue of *M*-tensors, bounds for eigenvalues of *M*-tensors except the minimum eigenvalue are also presented; finally, we give the Ky Fan theorem for *M*-tensors.
**MSC:** 15A18; 15A69; 65F15; 65F10

**Keywords:** *M*-tensors; nonnegative tensor; spectral radius; eigenvalues

## 1 Introduction

Eigenvalue problems of higher-order tensors have become an important topic of study in a new applied mathematics branch, numerical multilinear algebra, and they have a wide range of practical applications [1–7].

If there are a complex number $\lambda$ and a nonzero complex vector $x$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then $\lambda$ is called the eigenvalue of $\mathcal{A}$ and $x$ the eigenvector of $\mathcal{A}$ associated with $\lambda$, where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose $i$th component is

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_n}\right)_{1\le i\le n},$$
$$x^{[m-1]} := \left(x_i^{m-1}\right)_{1\le i\le n}.$$

This definition was introduced by Qi and Lim [8, 9] where they supposed that $\mathcal{A}$ is an order $m$ dimension $n$ symmetric tensor and $m$ is even. First, we introduce some results of nonnegative tensors [10–12], which are generalized from nonnegative matrices.

**Definition 1.1** The tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset \{1, 2, \ldots, n\}$ such that $a_{i_1,i_2,\ldots,i_m} = 0$, $\forall i_1 \in \mathbb{J}$, $\forall i_2, \ldots, i_m \notin \mathbb{J}$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ to be irreducible.

Let $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$, where $|\lambda|$ denotes the modulus of $\lambda$. We call $\rho(\mathcal{A})$ the spectral radius of tensor $\mathcal{A}$.

**Theorem 1.2** *If $\mathcal{A}$ is irreducible and nonnegative, then there exists a number $\rho(\mathcal{A}) > 0$ and a vector $x_0 > 0$ such that $\mathcal{A}x_0^{m-1} = \rho(\mathcal{A})x_0^{[m-1]}$. Moreover, if $\lambda$ is an eigenvalue with a nonnegative eigenvector, then $\lambda = \rho(\mathcal{A})$. If $\lambda$ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \rho(\mathcal{A})$.*

The authors in [13, 14] extended the notion of $M$-matrices to higher-order tensors and introduced the definition of an $M$-tensor.

**Definition 1.3** Let $\mathcal{A}$ be an $m$-order and $n$-dimensional tensor. $\mathcal{A}$ is called an $M$-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a real number $c > \rho(\mathcal{B})$, where $\mathcal{B}$ is the spectral radius of $\mathcal{B}$, such that

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

**Theorem 1.4** *Let $\mathcal{A}$ be an $M$-tensor and denote by $\tau(\mathcal{A})$ the minimal value of the real part of all eigenvalues of $\mathcal{A}$. Then $\tau(\mathcal{A}) > 0$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. Moreover, there exist a nonnegative tensor $\mathcal{B}$ and a real number $c > \rho(\mathcal{B})$ such that $\mathcal{A} = c\mathcal{I} - \mathcal{B}$. If $\mathcal{A}$ is irreducible, then $\tau(\mathcal{A})$ is the unique eigenvalue with a positive eigenvector.*

In this paper, let $N = \{1, 2, \ldots, n\}$, we define the $i$th row sum of $\mathcal{A}$ as $R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}$, and denote the largest and the smallest row sums of $\mathcal{A}$ by

$$R_{\max}(\mathcal{A}) = \max_{i=1,\ldots,n} R_i(\mathcal{A}), \qquad R_{\min}(\mathcal{A}) = \min_{i=1,\ldots,n} R_i(\mathcal{A}).$$

Furthermore, a real tensor of order $m$ dimension $n$ is called the unit tensor, if its entries are $\delta_{i_1\cdots i_m}$ for $i_1, \ldots, i_m \in N$, where

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

And we define $\sigma(\mathcal{A})$ as the set of all the eigenvalues of $\mathcal{A}$ and

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\cdots i_m}=0} |a_{ii_2\cdots i_m}|, \qquad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\cdots i_m}=0, \\ \delta_{ji_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = r_i(\mathcal{A}) - |a_{ij\cdots j}|.$$

In this paper, we continue this research on the eigenvalue problems for tensors. In Section 2, some bounds for the minimum eigenvalue of $M$-tensors are obtained, and proved to be tighter than those in Theorem 1.1 in [15]. In Section 3, some bounds for eigenvalues of $M$-tensors except the minimum eigenvalue are given. Moreover, the Ky Fan theorem for $M$-tensors is presented in Section 4.

