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# Pointwise approximation for a type of Bernstein-Durrmeyer operators

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**Abstract**
We give the direct and inverse approximation theorems for a new type of Bernstein-Durrmeyer operators with the modulus of smoothness.

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**Keywords:** Bernstein-Durrmeyer type operator; modulus of smoothness; $K$-functional; direct and inverse approximation theorems

## 1 Introduction

Durrmeyer [1] introduced the integral modification of the well-known Bernstein polynomials given by

$$D_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)\,dt,$$

where $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$. Derriennic [2] established some direct results in ordinary and simultaneous approximation for Durrmeyer operators. Then Durrmeyer type operators were studied widely [3–5]. Recently Gupta *et al.* [6] considered a family of Durrmeyer type operators:

$$P_{n,m}(f,x) = \begin{cases} n\sum_{k=1}^{n} p_{n,k}(x)\int_0^1 p_{n-1,k-1}(t)f(t)\,dt + p_{n,0}(x)f(0), & m=0; \\ \frac{n^{\underline{m}}}{(n+m-1)^{\underline{m-1}}}\sum_{k=0}^{n-m} p_{n-m,k}(x)\int_0^1 p_{n+m-1,k+m-1}(t)f(t)\,dt, & m>0, \end{cases}$$

where $m,n \in N_0$ with $m \le n$ and for any $a,b \in N_0$, $a^{\underline{b}} = a(a-1)\cdots(a-b+1)$, $a^{\underline{0}} = 1$ is the falling factorial; and we get the rate of convergence for these operators for a function having derivatives of bounded variation and the result in the simultaneous approximation. In the present note our main aim is to get the direct and inverse approximation theorem for this type of operators. Here we shall utilize modulus of smoothness and $K$-functional as the tools, which are defined by [7]

$$\omega_{\varphi^\lambda}^{2r}(f,t) = \sup_{0<h\le t}\ \sup_{x\pm hr\varphi^\lambda(x)\in[0,1]} \left|\triangle_{h\varphi^\lambda(x)}^{2r} f(x)\right|,$$

$$K_{\varphi^\lambda}\left(f,t^{2r}\right) = \inf_{g^{(2r-1)}\in A.C._{\mathrm{loc}}} \left\{\|f-g\| + t^{2r}\left\|\varphi^{2r\lambda}g^{(2r)}\right\|\right\},$$

where $\varphi(x) = \sqrt{x(1-x)}$, $0 \le \lambda \le 1$, $r \in N$. It is well known that $\omega_{\varphi^\lambda}^{2r}(f,t) \sim K_{\varphi^\lambda}(f,t^{2r})$, where $a \sim b$ means that there exists some constant $C > 0$ such that $C^{-1}b \le a \le Cb$. We denote $M_{n,m}(f,x) = \frac{(m+n)^{\underline{m}}}{n^{\underline{m}}} P_{n,m}(f,x)$ and state our main results as follows.

**Theorem 1** *For $f \in C[0,1]$, $0 < \lambda < 1$, $\varphi(x) = \sqrt{x(1-x)}$, one has*

$$\left|M_{n,m}(f,x) - f(x)\right| \le C\left(\omega_{\varphi^\lambda}^2\left(f, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\|f\|\right).$$

**Theorem 2** *Let $f \in C[0,1]$, $0 \le \lambda \le 1$, $\varphi(x) = \sqrt{x(1-x)}$, $\delta_n(x) = \max\{\varphi(x), \frac{1}{\sqrt{n}}\}$, $r \in N$, $0 < \alpha < 2r$, for $m > 0$, and from*

$$\left|M_{n,m}(f,x) - f(x)\right| = O\left(\left(n^{\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha\right),$$

*we get $\omega_{\varphi^\lambda}^{2r}(f,t) = O(t^\alpha)$.*

Throughout this paper $\|\cdot\| = \|\cdot\|_\infty$ and $C$ denotes a positive constant independent of $n$ and $x$ not necessarily the same at each occurrence.

