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Sharp function estimates and boundedness for commutators associated with general singular integral operator

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Abstract
In this paper, we establish the sharp maximal function estimates for the commutator associated with the singular integral operator with general kernel. As an application, we obtain the boundedness of the commutator on weighted Lebesgue, Morrey, and Triebel-Lizorkin spaces.

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1 Introduction and preliminaries

As the development of singular integral operators (see [1–3]), their commutators have been well studied. In [4–6], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [7]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [8–10], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [11, 12], the boundedness for the commutators generated by the singular integral operators and the weighted BMO and Lipschitz functions on $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [13], some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by BMO and Lipschitz functions are obtained (see [13, 14]). In this paper, we will study the commutators generated by the singular integral operators with general kernel and the weighted Lipschitz functions.

First, let us introduce some notation. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| f(y) - f_Q \right| dy,$$

where $f_Q = |Q|^{-1} \int_Q f(x)\,dx$. It is well known that (see [1, 2])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q \left| f(y) - c \right| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < 1$ and $1 \leq r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r\,dy \right)^{1/r}.$$

The A_p weight is defined by (see [1])

$$A_p = \left\{ w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\,dx \right)^{p-1} < \infty \right\},$$
$$1 < p < \infty,$$

and

$$A_1 = \left\{ w \in L^p_{\mathrm{loc}}(R^n) : M(w)(x) \leq Cw(x),\ \text{a.e.} \right\}.$$

The $A(p,r)$ weight is defined by (see [15]), for $1 < p, r < \infty$,

$$A(p,r) = \left\{ w > 0 : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^r\,dx \right)^{1/r} \left(\frac{1}{|Q|} \int_Q w(x)^{-p/(p-1)}\,dx \right)^{(p-1)/p} < \infty \right\}.$$

Given a non-negative weight function w. For $1 \leq p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x)\,dx \right)^{1/p} < \infty.$$

For $\beta > 0$, $p > 1$, and the non-negative weight function w, let $\dot{F}_p^{\beta,\infty}(w)$ be the weighted homogeneous Triebel-Lizorkin space (see [10]).

For $0 < \beta < 1$ and the non-negative weight function w, the weighted Lipschitz space $\mathrm{Lip}_\beta(w)$ is the space of functions b such that

$$\|b\|_{\mathrm{Lip}_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |b(y) - b_Q|\,dy < \infty.$$

Remark

(1) It is well known that, for $b \in \mathrm{Lip}_\beta(w)$, $w \in A_1$, and $x \in Q$,

$$|b_Q - b_{2^kQ}| \leq Ck\|b\|_{\mathrm{Lip}_\beta(w)} w(x) w(2^kQ)^{\beta/n}.$$

(2) Let $b \in \mathrm{Lip}_\beta(w)$ and $w \in A_1$. By [16], we know that spaces $\mathrm{Lip}_\beta(w)$ coincide and the norms $\|b\|_{\mathrm{Lip}_\beta(w)}$ are equivalent with respect to different values $1 \leq p \leq \infty$.

In this paper, we will study some singular integral operators as follows (see [13]).

Definition 1 Let $T : S \to S'$ be a linear operator such that T is bounded on $L^2(R^n)$ and there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)\,dy$$

for every bounded and compactly supported function f, where K satisfies the following: there is a sequence of positive constant numbers $\{C_k\}$ such that for any $k \geq 1$,

$$\int_{2|y-z|<|x-y|} \left(\left|K(x,y) - K(x,z)\right| + \left|K(y,x) - K(z,x)\right|\right) dx \leq C$$

and

$$\left(\int_{2^k|z-y|\leq|x-y|<2^{k+1}|z-y|} \left(\left|K(x,y) - K(x,z)\right| + \left|K(y,x) - K(z,x)\right|\right)^q dy\right)^{1/q} \\ \leq C_k\left(2^k|z-y|\right)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Let b be a locally integrable function on R^n. The commutator related to T is defined by

$$T_b(f)(x) = \int_{R^n} \left(b(x) - b(y)\right)K(x,y)f(y)\,dy.$$

Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 1 with $C_j = 2^{-j\delta}$ (see [4]).

