Braha *Journal of Inequalities and Applications* 2014, 2014:112
http://www.journalofinequalitiesandapplications.com/content/2014/1/112

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some geometric properties of $N(\lambda, p)$-spaces

Naim L Braha*

*Correspondence: nbraha@yahoo.com
Department of Mathematics and Computer Sciences, University of Prishtina, Avenue 'Mother Teresa' 5, Prishtinë, 10000, Kosovo
Department of Computer Sciences and Applied Mathematics, College Vizioni per Arsim, Rr, Ahmet Kaciku, Ferizaj, 70000, Kosovo

Abstract

In this paper, we introduce the sequence spaces $N(\lambda, p)$ and we show some geometric properties of that spaces. The main purpose of this paper is to show that $N(\lambda, p)$ is a Banach space and has the rotund property, the Kadec-Klee property, the uniform Opial property, the (β)-property, the k-NUC property and the Banach-Saks property of type p.

MSC: Primary 46A45; 40C05; secondary 46B45; 46A35

Keywords: normed sequence spaces; rotund property; Kadec-Klee property; uniform Opial property; (β)-property; k-NUC property; Banach-Saks property of type p

1 Introduction

By ω, we denote the space of all real valued sequences. Any vector subspace of ω is called a sequence space. We write l_∞, c, and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, l_1, and l_∞, we denote the spaces of all bounded, convergent, absolutely convergent and p-absolutely convergent series, respectively; where $1 < p < \infty$. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup\{p_k\} = H$ and $M = \max\{1, H\}$. Then, the linear space $l(p)$ was defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$l(p) = \left\{ x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty \right\} \quad (0 < p_k \le H < \infty)$$

which is complete paranormed space paranormed by

$$g(x) = \left(\sum_k |x_k|^{p_k} \right)^{\frac{1}{M}}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞.

In [4] was introduced the following numerical sequence $\lambda = (\lambda_k)_{k=0}^{\infty}$, which is a strictly increasing sequence of positive real numbers tending to infinity, as $k \to \infty$, that is

$$0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as } k \to \infty.$$

We will introduce the following sequence space:

$$N(\lambda, p) = \left\{ x = (x_n) \in \omega : \sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{i=1}^{k} \left| (\lambda_i - \lambda_{i-1}) x_i \right| \right)^{p_k} < \infty \right\}.$$

For $\lambda_k = k$, we obtain the Cesaro sequence space ces(p) (see [5]). If $\lambda_k = k$ and $p_k = p$, then $N(\lambda, p) = \text{ces}_p$ (see [6]). In case where $p_k = p$ for all $k \in \mathbb{N}$, then we will denote $N(\lambda, p) = N_p$. Some results related to the geometric properties of sequence spaces are given in [7–9].

2 Topological properties

Theorem 2.1 *The paranorm on $N(\lambda, p)$ is given by the relation*

$$h(x) = \left(\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{i=1}^{k} \left| (\lambda_i - \lambda_{i-1}) x_i \right| \right)^{p_k} \right)^{\frac{1}{M}},$$

where $M = \max\{1, H\}$ and $H = \sup p_k$.

3 Geometrical properties

In this section we will show some geometric properties of the $N(\lambda, p)$-spaces, such as the (β)-property, the k-NUC property, the Banach-Saks property of type p, and the (H)-property. It is well known that these properties are most important in Banach spaces (see [10, 11] and [1]).

Definition 3.1 A Banach space X is said to be k-nearly uniformly convex (k-NUC) if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any sequence $(x_n) \subset B(X)$ with $\text{sep}(x_n) \geq \epsilon$, there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that

$$\left\| \frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k} \right\| < 1 - \delta,$$

where $\text{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \epsilon$.

Definition 3.2 A Banach space X has property (β) if and only if for each $r > 0$ and $\epsilon > 0$, there exists a $\delta > 0$ such that for each element $x \in B(X)$ and each sequence $x_n \in B(X)$ with $\text{sep}(x_n) \geq \epsilon$, there is an index k for each

$$\left\| \frac{x + x_k}{2} \right\| \leq \delta.$$

Definition 3.3 A Banach space X is said to have the Banach-Saks property type p if every weakly null sequence (x_k) has a subsequence (x_{kl}) such that for some $C > 0$

$$\left\| \sum_{l=0}^{n} x_{kl} \right\| < C(n+1)^{\frac{1}{p}}$$

for all $n \in \mathbb{N}$.