## 2 Bounds for the minimum eigenvalue of *M*-tensors

**Theorem 2.1** *Let $\mathcal{A}$ be an irreducible $M$-tensor. Then*

$$\tau(\mathcal{A}) \leq \min\{a_{i\cdots i}\}, \tag{1}$$

$$R_{\min}(\mathcal{A}) \leq \tau(\mathcal{A}) \leq R_{\max}(\mathcal{A}). \tag{2}$$

*Proof* Let $x > 0$ be an eigenvector of $\mathcal{A}$ corresponding to $\tau(\mathcal{A})$, *i.e.*, $\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}$. For each $i \in N$, we can get

$$\big(a_{i\cdots i} - \tau(\mathcal{A})\big)x_i^{m-1} = -\sum_{\delta_{ii_2\cdots i_m}=0} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} \geq 0,$$

then

$$\tau(\mathcal{A}) \leq \min\{a_{i\cdots i}\}.$$

Assume that $x_s$ is the smallest component of $x$,

$$\big(a_{s\cdots s} - \tau(\mathcal{A})\big)x_s^{m-1} = -\sum_{\delta_{si_2\cdots i_m}=0} a_{si_2\cdots i_m} x_{i_2}\cdots x_{i_m} \geq 0.$$

That is,

$$\tau(\mathcal{A}) \leq \sum_{\delta_{si_2\cdots i_m}=0} a_{si_2\cdots i_m} + a_{s\cdots s},$$

so

$$\tau(\mathcal{A}) \leq R_{\max}(\mathcal{A}).$$

Similarly, if we assume that $x_t = \{\max x_i, i \in N\}$, then we can get

$$\tau(\mathcal{A}) \geq \sum_{\delta_{ti_2\cdots i_m}=0} a_{ti_2\cdots i_m} + a_{t\cdots t} \geq R_{\min}(\mathcal{A}).$$

Thus, we complete the proof. □

**Theorem 2.2** *Let $\mathcal{A}$ be an irreducible M-tensor. Then*

$$\begin{aligned}&\min_{i,j\in N, j\neq i} \frac{1}{2}\big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\big\}\\ &\quad\leq \tau(\mathcal{A}) \leq \max_{i,j\in N, j\neq i} \frac{1}{2}\big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\big\},\end{aligned} \tag{3}$$

*where*

$$\Delta_{i,j}(\mathcal{A}) = \big(a_{i\cdots i} - a_{j\cdots j} + r_i^j(\mathcal{A})\big)^2 - 4a_{ij\cdots j}r_j(\mathcal{A}).$$

*Proof* Because $\tau(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$, from Theorem 2.1 in [15], there are $i, j \in N$, $j \neq i$, such that

$$\big(\big|\tau(\mathcal{A}) - a_{i\cdots i}\big| - r_i^j(\mathcal{A})\big)\big|\tau(\mathcal{A}) - a_{j\cdots j}\big| \leq |a_{ij\cdots j}|r_j(\mathcal{A}).$$

From Theorem 2.1, we can get

$$\big(a_{i\cdots i} - \tau(\mathcal{A}) - r_i^j(\mathcal{A})\big)\big(a_{j\cdots j} - \tau(\mathcal{A})\big) \leq -a_{ij\cdots j}r_j(\mathcal{A}),$$

equivalently,

$$\tau(\mathcal{A})^2 - \big(a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A})\big)\tau(\mathcal{A}) + a_{j\cdots j}\big(a_{i\cdots i} - r_i^j(\mathcal{A})\big) + a_{ij\cdots j}r_j(\mathcal{A}) \le 0.$$