## 2 Lemmas

To prove the above theorems we need the following lemmas. First we define the moments, for any $s \in N_0$, $T_{n,m,s}(x) = M_{n,m}((t-x)^s, x)$.

**Lemma 3** ([6]) *The following claims hold.*

(1) *For any $s, m \in N_0$, $x \in [0,1]$, the following recurrence relation is satisfied:*

$$\begin{aligned}(n+m+s+1)T_{n,m,s+1}(x) &= x(1-x)\left[T'_{n,m,s}(x) + 2sT_{n,m,s-1}(x)\right] \\ &\quad + \left[(s+m) - x(1+2s+2m)\right]T_{n,m,s}(x),\end{aligned}$$

*where for $s = 0$, we denote $T_{n,m,-1}(x) = 0$.*

(2) *For any $m \in N_0$ and $x \in [0,1]$,*

$$\begin{aligned}&T_{n,m,0}(x) = 1, \qquad T_{n,m,1}(x) = \frac{m - x(1+2m)}{n+m+1}, \\ &T_{n,m,2}(x) = \frac{2nx(1-x) + m(1+m) - 2mx(2m+3) + 2x^2(2m^2+4m+1)}{(n+m+1)(n+m+2)}.\end{aligned}$$

(3) *For any $s, m \in N_0$, $x \in [0,1]$, $T_{n,m,s}(x) = O(n^{-[(s+1)/2]})$.*

**Remark** For $n$ sufficiently large and $x \in (0,1)$, it can be seen from Lemma 3 that

$$\frac{x(1-x)}{n} \le T_{n,m,2}(x) \le \frac{Cx(1-x)}{n}, \tag{2.1}$$

for any $C > 2$.

**Lemma 4** *For $f(x) \in C[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, $\delta_n(x) = \max\{\varphi(x), \frac{1}{\sqrt{n}}\}$, $0 \le \lambda \le 1$, $r \in N$, $m > 0$, we have*

$$\left|\varphi^{2r\lambda}(x)M_{n,m}^{(2r)}(f,x)\right| \le Cn^r\delta_n^{2r(\lambda-1)}\|f\|. \tag{2.2}$$

*Proof* To complete the proof we consider two cases of $x \in E_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ and $x \in E_n^c = [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1]$.

For $x \in E_n^c$, $\varphi^2(x) \le \frac{C}{n}$, $\delta_n^2(x) \sim \frac{1}{n}$. Using

$$M_{n,m}^{(2r)}(f,x) = \frac{(n-m)!}{(n-m-2r)!} \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \triangle^{2r} a_k(n),$$

where $a_k(n) = (n+m) \int_0^1 p_{n+m-1,k+m-1}(t) f(t)\, dt$, $\triangle a_k(n) = a_{k+1}(n) - a_k(n)$, $\triangle^r a_k(n) = \triangle(\triangle^{r-1} a_k(n))$; and $|\triangle^{2r} a_k(n)| \le C\|f\|$, $\frac{(n-m)!}{(n-m-2r)!} \le (n-m)^{2r} < n^{2r}$, one has

$$\left|\varphi^{2r\lambda}(x) M_{n,m}^{(2r)}(f,x)\right| \le Cn^{-r\lambda} n^{2r} \|f\| \le Cn^r \delta_n^{2r(\lambda-1)} \|f\|. \tag{2.3}$$

For $x \in E_n$, $\delta_n(x) \sim \varphi(x)$. From [7] we have

$$M_{n,m}^{(2r)}(f,x) = \varphi^{-4r}(x) \sum_{i=0}^{2r} Q_i^B(x,n) n^i \sum_{k=0}^{n-m} p_{n-m,k}(x) \left(\frac{k}{n-m} - x\right)^i a_k(n),$$

where $Q_i^B(x,n)$ a polynomial in $n\varphi^2(x)$ of degree $[(2r-i)/2]$ with non-constant bounded coefficients. Therefore,

$$\left|\varphi^{-4r}(x) Q_i^B(x,n) n^i\right| \le C\left(\frac{n}{\varphi^2(x)}\right)^{r+\frac{i}{2}}.$$