Definition 3.4 Let X be a real vector space. A functional $\sigma : X \to [0, \infty)$ is called a modular if

(1) $\sigma(x) = 0$ if and only if $x = \theta$,
(2) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$,
(3) $\sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$,

(4) the modular σ is called convex if $\sigma(\alpha x + \beta y) \leq \alpha\sigma(x) + \beta\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$, with $\alpha + \beta = 1$.

A modular σ is called:

(5) right continuous if $\lim_{\alpha\to 1^+} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$,
(6) left continuous if $\lim_{\alpha\to 1^-} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$,
(7) continuous if it is both right and left continuous,

where $X_\sigma = \{x \in X : \lim_{\alpha\to 0^+} \sigma(\alpha x) = 0\}$. We define σ_p on $N(\lambda, p)$ as follows:

$$\sigma_p(x) = \left(\sum_{k=1}^{\infty} \left(\frac{1}{\lambda_k}\sum_{i=0}^{k} \left|(\lambda_i - \lambda_{i-1})x_i\right|\right)^{p_k}\right),$$

where $\lambda_{-1} = 0$.

If $p_k \geq 1$, for all $k \in \mathbb{N}$, by the convexity of the function $t \to |t|^{p_k}$, for all $k \in \mathbb{N}$, σ_p defined above is a modular convex in the $N(\lambda, p)$.

Definition 3.5 A modular σ_p is said to satisfy the δ_2-conditions if for every $\epsilon > 0$, there exist constant $M > 0$ and $m > 0$ such that

$$\sigma_p(2t) \leq M\sigma_p(t) + \epsilon \tag{3.1}$$

for all $t \in X_{\sigma_p}$ with $\sigma_p(t) \leq m$.

Lemma 3.6 ([12]) *If σ_p satisfies the δ_2-conditions, then for any $A > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\left|\sigma_p(t + w) - \sigma_p(t)\right| < \epsilon \tag{3.2}$$

whenever $t, w \in X_{\sigma_p}$ with $\sigma_p(t) \leq A$ and $\sigma_p(w) \leq \delta$.

Theorem 3.7 ([12])

(1) *If σ_p satisfies the δ_2-conditions, then for any $x \in X_{\sigma_p}$, $\|x\| = 1$ if and only if $\sigma_p(x) = 1$.*
(2) *If σ_p satisfies the δ_2-conditions, then for any sequence $(x_n) \in X_{\sigma_p}$, $\|x_n\| \to 0$ if and only if $\sigma_p(x_n) \to 0$.*

Theorem 3.8 *If σ_p satisfies the δ_2-conditions, then for any $\epsilon \in (0, 1)$, there exists $\delta \in (0, 1)$ such that $\sigma_p(x) \leq 1 - \epsilon$ implies $\|x\| \leq 1 - \delta$.*

Proof The proof of the theorem follows directly from the above two facts. □

Theorem 3.9 *For any $x \in N(\lambda, p)$ and $\epsilon \in (0, 1)$, there exists $\delta \in (0, 1)$, such that $\sigma_p(x) \leq 1 - \epsilon$ implies $\|x\| \leq 1 - \delta$.*

Proof The proof of the theorem follows directly from Theorem 3.8. □

Proposition 3.10 *If $p_k \geq 1$, for all $k \in \mathbb{N}$, then the modular function σ_p, on $N(\lambda, p)$, satisfies the following conditions:*

(1) *If $0 < \alpha \leq 1$, then $\alpha^M \sigma_p(\frac{x}{\alpha}) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha\sigma_p(x)$.*
(2) *If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M \sigma_p(\frac{x}{\alpha})$.*

(3) *If* $\alpha \geq 1$, *then* $\sigma_p(x) \leq \alpha\sigma_p(\frac{x}{\alpha})$.
(4) *The modular function* $\sigma_p(x)$ *is continuous on* $N(\lambda, p)$.