Then, solving for $\tau(\mathcal{A})$,

$$\tau(\mathcal{A}) \ge \frac{1}{2}\big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\big\} \ge \min_{i,j\in N, j\neq i} \frac{1}{2}\big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\big\}.$$

Let $x > 0$ be an eigenvector of $\mathcal{A}$ corresponding to $\tau(\mathcal{A})$, *i.e.*, $\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}$, $x_s$ is the smallest component of $x$. For each $s, t \in N$, $s \neq t$, we can get

$$\big(a_{t\cdots t} - \tau(\mathcal{A})\big)x_t^{m-1} = -\sum_{\delta_{ti_2\cdots i_m}=0} a_{ti_2\cdots i_m}x_{i_2}\cdots x_{i_m} \ge r_t(\mathcal{A})x_s^{m-1}, \tag{4}$$

$$\big(a_{s\cdots s} - \tau(\mathcal{A})\big)x_s^{m-1} = -\sum_{\substack{\delta_{ti_2\cdots i_m}=0,\\ \delta_{si_2\cdots i_m}=0}} a_{ti_2\cdots i_m}x_{i_2}\cdots x_{i_m} - a_{st\cdots t}x_t^{m-1} \ge r_t^s(\mathcal{A})x_s^{m-1} - a_{st\cdots t}x_t^{m-1},$$

$$\big(a_{s\cdots s} - \tau(\mathcal{A}) - r_t^s(\mathcal{A})\big)x_s^{m-1} \ge -a_{st\cdots t}x_t^{m-1}. \tag{5}$$

Multiplying equations (4) and (5), we get

$$\big(a_{t\cdots t} - \tau(\mathcal{A})\big)\big(a_{s\cdots s} - \tau(\mathcal{A}) - r_t^s(\mathcal{A})\big) \ge -a_{st\cdots t}r_t(\mathcal{A}).$$

Then, solving for $\tau(\mathcal{A})$,

$$\tau(\mathcal{A}) \le \frac{1}{2}\big\{a_{t\cdots t} + a_{s\cdots s} - r_t^s(\mathcal{A}) - \Delta_{t,s}^{\frac{1}{2}}(\mathcal{A})\big\} \le \max_{i,j\in N, j\neq i} \frac{1}{2}\big\{a_{i\cdots i} + a_{j\cdots j} - r_i^j(\mathcal{A}) - \Delta_{i,j}^{\frac{1}{2}}(\mathcal{A})\big\}.$$

Thus, we complete the proof. □

We now show that the bounds in Theorem 2.2 are tight and sharper than those in Theorem 1.1 in [15] by the following example. Consider the $M$-tensor $\mathcal{A} = (a_{ijkl})$ of order 4 dimension 2 with entries defined as follows:

$$a_{1111} = 3, \qquad a_{1222} = -1,$$
$$a_{2111} = -2, \qquad a_{2222} = 2,$$

other $a_{ijkl} = 0$. By Theorem 1.1 in [15], we have

$$-2 \le \tau(\mathcal{A}) \le 4.$$

By Theorem 2.1, we have

$$0 \le \tau(\mathcal{A}) \le 2.$$

By Theorem 2.2, we have

$$\frac{1}{2}(5 - \sqrt{17}) \le \tau(\mathcal{A}) \le \frac{1}{2}(5 - \sqrt{5}).$$

In fact, $\tau(\mathcal{A}) = 1$. Hence, the bounds in Theorem 2.2 are tight and sharper than those in Theorem 1.1 in [15].

## 3 Bounds for eigenvalues of *M*-tensors except the minimum eigenvalue

In this section, we introduce the stochastic $M$-tensor, which is a generalization of the nonnegative stochastic tensor.

**Definition 3.1** An $M$-tensor $\mathcal{A}$ of order $m$ dimension $n$ is called stochastic provided

$$R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} \equiv 1, \quad i = 1,\ldots,n.$$

Obviously, when $\mathcal{A}$ is a stochastic $M$-tensor, 1 is the minimum eigenvalue of $\mathcal{A}$ and $e$ is an eigenvector corresponding to 1, where $e$ is an all-ones vector.