By Holder's inequality we get

$$\sum_{k=0}^{n-m} p_{n-m,k}(x)\left(\frac{k}{n-m} - x\right)^i \le \left(\sum_{k=0}^{n-m} p_{n-m,k}(x)\left(\frac{k}{n-m} - x\right)^{2i}\right)^{\frac{1}{2}} \le Cn^{-\frac{i}{2}} \varphi^i(x).$$

Consequently $|\varphi^{2r}(x) M_{n,m}^{(2r)}(f,x)| \le Cn^r \|f\|$, hence

$$\left|\varphi^{2r\lambda}(x) M_{n,m}^{(2r)}(f,x)\right| = \varphi^{2r(\lambda-1)}(x) \left|\varphi^{2r}(x) M_{n,m}^{(2r)}(f,x)\right| \le Cn^r \delta_n^{2r(\lambda-1)} \|f\|. \tag{2.4}$$

From (2.3) and (2.4), (2.2) holds. □

**Lemma 5** *For* $f^{(2r-1)}(x) \in A.C._{\text{loc}}$, $\|\varphi^{2r\lambda} f^{(2r)}\| < \infty$, $m > 0$, *we have*

$$\left|\varphi^{2r\lambda}(x) M_{n,m}^{(2r)}(f,x)\right| \le C\left\|\varphi^{2r\lambda} f^{(2r)}\right\|. \tag{2.5}$$

*Proof* From $p_{n-m,k}^{(2r)}(x) = \frac{(n-m)!}{(n-m-2r)!} \sum_{i=0}^{2r} (-1)^i \binom{2r}{i} p_{n-m-2r,k-(2r-i)}(x)$, we have

$$\begin{aligned} &M_{n,m}^{(2r)}(f,x) \\ &\quad = (n-m) \sum_{k=0}^{n-m} p_{n-m,k}^{(2r)}(x) \int_0^1 p_{n+m-1,k+m-1}(t) f(t)\, dt \\ &\quad = \frac{(n-m)(n-m)!}{(n-m-2r)!} \sum_{k=0}^{n-m-2r} \sum_{i=0}^{2r} (-1)^i \binom{2r}{i} p_{n-m-2r,k-(2r-i)}(x) \int_0^1 p_{n+m-1,k+m-1}(t) f(t)\, dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{(n-m)(n-m)!}{(n-m-2r)!} \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \int_0^1 \sum_{i=0}^{2r} (-1)^i \binom{2r}{i} p_{n+m-1,k+m+2r-i-1}(t) f(t)\,dt \\
&= \frac{(n-m)(n-m)!}{(n-m-2r)!} \frac{(n+m-1)!}{(n+m-1+2r)!} \\
&\quad \times \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \int_0^1 p_{n+m+2r-1,k+m+2r-1}(t) f^{(2r)}(t)\,dt.
\end{aligned}
$$

Let $I = (n-m)\sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) |\int_0^1 p_{n+m+2r-1,k+m+2r-1}(t) f^{(2r)}(t)\,dt|$. For $0 \le \lambda \le 1$ one has

$$
\begin{aligned}
&\varphi^{2r\lambda}(x) I \\
&\quad \le \|\varphi^{2r\lambda} f^{(2r)}\| \varphi^{2r\lambda}(x) \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x)(n-m) \int_0^1 p_{n+m+2r-1,k+m+2r-1}(t) \varphi^{-2r\lambda}(t)\,dt \\
&\quad \le \|\varphi^{2r\lambda} f^{(2r)}\| \left( \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \varphi^{2r}(x)(n-m) \int_0^1 p_{n+m+2r-1,k+m+2r-1}(t) \varphi^{-2r}(t)\,dt \right)^{\lambda}.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&p_{n-m-2r,k}(x)\varphi^{2r}(x) = \frac{(n-m-2r)!(k+r)!(n-m-k-r)!}{k!(n-m-2r-k)!(n-m)!} p_{n-m,k+r}(x) =: \alpha_{n,k} p_{n-m,k+r}(x), \\
&p_{n+m+2r-1,k+m+2r-1}(t)\varphi^{-2r}(t) \\
&\quad = \frac{(n+m+2r-1)!(k+m+r-1)!(n-k-r)!}{(k+m+2r-1)!(n-k)!(n+m-1)!} p_{n+m-1,k+m+r-1}(t) \\
&\quad =: \beta_{n,k} p_{n+m-1,k+m+r-1}(t)
\end{aligned}
$$