Proof The proof of the proposition is similar to Proposition 2.1 in [13]. □

Now we will define the following two norms (the first is known as the Luxemburg norm and the second as the Amemiya norm) in $N(\lambda, p)$:

$$\|x\|_L = \inf\left\{\alpha > 0 : \sigma_p\left(\frac{x}{\alpha}\right) \leq 1\right\} \tag{3.3}$$

and

$$\|x\|_A = \inf_{\alpha > 0} \frac{1}{\alpha}\left\{1 + \sigma_p(\alpha \cdot x)\right\}. \tag{3.4}$$

Proposition 3.11 *Let* $x \in N(\lambda, p)$. *Then the following relations are satisfied*:
(1) *If* $\|x\|_L < 1$, *then* $\sigma_p(x) \leq \|x\|_L$.
(2) *If* $\|x\|_L > 1$, *then* $\sigma_p(x) \geq \|x\|_L$.
(3) $\|x\|_L = 1$ *if and only if* $\sigma_p(x) = 1$.
(4) $\|x\|_L < 1$ *if and only if* $\sigma_p(x) < 1$.
(5) $\|x\|_L > 1$ *if and only if* $\sigma_p(x) > 1$.

Proof (1) Let $x \in N(\lambda, p)$ and $\|x\|_L < 1$. Let also $\epsilon > 0$ such that $0 < \epsilon < 1 - \|x\|_L$. On the other hand from the definition of the norm by relation (3.3) we find that there exists a $\alpha > 0$ such that $\|x\|_L + \epsilon > \alpha$ and $\sigma_p(\frac{x}{\alpha}) \leq 1$. From the above relations and property (1) of Proposition 3.10, we obtain

$$\frac{\|x\|_L + \epsilon}{\alpha} > 1$$

and

$$\sigma_p(x) \leq \frac{\|x\|_L + \epsilon}{\alpha}\sigma_p(x) = \frac{\|x\|_L + \epsilon}{\alpha}\sigma_p\left(\alpha \cdot \frac{x}{\alpha}\right) \leq \frac{\|x\|_L + \epsilon}{\alpha} \cdot \alpha \cdot \sigma_p\left(\frac{x}{\alpha}\right) \leq \|x\|_L + \epsilon.$$

The previous statement is valid for every $\epsilon > 0$, from which it follows that $\sigma_p(x) \leq \|x\|_L$.

(2) In this case we will choose $\epsilon > 0$ such that $0 < \epsilon < 1 - \frac{1}{\|x\|_L}$, and we obtain $1 < (1 - \epsilon)\|x\|_L < \|x\|_L$. Now using into consideration definition of the norm (3.3) and relation (1) of Proposition 3.10, we get

$$1 < \sigma_p\left(\frac{x}{(1-\epsilon)\|x\|_L}\right) \leq \frac{1}{(1-\epsilon)\|x\|_L}\sigma_p(x) \quad \Rightarrow \quad (1-\epsilon)\|x\|_L \leq \sigma_p(x)$$

for every $\epsilon \in (0, 1 - \frac{1}{\|x\|_L})$. Finally we have proved that $\|x\|_L \leq \sigma_p(x)$.

(3) Since $\sigma_p(x)$ is continuous function (see [14]), this property follows immediately.

(4) Follows from properties (1) and (3).

(5) Follows from properties (2) and (3). □

Theorem 3.12 $N(\lambda, p)$ *is a Banach space under the Luxemburg and Amemiya norms*.

Proof We will prove that $N(\lambda,p)$ is a Banach space under the Luxemburg norm. In a similar way we can prove that $N(\lambda,p)$ is a Banach space under the Amemiya norm. In what follows we need to show that every Cauchy sequence in $N(\lambda,p)$ is convergent according to the Luxemburg norm. Let $\{x_k^n\}$ be any Cauchy sequence in $N(\lambda,p)$ and $\epsilon \in (0,1)$. Thus there exists a natural number n_0, such that for any $n,m \geq n_0$ we get $\|x^{(n)} - x^{(m)}\|_L < \epsilon$. From Proposition 3.11 we get

$$\sigma_p\bigl(x^{(n)} - x^{(m)}\bigr) \leq \bigl\|x^{(n)} - x^{(m)}\bigr\|_L < \epsilon \tag{3.5}$$

for all $n,m \geq n_0$. This implies that

$$\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})\bigl(x_i^{(n)} - x_i^{(m)}\bigr)\bigr|\right)^{p_k} < \epsilon. \tag{3.6}$$

For each fixed k and for all $n,m \geq n_0$,

$$\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})\bigl(x_i^{(n)} - x_i^{(m)}\bigr)\bigr| < \epsilon.$$