**Theorem 3.2** *Let $\mathcal{A}$ be an order m dimension n irreducible M-tensor. Then there exists a diagonal matrix D with positive main diagonal entries such that*

$$\tau(\mathcal{A}) \cdot \mathcal{B} = \mathcal{A} \cdot D^{(1-m)} \cdot \overbrace{D \cdot \ldots \cdot D}^{m-1},$$

*where B is a stochastic irreducible M-tensor. Furthermore, B is unique, and the diagonal entries of D are exactly the components of the unique positive eigenvector corresponding to $\tau(\mathcal{A})$.*

*Proof* Let $x$ be the unique positive eigenvector corresponding to $\tau(\mathcal{A})$, *i.e.*,

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}.$$

Let $D$ be the diagonal matrix such that its diagonal entries are components of $x$, let us check the tensor $\mathcal{C} = \mathcal{A} \cdot D^{(1-m)} \cdot D \cdot \ldots \cdot D$. It is clear that for $i = 1,2,\ldots,n$,

$$\sum_{i_2,\ldots,i_m=1}^{n} \mathcal{C}_{ii_2\cdots i_m} = \left(\mathcal{C}e^{m-1}\right)_i = \left(\mathcal{A} \cdot D^{(1-m)} \cdot \overbrace{D \cdot \ldots \cdot D}^{m-1} e^{m-1}\right)_i = \tau(\mathcal{A}).$$

Hence $\mathcal{B} = \mathcal{C}/\tau(\mathcal{A})$ is the desired stochastic $M$-tensor. Since the positive eigenvector is unique, then $B$ is unique, and the diagonal entries of $D$ are exactly the components of the unique positive eigenvector corresponding to $\tau(\mathcal{A})$. □

**Theorem 3.3** *Let $\mathcal{A}$ be an order m dimension n stochastic irreducible nonnegative tensor,* $\omega = \min a_{i\cdots i}$, $\lambda \in \sigma(\mathcal{A})$. *Then*

$$|\lambda - \omega| \le 1 - \omega.$$

*Proof* Let $\lambda$ be an eigenvalue of the stochastic irreducible nonnegative tensor $\mathcal{A}$, $x$ is the eigenvector corresponding to $\lambda$, *i.e.*,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}.$$

Assume that $0 < |x_s| = \max_i |x_i|$, then we can get

$$(\lambda - a_{s\cdots s})x_s^{m-1} = \sum_{\delta_{si_2\cdots i_m}=0} a_{si_2\cdots i_m} x_{i_2}\cdots x_{i_m}.$$

Then

$$|\lambda - a_{s\cdots s}| \leq \sum_{\delta_{si_2\cdots i_m}=0} a_{si_2\cdots i_m} = r_s(\mathcal{A}) = 1 - a_{s\cdots s},$$

and therefore,

$$\begin{aligned} |\lambda - \omega| &\leq |\lambda - a_{s\cdots s} + a_{s\cdots s} - \omega| \\ &\leq |\lambda - a_{s\cdots s}| + |a_{s\cdots s} - \omega| \\ &\leq (1 - a_{s\cdots s}) + (a_{s\cdots s} - \omega) \\ &= 1 - \omega. \end{aligned} \tag{6}$$

Thus, we complete the proof. □

**Theorem 3.4** *Let $\mathcal{A}$ be an order m dimension n irreducible M-tensor*, $\Omega = \max a_{i\cdots i}$, $\lambda \in \sigma(\mathcal{A})$. *Then*

$$|\Omega - \lambda| \leq \Omega - \tau(\mathcal{A}).$$

*Proof* From Theorem 3.2, we may evidently take $\tau(\mathcal{A}) = 1$, and after performing a similarity transformation with a positive diagonal matrix, we may assume that $\mathcal{A}$ is stochastic. Then, for $\theta \in (0,1)$, the matrix $\mathcal{A}(\theta) = (1+\theta)\mathcal{I} - \theta\mathcal{A}$ is irreducible nonnegative stochastic, by Theorem 3.3, if $\lambda(\theta) \in \sigma(\mathcal{A}(\theta))$, $\omega(\theta) = \min a_{i\cdots i}(\theta)$, we can get

$$\left|\lambda(\theta) - \omega(\theta)\right| \leq 1 - \omega(\theta).$$

That is,

$$\left|1 + \theta - \theta\lambda - (1 + \theta - \theta \max a_{i\cdots i})\right| \leq 1 - (1 + \theta - \theta \max a_{i\cdots i}).$$