Definition 2 Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for } t \geq 0.$$

Let w be a non-negative weight function on R^n and f be a locally integrable function on R^n. Set, for $1 \leq p < \infty$,

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x\in R^n,\, d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} \left|f(y)\right|^p w(y)\,dy\right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized weighted Morrey space is defined by

$$L^{p,\varphi}\left(R^n, w\right) = \left\{f \in L^1_{\text{loc}}\left(R^n\right) : \|f\|_{L^{p,\varphi}(w)} < \infty\right\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n,w) = L^{p,\delta}(R^n,w)$, which is the classical Morrey spaces (see [17, 18]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n,w) = L^p(R^n,w)$, which is the weighted Lebesgue spaces (see [19]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [19–23]).

It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [5, 6]). In [6], Pérez and Trujillo-Gonzalez prove a sharp estimate for the multilinear commutator. The main purpose of this paper is to prove the sharp maximal inequalities for the commutator. As the application, we obtain the weighted L^p-norm inequality, and Morrey and Triebel-Lizorkin spaces' boundedness for the commutator.

2 Theorems

We shall prove the following theorems.

Theorem 1 *Let T be the singular integral operator as Definition* 1, *the sequence* $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $q' \le s < \infty$, *and* $b \in \mathrm{Lip}_\beta(w)$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}\big(T_b(f)\big)(\tilde{x}) \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n}\big(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}\big(T(f)\big)(\tilde{x})\big).$$

Theorem 2 *Let T be the singular integral operator as Definition* 1, *the sequence* $\{k2^{\beta k}C_k\} \in l^1$, $w \in A_1$, $0 < \beta < 1$, $q' \le s < \infty$, *and* $b \in \mathrm{Lip}_\beta(w)$. *Then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \inf_{c \in R^n} \frac{1}{|Q|^{1+\beta/n}} \int_Q \big|T_b(f)(x) - c\big|\,dx \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n}\big(M_s(f)(\tilde{x}) + M_s\big(T(f)\big)(\tilde{x})\big).$$

Theorem 3 *Let T be the singular integral operator as Definition* 1, *the sequence* $\{kC_k\} \in l^1$, $w \in A_1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$, *and* $b \in \mathrm{Lip}_\beta(w)$. *Then T_b is bounded from* $L^p(w)$ *to* $L^r(w^{r/p-r(1+\beta/n)})$.

Theorem 4 *Let T be the singular integral operator as Definition* 1, *the sequence* $\{kC_k\} \in l^1$, $0 < D < 2^n$, $w \in A_1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$, *and* $b \in \mathrm{Lip}_\beta(w)$. *Then T_b is bounded from* $L^{p,\varphi}(w)$ *to* $L^{r,\varphi}(w^{r/p-r(1+\beta/n)})$.

Theorem 5 *Let T be the singular integral operator as Definition* 1, *the sequence* $\{k2^{\beta k}C_k\} \in l^1$, $w \in A_1$, $0 < \beta < \min(1, n/q')$, $q' < p < n/\beta$, $1/r = 1/p - \beta/n$, *and* $b \in \mathrm{Lip}_\beta(w)$. *Then T_b is bounded from* $L^p(w)$ *to* $\dot{F}_r^{\beta,\infty}(w^{r/p-r(1+\beta/n)})$.

3 Proofs of theorems

To prove the theorems, we need the following lemmas.

Lemma 1 (see [13]) *Let T be the singular integral operator as Definition* 1, *the sequence* $\{C_k\} \in l^1$. *Then T is bounded on $L^p(w)$ for $w \in A_\infty$ with* $1 < p < \infty$.