Hence $(y_k^{(n)})_k = (\frac{1}{\lambda_k}\sum_{i=1}^{k}|(\lambda_i - \lambda_{i-1})x_i^{(n)}|)_k$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a complete normed space, there exists a $(y_k)_k = (\frac{1}{\lambda_k}\sum_{i=1}^{k}|(\lambda_i - \lambda_{i-1})x_i|)_k \in \mathbb{R}$ such that $(y_k^{(n)}) \to y_k$ as $n \to \infty$. Therefore, as $n \to \infty$, by relation (3.6) we have

$$\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})\bigl(x_i - x_i^{(m)}\bigr)\bigr|\right)^{p_k} < \epsilon$$

for all $m \geq n_0$. In the sequel we will show that (y_k) is a sequence form $N(\lambda,p)$. From Proposition 3.10 and relation (3.5) we have

$$\lim_{n\to\infty}\sigma_p\bigl(x^{(n)} - x^{(m)}\bigr) = \sigma_p\bigl(x - x^{(m)}\bigr) \leq \bigl\|x - x^{(m)}\bigr\|_L < \epsilon$$

for all $m \geq n_0$. This implies that $(x^{(n)}) \to x$ as $m \to \infty$. So we have $x = x^{(n)} - (x^{(n)} - x) \in N(\lambda,p)$. This proves that $N(\lambda,p)$ is a complete normed space under the Luxemburg norm. □

In what follows we will show results related to the Luxemburg norm, and for this reason we will denote it just $\|\cdot\|$.

Theorem 3.13 *The space $N(\lambda,p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof Let $N(\lambda,p)$ be rotund and choose $k \in \mathbb{N}$ such that $p_k = 1$. Consider the two sequences given by

$$x = \left(0,0,\ldots,0,\frac{\lambda_k}{2^k\cdot|\lambda_k - \lambda_{k-1}|},0,0,\ldots\right)$$

and

$$y = \left(0,0,\ldots,0,\frac{2\lambda_k}{3^k\cdot|\lambda_k - \lambda_{k-1}|},0,0,\ldots\right).$$

Then obviously $x \neq y$ and

$$\sigma_p(x) = \sigma_p(y) = \sigma_p\left(\frac{x+y}{2}\right) = 1.$$

Then from Proposition 3.11, property (3), it follows that $x, y, \frac{x+y}{2} \in S[N(\lambda,p)]$, which leads to the contradiction that the sequence space $N(\lambda,p)$ is not rotund. Hence $p_k > 1$, for all $k \in \mathbb{N}$.

Conversely, let $x \in S[N(\lambda,p)]$ and $y, z \in S[N(\lambda,p)]$ such that $x = \frac{y+z}{2}$. By the convexity of σ_p and property (3) from Proposition 3.11, we have

$$1 = \sigma_p(x) \leq \frac{\sigma_p(y) + \sigma_p(z)}{2} \leq \frac{1}{2} + \frac{1}{2} = 1,$$

which gives $\sigma_p(y) = \sigma_p(z) = 1$ and

$$\sigma_p(x) = \frac{\sigma_p(y) + \sigma_p(z)}{2}. \tag{3.7}$$

From the previous relation we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})x_i\bigr|\right)^{p_k}\\&\quad = \frac{1}{2}\left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})y_i\bigr|\right)^{p_k} + \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})z_i\bigr|\right)^{p_k}\right).\end{aligned}$$

Since $x = \frac{y+z}{2}$, we get

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})(y_i + z_i)\bigr|\right)^{p_k}\\&\quad = \frac{1}{2}\left(\sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})y_i\bigr|\right)^{p_k} + \sum_{k=1}^{\infty}\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})z_i\bigr|\right)^{p_k}\right).\end{aligned}$$

This implies that

$$\begin{aligned}&\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})(y_i + z_i)\bigr|\right)^{p_k}\\&\quad = \frac{1}{2}\left(\left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})y_i\bigr|\right)^{p_k} + \left(\frac{1}{\lambda_k}\sum_{i=1}^{k}\bigl|(\lambda_i - \lambda_{i-1})z_i\bigr|\right)^{p_k}\right).\end{aligned} \tag{3.8}$$

From the previous relation we get $y_i = z_i$ for all $i \in \mathbb{N}$, respectively, $z = y$. That is, the sequence space $N(\lambda,p)$ is rotund. □

In what follows we will give two facts without proof because their proofs follow directly from Proposition 3.10 and Proposition 3.11.