Then

$$|\Omega - \lambda| \leq \Omega - 1.$$

Transforming back to $\mathcal{A}$, we get

$$|\Omega - \lambda| \leq \Omega - \tau(\mathcal{A}).$$

Thus, we complete the proof. □

## 4 Ky Fan theorem for *M*-tensors

In this section we give the Ky Fan theorem for $M$-tensors. Denote by $\mathbb{Z}$ the set of $m$-order and $n$-dimensional real tensors whose off-diagonal entries are nonpositive.

**Theorem 4.1** *Let $\mathcal{A}, \mathcal{B} \in \mathbb{Z}$, assume that $\mathcal{A}$ is an $M$-tensor and $\mathcal{B} \geq \mathcal{A}$. Then $\mathcal{B}$ is an $M$-tensor, and*

$$\tau(\mathcal{A}) \leq \tau(\mathcal{B}).$$

*Proof* If $x > 0$, from assume that $\mathcal{A}$ is an $M$-tensor and condition (D4) in [14], we know

$$\mathcal{A}x^{m-1} > 0.$$

Because $\mathcal{B} \geq \mathcal{A}$, we can get

$$\mathcal{B}x^{m-1} \geq \mathcal{A}x^{m-1} > 0,$$

then $\mathcal{B}$ is an $M$-tensor.

Let $a = \max_{1 \leq i \leq n} \mathcal{B}_{i \cdots i}$, from Theorem 3.1 and Corollary 3.2 in [13], assume that

$$\mathcal{B} = a\mathcal{I} - \mathcal{C}_{\mathcal{B}}, \qquad \mathcal{A} = a\mathcal{I} - \mathcal{C}_{\mathcal{A}},$$

where $\mathcal{C}_{\mathcal{A}}$, $\mathcal{C}_{\mathcal{B}}$ are nonnegative tensors.

Because $\mathcal{A}, \mathcal{B} \in \mathbb{Z}$ and $\mathcal{B} \geq \mathcal{A}$, then we can get

$$\mathcal{C}_{\mathcal{A}} \geq \mathcal{C}_{\mathcal{B}}.$$

From Lemma 3.5 in [12], we can get

$$\rho(\mathcal{C}_{\mathcal{A}}) \geq \rho(\mathcal{C}_{\mathcal{B}}).$$

Therefore,

$$\tau(\mathcal{A}) \leq \tau(\mathcal{B}).$$

Thus, we complete the proof. □

**Theorem 4.2** *Let $\mathcal{A}$, $\mathcal{B}$ be of order $m$ dimension $n$, suppose that $\mathcal{B}$ is an $M$-tensor and $|b_{i_1 \cdots i_m}| \geq |a_{i_1 \cdots i_m}|$ for all $i_1 \neq \cdots \neq i_m$. Then, for any eigenvalue $\lambda$ of $\mathcal{A}$, there exists $i \in 1, \ldots, n$ such that $|\lambda - a_{i \cdots i}| \leq b_{i \cdots i} - \tau(\mathcal{B})$.*