Lemma 2 (see [12, 16]) *For any cube Q,* $b \in \mathrm{Lip}_\beta(w)$, $0 < \beta < 1$, *and* $w \in A_1$, *we have*

$$\sup_{x \in Q}\big|b(x) - b_Q\big| \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(Q)^{1+\beta/n}|Q|^{-1}.$$

Lemma 3 (see [10]) *For* $0 < \beta < 1$, $1 < p < \infty$, *and* $w \in A_\infty$, *we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}(w)} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q \left|f(x) - f_Q\right| dx \right\|_{L^p(w)}$$
$$\approx \left\| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q \left|f(x) - c\right| dx \right\|_{L^p(w)}.$$

Lemma 4 (see [1]) *Let* $0 < p < \infty$ *and* $w \in \bigcup_{1 \le r < \infty} A_r$. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$\int_{R^n} M(f)(x)^p w(x)\, dx \le C \int_{R^n} M^\#(f)(x)^p w(x)\, dx.$$

Lemma 5 (see [15, 24]) *Suppose that* $0 \le \eta < n$, $1 < s < p < n/\eta$, $1/r = 1/p - \eta/n$, *and* $w \in A(p,r)$. *Then*

$$\left\| M_{\eta,s}(f) \right\|_{L^r(w^r)} \le C \|f\|_{L^p(w^p)}.$$

Lemma 6 (see [1, 25]) *If* $w \in A_p$, *then* $w\chi_Q \in A_p$ *for* $1 \le p \le \infty$ *and any cube* Q.

Lemma 7 *Let* $1 < r < \infty$, $0 < \eta < \infty$, $0 < D < 2^n$, $w \in A_\infty$, *and* $L^{r,\varphi}(R^n, w)$ *be the weighted Morrey space as Definition* 2. *Then, for any smooth function* f *for which the left-hand side is finite,*

$$\left\| M(f) \right\|_{L^{r,\varphi}(w)} \le C \left\| M^\#(f) \right\|_{L^{r,\varphi}(w)}.$$

Proof Notice that $w\chi_Q \in A_\infty$ for any cube $Q = Q(x,d)$ by [19] and Lemma 6; thus, for $f \in L^{r,\varphi}(R^n, w)$ and any cube Q, we have, by Lemma 4,

$$\int_Q M(f)(x)^r w(x)\, dx = \int_{R^n} M(f)(x)^r w(x) \chi_Q(x)\, dx \le C \int_{R^n} M^\#(f)(x)^r w(x) \chi_Q(x)\, dx$$
$$= C \int_Q M^\#(f)(x)^r w(x)\, dx,$$

thus

$$\left(\frac{1}{\varphi(d)} \int_{Q(x,d)} M(f)(x)^r w(x)\, dx \right)^{1/r} \le C \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} M^\#(f)(x)^r w(x)\, dx \right)^{1/r}$$

and

$$\left\| M(f) \right\|_{L^{r,\varphi}(w)} \le C \left\| M^\#(f) \right\|_{L^{r,\varphi}(w)}.$$

This finishes the proof. □

Lemma 8 *Let* $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$, T *be the singular integral operator as Definition* 1 *and* $L^{p,\varphi}(R^n, w)$ *be the weighted Morrey space as Definition* 2. *Then*

$$\left\| T(f) \right\|_{L^{p,\varphi}(w)} \le C \|f\|_{L^{p,\varphi}(w)}.$$

Lemma 9 *Let* $1 \le s < p < n/\eta$, $1/r = 1/p - \eta/n$, $w \in A(p,r)$, *and* $L^{p,\varphi}(R^n, w)$ *be the weighted Morrey space as Definition* 2. *Then*

$$\|M_{\eta,s}(f)\|_{L^{r,\varphi}(w^r)} \le C\|f\|_{L^{p,\varphi}(w^p)}.$$

The proofs of the two lemmas are similar to that of Lemma 7 by Lemmas 1 and 5, we omit the details.