Theorem 3.14 *Let $x \in N(\lambda,p)$. Then the following statements hold*:

(i) *For* $0 < \alpha < 1$ *and* $\|x\| > \alpha$ *we get* $\sigma_p(x) > \alpha^M$.
(ii) *If* $\alpha \geq 1$ *and* $\|x\| < \alpha$, *then we have* $\sigma_p(x) < \alpha^M$.

Theorem 3.15 *Let* (x_n) *be a sequence in* $N(\lambda, p)$. *Then the following statements hold*:
(i) $\lim_{n\to\infty} \|x_n\| = 1$ *implies* $\lim_{n\to\infty} \sigma_p(x_n) = 1$.
(ii) $\lim_{n\to\infty} \sigma_p(x_n) = 0$ *implies* $\lim_{n\to\infty} \|x_n\| = 0$.

Theorem 3.16 *Let* $x \in N(\lambda, p)$ *and* $(x^{(n)}) \subset N(\lambda, p)$. *If* $\sigma_p(x^{(n)}) \to \sigma_p(x)$ *as* $n \to \infty$ *and* $x_k^{(n)} \to x_k$ *as* $n \to \infty$ *for all* $k \in \mathbb{N}$, *then* $x^{(n)} \to x$ *as* $n \to \infty$.

Proof The proof of the theorem is similar to Theorem 2.9 in [13]. □

Theorem 3.17 *The sequence space* $N(\lambda, p)$ *has the Kadec-Klee property.*

Proof It is enough to prove that every weakly convergent sequence on the unit sphere is convergent in norm. Let $x \in N(\lambda, p)$ and $(x^{(n)}) \in N(\lambda, p)$ such that $\|x^{(n)}\| \to 1$ and $x^{(n)} \xrightarrow{w} x$ be given. From the properties of Theorem 3.15 it follows that $\sigma_p(x^{(n)}) \to 1$ as $n \to \infty$. On the other hand, from Proposition 3.11, we get $\sigma_p(x) = 1$. Therefore we have $\sigma_p(x^{(n)}) \to \sigma_p(x)$, as $n \to \infty$. Since $x^{(n)} \xrightarrow{w} x$ and $p_k(x) = x_k$ is a continuous functional, $x_k^{(n)} \to x_k$ as $n \to \infty$ and for $k \in \mathbb{N}$. Now the proof of the theorem follows from Theorem 3.16. □

Theorem 3.18 *For any* $1 < p < \infty$, *the space* N_p *has the uniform Opial property.*

We omit this proof.

To prove the following theorem we will use the same technique given in [15] and will consider that $\lim_n \inf p_n > 1$.

Theorem 3.19 *The Banach space* $N(\lambda, p)$ *has the k-NUC property for every* $k \geq 2$.

Proof Let $\epsilon > 0$ and $(x_n) \subset B(N(\lambda, p))$ with $\operatorname{sep}(x_n) \geq \epsilon$. For each $m \in \mathbb{N}$, let

$$x_n^m = \bigl(\overbrace{0, 0, \ldots, 0}^{m-1}, x_n(m), x(m+1), \ldots\bigr). \tag{3.9}$$

Since for each $i \in \mathbb{N}$, $(x_n(i))_{n=1}^\infty$ is bounded, by the diagonal method (see [16]), we find that for each $m \in \mathbb{N}$, we can find a subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathbb{N}$, $1 \leq i \leq m$. Therefore, there exists an increasing sequence of positive integers (t_m) such that $\operatorname{sep}((x_{n_j}^m)_{j>t_m}) \geq \epsilon$. Hence, there is a sequence of positive integers $(r_m)_{m\in\mathbb{N}}$ with $r_1 < r_2 < r_3 < \cdots$ such that $\|x_{r_m}^m\| \geq \frac{\epsilon}{2}$ for all $m \in \mathbb{N}$. Then by Theorem 3.15, we may assume that there exists $\eta > 0$ such that

$$\sigma_p\bigl(x_{r_m}^m\bigr) \geq \eta \quad \text{for all } m \in \mathbb{N}. \tag{3.10}$$

