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Bernstein-Jackson-type inequalities and Besov spaces associated with unbounded operators

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Abstract
Besov-type interpolation spaces and appropriate Bernstein-Jackson inequalities, generated by unbounded linear operators in a Banach space, are considered. In the case of the operator of differentiation these spaces and inequalities exactly coincide with the classical ones. Inequalities are applied to a best approximation problem in a Banach space, particularly, to spectral approximations of regular elliptic operators.
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1 Introduction and preliminaries

The classical Jackson and Bernstein inequalities express a relation between smoothness modules of functions and properties of their best approximations by polynomials or entire functions of exponential type that can be characterized with the help of Besov norms [1, Sections 1.5, 7.2]. These results are extended to approximations of smooth functions by wavelets (see *e.g.* [2–4]), and to approximations of linear operators in Banach spaces by operators with finite ranks [5], and other similar approximations.

The motivation of our work is to extend the Bernstein-Jackson inequalities to cases of best spectral approximations in a Banach space. An analog of Bernstein-Jackson inequalities in the case of approximations in the space $L_p(G)$ on a Lie group G by spectral subspaces $\mathcal{M}_p(h) = \{f \in L_p(G) : E(\lambda)f = f \text{ if } \lambda \geq h > 0\}$ of the group sublaplacian Δ_G, where $\Delta_G = \int_0^\infty \lambda \, dE(\lambda)$ is its spectral resolution, is established in [6, 7]. Spectral subspaces are analogous subspaces of entire functions of exponential type. The appropriate Besov space is characterized by the functional of best approximation $E_p(h,f) = \inf_{g\in\mathcal{M}_p(h)} \|f - g\|_{L_p(G)}$.

This approach is a prototype of our generalizations. We consider a closed operator A in a Banach space $\mathfrak{X}$ instead of Δ_G and replace the spectral subspaces $\mathcal{M}_p(h)$ by invariant subspaces of exponential type entire vectors of A. Note that similar subspaces of exponential type entire vectors have appeared in [8–11].

Our goal is to investigate a best approximation problem by invariant subspaces of exponential type entire vectors of an arbitrary unbounded closed linear operator A in a Banach space $\mathfrak{X}$. As a basic tool, we use an analog of approximate Bernstein and Jackson inequalities and an abstract quasi-normed Besov-type interpolation space $\mathcal{B}^{\alpha}_{p,\tau}(A)$, associated with exponential type entire vectors of A, which sharply characterizes the behavior of the best spectral approximation.

Using the quasi-norm of $\mathcal{B}^{\alpha}_{p,\tau}(A)$, the main result is formulated in Theorem 5 as two inequalities, estimating the minimal distance from a given element to a subspace of exponential type vectors with fixed indices. In the case of the operator of differentiation in $L_p(\mathbb{R})$, the spaces $\mathcal{B}^{\alpha}_{p,\tau}(A)$ coincide with the classical Besov-type spaces (Theorem 7) and the estimations reduce to the known Bernstein and Jackson inequalities (Theorem 8). A new application to spectral approximations of elliptic operators is shown in Section 6 (see also Theorem 6).

In a Banach complex space $(\mathfrak{X}, \|\cdot\|)$ we consider a closed unbounded linear operator with the norm dense domain $\mathcal{C}^1(A)$,

$$A\colon \mathcal{C}^1(A) \subset \mathfrak{X} \longrightarrow \mathfrak{X}.$$

Let $\mathcal{C}^{k+1}(A) = \{x \in \mathcal{C}^k(A)\colon A^k x \in \mathcal{C}^k(A)\}$ and $\mathcal{C}^\infty(A) = \bigcap\{\mathcal{C}^k(A)\colon k \in \mathbb{N}\}$. We call the elements

$$\mathcal{E}(A) = \bigcup_{t>0} \bigcap_{k\in\mathbb{Z}_+} \{x \in \mathcal{C}^\infty(A)\colon \|A^k x\| \le ct^k\}$$

exponential type entire vectors of A, where the constant $c = c(x, A)$ is independent on $k \in \mathbb{Z}_+$ and A^0 is the unit operator on $\mathfrak{X}$. Clearly, every exponential type entire vector also is an analytic vector of A in the well-known Nelson sense.