Proof of Theorem 1 It suffices to prove, for $f \in C_0^\infty(R^n)$ and some constant C_0, that the following inequality holds:

$$\frac{1}{|Q|}\int_Q \left|T_b(f)(x) - C_0\right| dx \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n}\left(M_{\beta,s}(f)(\tilde{x}) + M_{\beta,s}\left(T(f)\right)(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$T_b(f)(x) = \left(b(x) - b_{2Q}\right)T(f)(x) - T\left((b - b_{2Q})f_1\right)(x) - T\left((b - b_{2Q})f_2\right)(x).$$

Then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|T_b(f)(x) - T\left((b_{2Q} - b)f_2\right)(x_0)\right| dx \\
&\quad\le \frac{1}{|Q|}\int_Q \left|\left(b(x) - b_{2Q}\right)T(f)(x)\right| dx + \frac{1}{|Q|}\int_Q \left|T\left((b - b_{2Q})f_1\right)(x)\right| dx \\
&\qquad + \frac{1}{|Q|}\int_Q \left|T\left((b - b_{2Q})f_2\right)(x) - T\left((b - b_{2Q})f_2\right)(x_0)\right| dx \\
&\quad= I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, by Hölder's inequality and Lemma 2, we obtain

$$\begin{aligned}
I_1 &\le \frac{C}{|Q|}\sup_{x\in 2Q}\left|b(x) - b_{2Q}\right||Q|^{1-1/s}\left(\int_Q \left|T(f)(x)\right|^s dx\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)}\frac{w(2Q)^{1+\beta/n}}{|2Q|}|Q|^{-1/s}|Q|^{1/s-\beta/n}\left(\frac{1}{|Q|^{1-s\beta/n}}\int_Q \left|T(f)(x)\right|^s dx\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n} M_{\beta,s}\left(T(f)\right)(\tilde{x}) \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}\left(T(f)\right)(\tilde{x}).
\end{aligned}$$

For I_2, by the boundedness of T, we get

$$\begin{aligned}
I_2 &\le \left(\frac{1}{|Q|}\int_{R^n}\left|T\left((b - b_{2Q})f_1\right)(x)\right|^s dx\right)^{1/s} \\
&\le C\left(\frac{1}{|Q|}\int_{R^n}\left|\left(b(x) - b_{2Q}\right)f_1(x)\right|^s dx\right)^{1/s} \\
&\le C|Q|^{-1/s}\sup_{x\in 2Q}\left|b(x) - b_{2Q}\right||2Q|^{1/s-\beta/n}\left(\frac{1}{|2Q|^{1-s\beta/n}}\int_{2Q}\left|f(x)\right|^s dx\right)^{1/s}
\end{aligned}$$

$$\begin{aligned}
&\le C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\frac{w(Q)}{|Q|}\right)^{1+\beta/n} M_{\beta,s}(f)(\tilde{x}) \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}(f)(\tilde{x}).
\end{aligned}$$