Let $\alpha > 0$ be such that $1 < \alpha < \liminf_n p_n$. For fixed integer $k \geq 2$, let $\epsilon_1 = (\frac{(k^{\alpha-1}-1)}{(k-1)k^\alpha}) \cdot \frac{\eta}{2}$. Then by Lemma 3.6, there is a $\delta > 0$ such that

$$\bigl|\sigma_p(u+v) - \sigma_p(u)\bigr| \leq \epsilon_1, \tag{3.11}$$

whenever $\sigma_p(u) \le 1$ and $\sigma_p(v) \le \delta$. Since by Proposition 3.11, property (1), we get $\sigma_p(x_n) \le 1$, $\forall n \in \mathbb{N}$. Then there exist positive integers m_i $(i = 1, 2, \dots, k-1)$ with $m_1 < m_2 < \cdots < m_{k-1}$ such that $\sigma_p(x_{p_i}^{m_i}) \le \delta$ and $\alpha \le p_j$ for all $j \ge m_{k-1}$. Define $m_k = m_{k-1} + 1$. By (3.10), we have $\sigma_p(x_{r_{m_k}}^{m_k}) \ge \eta$. Let $s_i = i$ for $1 \le i \le k-1$ and $s_k = r_{m_k}$. From relations (3.10), (3.11), and the convexity of the function $f_i(u) = |u|^{p_i}$ $(i \in \mathbb{N})$, we have

$$
\begin{aligned}
&\sigma_p\left(\frac{x_{s_1} + x_{s_2} + \cdots + x_{s_k}}{k}\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&= \sum_{n=1}^{m_1}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n};
\end{aligned}
$$

from (3.11) we get

$$
\begin{aligned}
&\sum_{n=1}^{m_1}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_1}(i) + x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_2}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n} + \epsilon_1 \le
\end{aligned}
$$

from the convexity of $f_i(u) = |u|^{p_i}$ $(i \in \mathbb{N})$, it follows that

$$
\begin{aligned}
&\le \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{\lambda_k}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n} + \epsilon_1 \\
&\le \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{\lambda_k}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_2}(i) + x_{s_3}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n} \\
&\quad + \sum_{n=m_2+1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})\frac{x_{s_3}(i) + x_{s_4}(i) + \cdots + x_{s_k}(i)}{k}\right|\right)^{p_n} + 2\epsilon_1 \\
&\le \sum_{n=1}^{m_1}\frac{1}{k}\sum_{j=1}^{k}\left(\frac{1}{\lambda_k}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})x_{s_j}(i)\right|\right)^{p_n} \\
&\quad + \sum_{n=m_1+1}^{m_2}\frac{1}{k}\sum_{j=2}^{k}\left(\frac{1}{\lambda_k}\sum_{i=1}^{n}\left|(\lambda_i - \lambda_{i-1})x_{s_j}(i)\right|\right)^{p_n}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=m_2+1}^{m_3} \frac{1}{k} \sum_{j=3}^{k} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) x_{s_j}(i) \right| \right)^{p_n} \\
&+ \sum_{n=m_3+1}^{m_4} \frac{1}{k} \sum_{j=4}^{k} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) x_{s_j}(i) \right| \right)^{p_n} + \cdots \\
&+ \sum_{n=m_{k-1}+1}^{m_k} \frac{1}{k} \sum_{j=k-1}^{k} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) x_{s_j}(i) \right| \right)^{p_n} \\
&+ \sum_{n=m_k+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) \frac{x_{s_k}(i)}{k} \right| \right)^{p_n} + (k-1)\epsilon_1 \\
\leq{}& \frac{\sigma_p(x_{s_1}) + \sigma_p(x_{s_2}) + \cdots + \sigma_p(x_{s_{k-1}})}{k} + \frac{1}{k} \sum_{n=1}^{m_k} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) x_{s_k}(i) \right| \right)^{p_n} \\
&+ \sum_{n=m_k+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) \frac{x_{s_k}(i)}{k} \right| \right)^{p_n} + (k-1)\epsilon_1 \\
\leq{}& \frac{k-1}{k} + \frac{1}{k} \sum_{n=1}^{m_k} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) x_{s_k}(i) \right| \right)^{p_n} \\
&+ \frac{1}{k^\alpha} \cdot \sum_{n=m_k+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) x_{s_k}(i) \right| \right)^{p_n} + (k-1)\epsilon_1 \\
\leq{}& 1 - \frac{1}{k} + \frac{1}{k} \left[1 - \sum_{n=m_k+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) \frac{x_{s_k}(i)}{k} \right| \right)^{p_n} \right] \\
&+ \frac{1}{k^\alpha} \cdot \sum_{n=m_k+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) \frac{x_{s_k}(i)}{k} \right| \right)^{p_n} + (k-1)\epsilon_1 \\
\leq{}& 1 + (k-1)\epsilon_1 - \left(\frac{k^{\alpha-1} - 1}{k^\alpha} \right) \sum_{n=m_k+1}^{\infty} \left(\frac{1}{\lambda_k} \sum_{i=1}^{n} \left| (\lambda_i - \lambda_{i-1}) x_{s_k}(i) \right| \right)^{p_n} \\
\leq{}& 1 + (k-1)\epsilon_1 - \left(\frac{k^{\alpha-1} - 1}{k^\alpha} \right) \eta \\
={}& 1 - \left(\frac{k^{\alpha-1} - 1}{k^\alpha} \right) \frac{\eta}{2}.
\end{aligned}
$$