Throughout this article we assume that the norm density condition $\overline{\mathcal{E}(A)} = \mathfrak{X}$ holds and that the operators A^k ($k \in \mathbb{N}$) are closed in $\mathfrak{X}$. In many important cases for applications these assumptions hold. Particularly, we have the cases:

(i) if A has a real spectrum and $\forall\varepsilon > 0$ the integral $\int_0^\varepsilon \ln\ln M(r)\,dr$ with $M(r) = \sup_{|\Im\lambda|\ge r} \|(\lambda - A)^{-1}\|$ is convergent (see [8]);

(ii) if A generates we have an one-parameter group e^{tA} with the convergent integral $\int_{\mathbb{R}} \|e^{tA}\|(1 + |t|^2)^{-1}\,dt$ [8];

(iii) if A generates we have a bounded C_0-group e^{tA} on $\mathfrak{X}$ (see [10]).

If $\mathfrak{X} = L_p(\mathbb{R})$ ($1 \le p \le \infty$) and $A = D$ is the differentiation operator on $\mathbb{R}$ then $\mathcal{E}(A)$ is the space of entire functions of exponential type, belonging to $L_p(\mathbb{R})$. In this case the inequality $\|A^k x\| \le ct^k$ reduces to the Bernstein inequality. If the spectrum $\sigma(A)$ of an operator A is discrete then the subspace $\mathcal{E}(A)$ exactly coincides with the linear span of all its spectral subspaces in $\mathfrak{X}$ (see [12]).

Recall the real interpolation method (for more details see [1, 13]). Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be a quasi-normed complex spaces. Given the pair positive numbers $\{0 < \vartheta < 1, 1 \le p \le \infty\}$ or $\{0 < \vartheta \le 1, p = \infty\}$ the interpolation vector space can be defined as the set $(X, Y)_{\vartheta,p} = \{a \in X + Y\colon |a|_{(X,Y)_{\vartheta,p}} < \infty\}$ endowed with the quasi-norm

$$|a|_{(X,Y)_{\vartheta,p}} = \begin{cases} (\int_0^\infty [\tau^{-\vartheta} K(\tau, a; X, Y)]^p \frac{d\tau}{\tau})^{1/p}, & p < \infty, \\ \sup_{0<\tau<\infty} \tau^{-\vartheta} K(\tau, a; X, Y), & p = \infty, \end{cases} \tag{1}$$

where $K(\tau, a; X, Y) := \inf_{a=x+y}(|x|_X + \tau|y|_Y)$ is called a K-functional [1, Section 3.11]. Clearly, $X \cap Y \subset (X, Y)_{\vartheta,p} \subset X + Y$.

2 Scales of invariant subspaces

Let $0 < t < \infty$ and $1 \le p \le \infty$. Consider the mapping

$$\mathcal{C}^\infty(A) \ni x \longrightarrow \{(A/t)^k x \colon k \in \mathbb{Z}_+\}$$

which image is formed by sequences of elements of a Banach space $\mathfrak{X}$. For any pair of indices t, p we define the normed spaces $\mathcal{E}_p^t(A) = \{x \in \mathcal{C}^\infty(A) \colon \|x\|_{\mathcal{E}_p^t} < \infty\}$, where

$$\|x\|_{\mathcal{E}_p^t} = \begin{cases} (\sum_{k\in\mathbb{Z}_+} \|(A/t)^k x\|^p)^{1/p}, & 1 \le p < \infty, \\ \sup_{k\in\mathbb{Z}_+} \|(A/t)^k x\|, & p = \infty. \end{cases}$$

Theorem 1 (i) *The contractive inclusions* $\mathcal{E}_p^t(A) \looparrowright \mathcal{E}_p^\tau(A) \looparrowright \mathcal{E}_\infty^\tau(A) \looparrowright \mathfrak{X}$ *with* $\tau > t$ *hold.*

(ii) *Every space* $\mathcal{E}_p^t(A)$ *is* A*-invariant and the restriction* $A|_{\mathcal{E}_p^t}$ *is a bounded operator over* $\mathcal{E}_p^t(A)$ *with the norm* $\|A|_{\mathcal{E}_p^t}\|_{\mathcal{E}_p^t} \le t$.

(iii) *The spectrum of* A *has the property* $\sigma(A|_{\mathcal{E}_p^t}) \subset \sigma(A)$.