and $\alpha_{n,k}\beta_{n,k} \le C$, we get $\varphi^{2r\lambda}(x) I \le \|\varphi^{2r\lambda} f^{(2r)}\|$. This completes the proof of Lemma 5. □

**Lemma 6** ([8]) *For $0 < t < \frac{1}{16r}$, $\frac{rt}{2} < x < 1 - \frac{rt}{2}$, $0 \le \beta \le 2r$, we have*

$$
\int \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} \varphi^{-\beta}(x + u_1 + \cdots + u_{2r})\,du_1 \cdots du_{2r} \le C t^{2r} \varphi^{-\beta}(x).
$$

## 3 Proof of the theorems

In this section we will give the proof of Theorem 1 and Theorem 2.

*Proof of Theorem* 1 By the definition of $K_{\varphi^\lambda}(f, t^{2r})$ and the equivalence between $\omega_{\varphi^\lambda}^{2r}(f,t)$ and $K_{\varphi^\lambda}(f, t^{2r})$, for the fixed $n$ and $x$, we can choose $g = g_{n,x}$ such that

$$
\|f-g\| + \frac{1}{n}\|\varphi^{2\lambda} g''\| \le C\omega_{\varphi^\lambda}^2\left(f, \frac{1}{\sqrt{n}}\right). \tag{3.1}
$$

We know that

$$
|M_{n,m}(f,x) - f(x)| \le 2\|f-g\| + |M_{n,m}(g,x) - g(x)|, \tag{3.2}
$$

and we have to estimate the second term on the right side of (3.2). By Taylor's formula, $g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du$, and Lemma 3 we have

$$\begin{aligned}\left|M_{n,m}(g,x) - g(x)\right| &\le \left|g'(x)\right|\left|M_{n,m}\big((t-x),x\big)\right| + \left|M_{n,m}\left(\int_x^t (t-u)g''(u)\,du, x\right)\right| \\ &\le C\frac{1}{n}\left|g'(x)\right| + \left|M_{n,m}\left(\int_x^t (t-u)g''(u)\,du, x\right)\right| \\ &=: I_1 + I_2. \end{aligned} \tag{3.3}$$

We consider $I_1$ first. For $0 \le x \le \frac{1}{2}$,

$$\left|g'(x) - g'\left(\frac{1}{2}\right)\right| = \left|\int_x^{\frac{1}{2}} g''(u)\,du\right| \le \left\|x^\lambda g''\right\| \int_x^{\frac{1}{2}} \frac{1}{u^\lambda}\,du \le C\left\|\varphi^{2\lambda} g''\right\|,$$

together with $|g'(\frac{1}{2})| \le C(\|g''\|_{L_\infty[\frac{1}{4},\frac{3}{4}]} + \|g\|_{L_\infty[\frac{1}{4},\frac{3}{4}]}) \le C(\|\varphi^{2\lambda}g''\| + \|g\|)$, one has

$$\left|g'(x)\right| \le C\|g\| + \left\|\varphi^{2\lambda}g''\right\|. \tag{3.4}$$

It is similar for $\frac{1}{2} < x \le 1$.