For I_3, recalling that $s > q'$, we have

$$\begin{aligned}
I_3 &\le \frac{1}{|Q|}\int_Q \int_{(2Q)^c} \left|b(y)-b_{2Q}\right|\left|f(y)\right|\left|K(x,y)-K(x_0,y)\right| \,dy\,dx \\
&\le \frac{1}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d} \left|K(x,y)-K(x_0,y)\right|\left|b(y)-b_{2^{k+1}Q}\right|\left|f(y)\right| \,dy\,dx \\
&\quad + \frac{1}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{2^k d\le|y-x_0|<2^{k+1}d} \left|K(x,y)-K(x_0,y)\right|\left|b_{2^{k+1}Q}-b_{2Q}\right|\left|f(y)\right| \,dy\,dx \\
&\le \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty}\left(\int_{2^k d\le|y-x_0|<2^{k+1}d} \left|K(x,y)-K(x_0,y)\right|^q \,dy\right)^{1/q} \\
&\quad \times \sup_{y\in 2^{k+1}Q}\left|b(y)-b_{2^{k+1}Q}\right|\left(\int_{2^{k+1}Q}\left|f(y)\right|^{q'}\,dy\right)^{1/q'} dx \\
&\quad + \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty}\left|b_{2^{k+1}Q}-b_{2Q}\right|\left(\int_{2^k d\le|y-x_0|<2^{k+1}d} \left|K(x,y)-K(x_0,y)\right|^q \,dy\right)^{1/q} \\
&\quad \times \left(\int_{2^{k+1}Q}\left|f(y)\right|^{q'}\,dy\right)^{1/q'} dx \\
&\le C\sum_{k=1}^{\infty} C_k\left(2^k d\right)^{-n/q'}\frac{w(2^{k+1}Q)^{1+\beta/n}}{|2^{k+1}Q|}\|b\|_{\mathrm{Lip}_\beta(w)}\left|2^{k+1}Q\right|^{1/q'-1/s}\left|2^{k+1}Q\right|^{1/s-\beta/n} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|^{1-s\beta/n}}\int_{2^{k+1}Q}\left|f(y)\right|^s\,dy\right)^{1/s} \\
&\quad + C\sum_{k=1}^{\infty} k\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) w\left(2^k Q\right)^{\beta/n} C_k\left(2^k d\right)^{-n/q'}\left|2^{k+1}Q\right|^{1/q'-1/s}\left|2^{k+1}Q\right|^{1/s-\beta/n} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|^{1-s\beta/n}}\int_{2^{k+1}Q}\left|f(y)\right|^s\,dy\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)}\sum_{k=1}^{\infty} C_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{1+\beta/n} M_{\beta,s}(f)(\tilde{x}) \\
&\quad + C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})\sum_{k=1}^{\infty} kC_k\left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|}\right)^{\beta/n} M_{\beta,s}(f)(\tilde{x}) \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}(f)(\tilde{x})\sum_{k=1}^{\infty}(k+1)C_k \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_{\beta,s}(f)(\tilde{x}).
\end{aligned}$$

These complete the proof of Theorem 1. □

Proof of Theorem 2 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl|T_b(f)(x) - C_0\bigr| \, dx \\
&\quad \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} \bigl(M_s(f)(\tilde{x}) + M_s\bigl(T(f)\bigr)(\tilde{x})\bigr).
\end{aligned}$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$T_b(f)(x) = \bigl(b(x) - b_{2Q}\bigr)T(f)(x) - T\bigl((b - b_{2Q})f_1\bigr)(x) - T\bigl((b - b_{2Q})f_2\bigr)(x).$$

Then

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl|T_b(f)(x) - T\bigl((b_{2Q} - b)f_2\bigr)(x_0)\bigr| \, dx \\
&\quad \le \frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl|\bigl(b(x) - b_{2Q}\bigr)T(f)(x)\bigr| \, dx \\
&\qquad + \frac{1}{|Q|} \int_Q \bigl|T\bigl((b - b_{2Q})f_1\bigr)(x)\bigr| \, dx \\
&\qquad + \frac{1}{|Q|^{1+\beta/n}} \int_Q \bigl|T\bigl((b - b_{2Q})f_2\bigr)(x) - T\bigl((b - b_{2Q})f_2\bigr)(x_0)\bigr| \, dx \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

By using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned}
I_1 &\le \frac{C}{|Q|^{1+\beta/n}} \sup_{x \in 2Q} \bigl|b(x) - b_{2Q}\bigr| |Q|^{1-1/s} \biggl(\int_Q \bigl|T(f)(x)\bigr|^s \, dx\biggr)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} \frac{w(2Q)^{1+\beta/n}}{|2Q|} |Q|^{-1/s} |Q|^{1/s-\beta/n} \biggl(\frac{1}{|Q|} \int_Q \bigl|T(f)(x)\bigr|^s \, dx\biggr)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} \biggl(\frac{w(Q)}{|Q|}\biggr)^{1+\beta/n} M_s\bigl(T(f)\bigr)(\tilde{x}) \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_s\bigl(T(f)\bigr)(\tilde{x}), \\
I_2 &\le \frac{1}{|Q|^{1+\beta/n}} |Q|^{1-1/s} \biggl(\int_{R^n} \bigl|T\bigl((b - b_{2Q})f_1\bigr)(x)\bigr|^s \, dx\biggr)^{1/s} \\
&\le C \frac{1}{|Q|^{1+\beta/n}} |Q|^{1-1/s} \biggl(\int_{R^n} \bigl|\bigl(b(x) - b_{2Q}\bigr)f_1(x)\bigr|^s \, dx\biggr)^{1/s} \\
&\le C \frac{1}{|Q|^{1+\beta/n}} |Q|^{1-1/s} \sup_{x \in 2Q} \bigl|b(x) - b_{2Q}\bigr| |2Q|^{1/s} \biggl(\frac{1}{|2Q|} \int_{2Q} \bigl|f(x)\bigr|^s \, dx\biggr)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} \biggl(\frac{w(Q)}{|Q|}\biggr)^{1+\beta/n} M_s(f)(\tilde{x}) \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_s(f)(\tilde{x}),
\end{aligned}$$