*Proof* We first suppose that $\mathcal{B}$ is an $M$-tensor, $\tau(\mathcal{B})$ is an eigenvalue of $\mathcal{B}$ with a positive corresponding eigenvector $v$. Denote

$$W = \operatorname{diag}(v_1, \ldots, v_n),$$

where $v_i$ is the $i$th component of $v$. Let

$$\mathcal{C} = \mathcal{A} \cdot W^{1-m} \overbrace{W \cdot \ldots \cdot W}^{[m-1]}$$

and let $\lambda$ be an eigenvalue of $\mathcal{A}$ with $x$, a corresponding eigenvector, *i.e.*, $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$. Then, as in the proof of Theorem 4.1 in [12], we have

$$\mathcal{C}\left(W^{-1}x\right)^{m-1} = \lambda\left(W^{-1}x\right)^{m-1}.$$

By the definition of $\mathcal{C}$, we have $c_{i\cdots i} = a_{i\cdots i}$, $i = 1, \ldots, n$. Applying the first conclusion of Theorem 6 of [8], we can get

$$\begin{aligned} |\lambda - c_{i\cdots i}| &\leq \sum_{\delta_{ii_2\cdots i_m}=0} |c_{ii_2\cdots i_m}| \\ &= v_i^{1-m} \sum |a_{ii_2\cdots i_m}| v_{i_2} \cdots v_{i_m} \\ &\leq v_i^{1-m} \sum |b_{ii_2\cdots i_m}| v_{i_2} \cdots v_{i_m} \\ &= v_i^{1-m} \left( b_{i\cdots i} v^{m-1} - \sum_{i_1,\ldots,i_m=1} b_{ii_2\cdots i_m} v_{i_2} \cdots v_{i_m} \right) \\ &= b_{i\cdots i} - \tau(\mathcal{B}). \end{aligned} \tag{7}$$

Thus, we complete the proof. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

**Acknowledgements**
This research is supported by NSFC (61170311), Chinese Universities Specialized Research Fund for the Doctoral Program (20110185110020), Sichuan Province Sci. & Tech. Research Project (12ZC1802). The first author is supported by the Fundamental Research Funds for Central Universities.

Received: 3 January 2014 Accepted: 27 February 2014 Published: 13 Mar 2014

**References**
1. Chang, KC, Pearson, K, Zhang, T: On eigenvalue problems of real symmetric tensors. J. Math. Anal. Appl. **350**, 416-422 (2009)
2. Chang, KC, Qi, L, Zhou, G: Singular values of a real rectangular tensor. J. Math. Anal. Appl. **37**, 284-294 (2010)
3. Chang, KC, Pearson, K, Zhang, T: Primitivity, the convergence of the NZQ method, and the largest eigenvalue for nonnegative tensors. SIAM J. Matrix Anal. Appl. **32**, 806-819 (2011)
4. Qi, L: Eigenvalues and invariants of tensor. J. Math. Anal. Appl. **325**, 1363-1377 (2007)
5. Qi, L: Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl. **439**, 228-238 (2013)
6. Liu, Y, Zhou, G, Ibrahim, NF: An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor. J. Comput. Appl. Math. **235**, 286-292 (2010)
7. Ng, M, Qi, L Zhou, G: Finding the largest eigenvalue of a non-negative tensor. SIAM J. Matrix Anal. Appl. **31**, 1090-1099 (2009)
8. Qi, L: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. **40**, 1302-1324 (2005)
9. Lim, LH: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, CAMSAP 05, pp. 129-132 (2005)
10. Chang, KC, Zhang, T, Pearson, K: Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci. **6**, 507-520 (2008)
11. Yang, Y, Yang, Q: Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl. **31**, 2517-2530 (2010)

12. Yang, Y, Yang, Q: Further results for Perron-Frobenius theorem for nonnegative tensors. II. SIAM J. Matrix Anal. Appl. **32**, 1236-1250 (2011)
13. Zhang, L, Qi, L, Zhou, G *M*-tensors and the Positive Definiteness of a Multivariate Form. arXiv:1202.6431v1
14. Ding, W, Qi, L, Wei, Y: *M*-tensors and nonsingular *M*-tensors. Linear Algebra Appl. **439**, 3264-3278 (2013)
15. Li, C, Li, Y, Kong, X: New eigenvalue inclusion sets for tensors. Numer. Linear Algebra Appl. **21**, 39-50 (2014)

10.1186/1029-242X-2014-114
**Cite this article as:** He and Huang: Inequalities for *M*-tensors. *Journal of Inequalities and Applications* 2014, 2014:114