Now we address $I_2$. By the process of (9.6.1) in [7]

$$\left|R_{2r}(f,u,x)\right| \le \frac{|u-x|^{2r-1}}{\varphi^{2r}(x)}\left|\int_u^x \varphi^{2r}(v)\left|f^{(2r)}(v)\right|dv\right|,$$

we get $\frac{|t-u|}{\varphi^{2\lambda}(u)} \le \frac{|t-x|}{\varphi^{2\lambda}(x)}$, and combining with (2.1) we deduce

$$I_2 \le \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)} M_{n,m}\big((t-x)^2,x\big) \le C\frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)}\frac{\varphi^2(x)}{n} \le C\frac{1}{n}\left\|\varphi^{2\lambda}g''\right\|. \tag{3.5}$$

By (3.2)-(3.5), we complete the proof of Theorem 1. □

*Proof of Theorem* 2 For convenience let $\gamma_{n,\lambda}(x) = n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)$. If $M_{n,m}(f,x) - f(x) = O(\gamma_{n,\lambda}^\alpha(x))$, for every $n : n > 2r$, we have

$$\begin{aligned}\left|\triangle_{t\varphi^\lambda(x)}^{2r} f(x)\right| &\le \left|\triangle_{t\varphi^\lambda(x)}^{2r}\big(M_{n,m}(f,x) - f(x)\big)\right| + \left|\triangle_{t\varphi^\lambda(x)}^{2r} M_{n,m}(f,x)\right| \\ &\le C\gamma_{n,\lambda}^\alpha(x) + \int \cdots \int_{-\frac{t\varphi^\lambda(x)}{2}}^{\frac{t\varphi^\lambda(x)}{2}} \left|M_{n,m}^{(2r)}\left(f, x + \sum_{j=1}^{2r} u_j\right)\right| du_1 \cdots du_{2r} \\ &\le C\gamma_{n,\lambda}^\alpha(x) + \int \cdots \int_{-\frac{t\varphi^\lambda(x)}{2}}^{\frac{t\varphi^\lambda(x)}{2}} \left|M_{n,m}^{(2r)}\left(f - g, x + \sum_{j=1}^{2r} u_j\right)\right| du_1 \cdots du_{2r} \\ &\quad + \int \cdots \int_{-\frac{t\varphi^\lambda(x)}{2}}^{\frac{t\varphi^\lambda(x)}{2}} \left|M_{n,m}^{(2r)}\left(g, x + \sum_{j=1}^{2r} u_j\right)\right| du_1 \cdots du_{2r} \\ &:= C\gamma_{n,\lambda}^\alpha(x) + J_1 + J_2. \end{aligned} \tag{3.6}$$

Combining Lemma 4, Lemma 5, and Lemma 6, we have

$$J_1 \le Ct^{2r}\gamma_{n,\lambda}^{-2r}(x)\|f-g\|, \tag{3.7}$$

$$J_2 \le Ct^{2r}\left\|\varphi^{2r\lambda}g^{(2r)}\right\|. \tag{3.8}$$

Utilizing (3.6), (3.7), and (3.8), choosing the appropriate $g$, we obtain

$$\left|\triangle_{t\varphi^\lambda(x)}^{2r}f(x)\right| \le C\left(\gamma_{n,\lambda}^{\alpha}(x) + t^{2r}\gamma_{n,\lambda}^{-2r}(x)\omega_{\varphi^\lambda}^{2r}\left(f,\gamma_{n,\lambda}(x)\right)\right).$$

For every fixed $h: 0 < h < \frac{1}{16r}$ and every $x: x \ge rt$, we can choose $n$ such that $\gamma_{n,\lambda}(x) \le h < 2\gamma_{n,\lambda}(x)$. Then

$$\left|\triangle_{t\varphi^\lambda(x)}^{2r}f(x)\right| \le C\left(h^\alpha + \left(\frac{t}{h}\right)^{2r}\omega_{\varphi^\lambda}^{2r}(f,h)\right).$$

So,

$$\omega_{\varphi^\lambda}^{2r}(f,t) \le C\left(h^\alpha + \left(\frac{t}{h}\right)^{2r}\omega_{\varphi^\lambda}^{2r}(f,h)\right),$$

which yields the assertion of Theorem 2 by the Berens-Lorentz lemma. □

**Competing interests**
The author declares that they have no competing interests.

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