$$
\begin{aligned}
I_3 &\le \frac{1}{|Q|^{1+\beta/n}} \int_Q \int_{(2Q)^c} \bigl|b(y)-b_{2Q}\bigr| \bigl|f(y)\bigr| \bigl|K(x,y)-K(x_0,y)\bigr|\, dy\, dx \\
&\le \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl|K(x,y)-K(x_0,y)\bigr| \bigl|b(y)-b_{2^{k+1}Q}\bigr| \bigl|f(y)\bigr|\, dy\, dx \\
&\quad + \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl|K(x,y)-K(x_0,y)\bigr| |b_{2^{k+1}Q}-b_{2Q}| \bigl|f(y)\bigr|\, dy\, dx \\
&\le \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} \biggl(\int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl|K(x,y)-K(x_0,y)\bigr|^q\, dy \biggr)^{1/q} \\
&\quad \times \sup_{y \in 2^{k+1}Q} \bigl|b(y)-b_{2^{k+1}Q}\bigr| \biggl(\int_{2^{k+1}Q} \bigl|f(y)\bigr|^{q'}\, dy \biggr)^{1/q'} dx \\
&\quad + \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^{\infty} |b_{2^{k+1}Q}-b_{2Q}| \biggl(\int_{2^k d \le |y-x_0| < 2^{k+1} d} \bigl|K(x,y)-K(x_0,y)\bigr|^q\, dy \biggr)^{1/q} \\
&\quad \times \biggl(\int_{2^{k+1}Q} \bigl|f(y)\bigr|^{q'}\, dy \biggr)^{1/q'} dx \\
&\le C|Q|^{-\beta/n} \sum_{k=1}^{\infty} C_k \bigl(2^k d\bigr)^{-n/q'} \frac{w(2^{k+1}Q)^{1+\beta/n}}{|2^{k+1}Q|} \|b\|_{\mathrm{Lip}_\beta(w)} \bigl|2^{k+1}Q\bigr|^{1/q'} \\
&\quad \times \biggl(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \bigl|f(y)\bigr|^s\, dy \biggr)^{1/s} \\
&\quad + C|Q|^{-\beta/n} \sum_{k=1}^{\infty} k \|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) w\bigl(2^k Q\bigr)^{\beta/n} C_k \bigl(2^k d\bigr)^{-n/q'} \bigl|2^{k+1}Q\bigr|^{1/q'} \\
&\quad \times \biggl(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \bigl|f(y)\bigr|^s\, dy \biggr)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} \sum_{k=1}^{\infty} 2^{\beta k} C_k \biggl(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \biggr)^{1+\beta/n} M_s(f)(\tilde{x}) \\
&\quad + C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x}) \sum_{k=1}^{\infty} k 2^{\beta k} C_k \biggl(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \biggr)^{\beta/n} M_s(f)(\tilde{x}) \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_s(f)(\tilde{x}) \sum_{k=1}^{\infty} (k+1) 2^{\beta k} C_k \\
&\le C\|b\|_{\mathrm{Lip}_\beta(w)} w(\tilde{x})^{1+\beta/n} M_s(f)(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 2. □