Now from Theorem 3.9, there exists a $\vartheta > 0$ such that

$$
\left\| \frac{x_{s_1} + x_{s_2} + \cdots + x_{s_k}}{k} \right\| < 1 - \vartheta. \qquad \square
$$

The proof of the following results we omit.

Theorem 3.20 *The Banach space $N(\lambda, p)$ has the (β)-property.*

Theorem 3.21 *The Banach space $N(\lambda, p)$ has the Banach-Saks property of type p.*

Competing interests

The author declares that they have no competing interests.

Received: 23 December 2013 Accepted: 25 February 2014 Published: 12 Mar 2014

References

1. Maddox, IJ: Spaces of strongly summable sequences. Q. J. Math. **18**(2), 345-355 (1967)
2. Simons, S: The sequence spaces $l(p_\nu)$ and $m(p_\nu)$. Proc. Lond. Math. Soc. **15**(3), 422-436 (1965)
3. Nakano, H: Modulared sequence spaces. Proc. Jpn. Acad. **27**(2), 508-512 (1951)
4. Mursaleen, M, Noman, AK: On the spaces of λ-convergent and bounded sequences. Thai J. Math. **8**(2), 311-329 (2010)
5. Suantai, S: On the *H*-property of some Banach sequence spaces. Arch. Math. **39**, 309-316 (2003)
6. Shiue, JS: On the Cesaro sequence space. Tamkang J. Math. **2**, 19-25 (1970)
7. Basar, F, Mursaleen, M, Altay, B: Some generalizations of the spaces bv_p of p-bounded variation sequences. Nonlinear Anal. **68**, 273-287 (2008)
8. Mursaleen, M, Çolak, R, Et, M: Some geometric inequalities in a new Banach sequence space. J. Inequal. Appl. **2007**, Article ID 86757 (2007). doi:10.1155/2007/86757
9. Mursaleen, M, Noman, AK: On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ I. Filomat **25**(2), 33-51 (2011)
10. Chen, S: Geometry of Orlicz spaces. Diss. Math. **356**, 1-224 (1996)
11. Diestel, J: Geometry of Banach Spaces - Selected Topics. Springer, Berlin (1984)
12. Cui, Y, Hudzik, H: On the uniform Opial property in some modular sequence spaces. Funct. Approx. Comment. Math. **26**, 93-102 (1998)
13. Nergiz, H, Basar, F: Some geometric properties of the domain of the double sequential band matrix $B(\tilde{r},\tilde{s})$ in the sequence space $\ell(p)$. Abstr. Appl. Anal. **2013**, Article ID 421031 (2013)
14. Maligranda, L: Orlicz Spaces and Interpolation. Inst. Math. Polish Academy of Sciences, Poznan (1985)
15. Sanhan, W, Suantai, S: On k-nearly uniform convex property in generalized Cesàro sequence spaces. Int. J. Math. Math. Sci. **2003**(57), 3599-3607 (2003)
16. Albiac, F, Kalton, NJ: Topics in Banach Space Theory. Graduate Texts in Mathematics, vol. 233. Springer, New York (2006)

10.1186/1029-242X-2014-112
Cite this article as: Braha: Some geometric properties of $N(\lambda, p)$-spaces. *Journal of Inequalities and Applications* 2014, 2014:112