Proof of Theorem 3 Choose $q' < s < p$ in Theorem 1, notice that $w^{r/p-r(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p,r)$; we have, by Lemmas 1, 4, and 5,

$$
\begin{aligned}
&\bigl\|T_b(f)\bigr\|_{L^q(w^{r/p-r(1+\beta/n)})} \\
&\quad \le \bigl\|M\bigl(T_b(f)\bigr)\bigr\|_{L^r(w^{r/p-r(1+\beta/n)})} \\
&\quad \le C\bigl\|M^{\#}\bigl(T_b(f)\bigr)\bigr\|_{L^r(w^{r/p-r(1+\beta/n)})}
\end{aligned}
$$

$$\begin{aligned}
&\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|M_{\beta,s}\big(T(f)\big)w^{1+\beta/n}\right\|_{L^r(w^{r/p-r(1+\beta/n)})}+\left\|M_{\beta,s}(f)w^{1+\beta/n}\right\|_{L^r(w^{r/p-r(1+\beta/n)})}\right)\\
&= C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|M_{\beta,s}\big(T(f)\big)\right\|_{L^r(w^{r/p})}+\left\|M_{\beta,s}(f)\right\|_{L^r(w^{r/p})}\right)\\
&\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|T(f)\right\|_{L^p(w)}+\|f\|_{L^p(w)}\right)\\
&\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4 Choose $q' < s < p$ in Theorem 1, notice that $w^{r/p-r(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p,r)$; we have, by Lemmas 7-9,

$$\begin{aligned}
&\left\|T_b(f)\right\|_{L^{r,\varphi}(w^{r/p-r(1+\beta/n)})}\\
&\quad\leq \left\|M\big(T_b(f)\big)\right\|_{L^{r,\varphi}(w^{r/p-r(1+\beta/n)})}\\
&\quad\leq C\left\|M^\#\big(T_b(f)\big)\right\|_{L^{r,\varphi}(w^{r/p-r(1+\beta/n)})}\\
&\quad\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|M_{\beta,s}\big(T(f)\big)w^{1+\beta/n}\right\|_{L^{s,\varphi}(w^{r/p-r(1+\beta/n)})}+\left\|M_{\beta,s}(f)w^{1+\beta/n}\right\|_{L^{r,\varphi}(w^{r/p-r(1+\beta/n)})}\right)\\
&\quad= C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|M_{\beta,s}\big(T(f)\big)\right\|_{L^{r,\varphi}(w^{r/p})}+\left\|M_{\beta,s}(f)\right\|_{L^{r,\varphi}(w^{r/p})}\right)\\
&\quad\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|T(f)\right\|_{L^{p,\varphi}(w)}+\|f\|_{L^{p,\varphi}(w)}\right)\\
&\quad\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\|f\|_{L^{p,\varphi}(w)}.
\end{aligned}$$

This completes the proof of Theorem 4. □

Proof Theorem 5 Choose $q' < s < p$ in Theorem 2, notice that $w^{r/p-r(1+\beta/n)} \in A_\infty$ and $w^{1/p} \in A(p,r)$. By using Lemma 3, we obtain

$$\begin{aligned}
&\left\|T_b(f)\right\|_{\dot{F}_r^{\beta,\infty}(w^{r/p-r(1+\beta/n)})}\\
&\quad\leq C\left\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T_b(f)(x)-T\big((b_{2Q}-b)f_2\big)(x_0)\right|dx\right\|_{L^r(w^{r/p-r(1+\beta/n)})}\\
&\quad\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|M_s\big(T(f)\big)w^{1+\beta/n}\right\|_{L^r(w^{r/p-r(1+\beta/n)})}+\left\|M_s(f)w^{1+\beta/n}\right\|_{L^r(w^{r/p-r(1+\beta/n)})}\right)\\
&\quad= C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|M_s\big(T(f)\big)\right\|_{L^r(w^{r/p})}+\left\|M_s(f)\right\|_{L^r(w^{r/p})}\right)\\
&\quad\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\left(\left\|T(f)\right\|_{L^p(w)}+\|f\|_{L^p(w)}\right)\\
&\quad\leq C\|b\|_{\mathrm{Lip}_\beta(w)}\|f\|_{L^p(w)}.
\end{aligned}$$

This completes the proof of the theorem. □

Competing interests
The author declares that they have no competing interests.

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References

1. Garcia-Cuerva, J, Rubio de Francia, JL: Weighted Norm Inequalities and Related Topics, vol. 16. North-Holland, Amsterdam (1985)
2. Stein, EM: Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton (1993)
3. Torchinsky, A: Real Variable Methods in Harmonic Analysis. Pure and Applied Math., vol. 123. Academic Press, New York (1986)
4. Coifman, RR, Rochberg, R, Weiss, G: Factorization theorems for Hardy spaces in several variables. Ann. Math. **103**, 611-635 (1976)
5. Pérez, C: Endpoint estimate for commutators of singular integral operators. J. Funct. Anal. **128**, 163-185 (1995)
6. Pérez, C, Trujillo-Gonzalez, R: Sharp weighted estimates for multilinear commutators. J. Lond. Math. Soc. **65**, 672-692 (2002)
7. Chanillo, S: A note on commutators. Indiana Univ. Math. J. **31**, 7-16 (1982)
8. Chen, WG: Besov estimates for a class of multilinear singular integrals. Acta Math. Sin. **16**, 613-626 (2000)
9. Janson, S: Mean oscillation and commutators of singular integral operators. Ark. Mat. **16**, 263-270 (1978)
10. Paluszynski, M: Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss. Indiana Univ. Math. J. **44**, 1-17 (1995)
11. Bloom, S: A commutator theorem and weighted *BMO*. Trans. Am. Math. Soc. **292**, 103-122 (1985)
12. Hu, B, Gu, J: Necessary and sufficient conditions for boundedness of some commutators with weighted Lipschitz spaces. J. Math. Anal. Appl. **340**, 598-605 (2008)
13. Chang, DC, Li, JF, Xiao, J: Weighted scale estimates for Calderón-Zygmund type operators. Contemp. Math. **446**, 61-70 (2007)
14. Lin, Y: Sharp maximal function estimates for Calderón-Zygmund type operators and commutators. Acta Math. Sci. Ser. A Chin. Ed. **31**, 206-215 (2011)
15. Muckenhoupt, B, Wheeden, RL: Weighted norm inequalities for fractional integral. Trans. Am. Math. Soc. **192**, 261-274 (1974)
16. Garcia-Cuerva, J: Weighted H^p spaces. Diss. Math. **162**, 63 (1979)
17. Peetre, J: On convolution operators leaving $L^{p,\lambda}$-spaces invariant. Ann. Mat. Pura Appl. **72**, 295-304 (1966)
18. Peetre, J: On the theory of $L^{p,\lambda}$-spaces. J. Funct. Anal. **4**, 71-87 (1969)
19. Chiarenza, F, Frasca, M: Morrey spaces and Hardy-Littlewood maximal function. Rend. Mat. **7**, 273-279 (1987)
20. Di Fazio, G, Ragusa, MA: Commutators and Morrey spaces. Boll. Unione Mat. Ital., A (7) **5**, 323-332 (1991)
21. Di Fazio, G, Ragusa, MA: Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients. J. Funct. Anal. **112**, 241-256 (1993)
22. Liu, LZ: Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators. Acta Math. Sci. **25**(1), 89-94 (2005)
23. Mizuhara, T: Boundedness of some classical operators on generalized Morrey spaces. In: Harmonic Analysis (Sendai, Japan, 1990). ICM-90 Satellite Conference Proceedings, pp. 183-189 (1991)
24. Lin, Y: Strongly singular Calderón-Zygmund operator and commutator on Morrey type spaces. Acta Math. Sin. **23**, 2097-2110 (2007)
25. Coifman, R, Rochberg, R: Another characterization of *BMO*. Proc. Am. Math. Soc. **79**, 249-254 (1980)

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