(iv) *Every space* $\mathcal{E}_p^t(A)$ *is complete.*

Proof (i) The inequalities $\|x\| \le \|x\|_{\mathcal{E}_\infty^t}$ and $\|x\|_{\mathcal{E}_\infty^t}^p \le \|x\|_{\mathcal{E}_p^t}^p$ yield the contractive inclusions $\mathcal{E}_\infty^t(A) \looparrowright \mathfrak{X}$ and $\mathcal{E}_p^t(A) \looparrowright \mathcal{E}_\infty^t(A)$, respectively. If $x \in \mathcal{E}_p^t(A)$ then $\|A^k x\|^p \le t^{pk}\|x\|_{\mathcal{E}_p^t}^p$ and $\|A^k x\|^{p/k} \le t^p \|x\|_{\mathcal{E}_p^t}^{p/k}$ for all $k \in \mathbb{Z}_+$. It follows that $\limsup_{k\to\infty} \|A^k x\|^{p/k} \le t^p$. Therefore, for any $\tau > t$ the series $\|x\|_{\mathcal{E}_p^\tau}^p = \sum_k \|(A/\tau)^k x\|^p$ is convergent. As a result, $x \in \mathcal{E}_p^\tau(A)$. Moreover, $\|x\|_{\mathcal{E}_p^\tau}^p \le \|x\|_{\mathcal{E}_p^t}^p$ for all $x \in \mathcal{E}_p^t(A)$ and $\tau > t$.

(ii) Using $A(A/t)^k x = t(A/t)^{k+1}x$, we obtain $\|Ax\|_{\mathcal{E}_\infty^t} \le t\|x\|_{\mathcal{E}_\infty^t}$ and $\|Ax\|_{\mathcal{E}_p^t}^p \le t^p\|x\|_{\mathcal{E}_p^t}^p$ when $1 \le p < \infty$.

(iii) For any $\lambda \in \rho(A)$ and $x \in \mathcal{E}_p^t(A)$ the equality $(A/t)^k(\lambda - A)^{-1}x = (\lambda - A)^{-1}(A/t)^k x$ holds. It follows that $\|(\lambda - A)^{-1}x\|_{\mathcal{E}_p^t} \le \|(\lambda - A)^{-1}\|\|x\|_{\mathcal{E}_p^t}$ for all $x \in \mathcal{E}_p^t$. Hence, λ belongs to the resolvent set $\rho(A|_{\mathcal{E}_p^t})$.

(iv) Let us use the inequality $\|x\|_{\mathcal{E}_p^t} \ge \|(A/t)^k x\|$ with $x \in \mathcal{E}_p^t(A)$, $k \in \mathbb{Z}_+$. It follows that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{E}_p^t(A)$ then $(x_n)_{n\in\mathbb{N}}$ and $\{(A/t)^k x_n \colon n \in \mathbb{N}\}$ are Cauchy sequences in the space $\mathfrak{X}$ for all $k \in \mathbb{Z}_+$. The completeness of $\mathfrak{X}$ implies that there exist $x, y \in \mathfrak{X}$ such that $x_n \to x$ and $(A/t)^k x_n \to y$ by norm of $\mathfrak{X}$. The graph of A^k is closed in $\mathfrak{X} \times \mathfrak{X}$, therefore $y = (A/t)^k x$ and $x \in \mathcal{C}^k(A)$. It is true for all $k \in \mathbb{Z}_+$, so $x \in \mathcal{C}^\infty(A)$ and $(A/t)^k x_n \to (A/t)^k x$ by norm of $\mathfrak{X}$ for all $k \in \mathbb{Z}_+$.

We reason standardly: $\forall\varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N} : \|x_n - x_m\|_{\mathcal{E}_p^t} < \varepsilon$, $\forall n, m \ge n_\varepsilon$. It follows that $\|(A/t)^k(x_n - x_m)\| < \varepsilon$, $\forall n, m \ge n_\varepsilon$, $k \in \mathbb{Z}_+$. So, $\forall k \in \mathbb{Z}_+$ $\exists m_{\varepsilon,k} \ge n_\varepsilon : \|(A/t)^k(x_m - x_n)\| < \varepsilon/2^k$ and $\|(A/t)^k(x_m - x)\| < \varepsilon/2^k$ for $m \ge m_{\varepsilon,k}$. Hence, from $\|(A/t)^k x\| \le \|(A/t)^k x_{n_\varepsilon}\| + \|(A/t)^k(x_m - x_{n_\varepsilon})\| + \|(A/t)^k(x_m - x)\|$ it follows that $\|(A/t)^k x\| \le \|(A/t)^k x_{n_\varepsilon}\| + 2\varepsilon/2^k$ for all $k \in \mathbb{Z}_+$. We may use the fact that the scalar sequences $a = (a_k)_{k\in\mathbb{N}}$ with $a_k = \|(A/t)^k x_{n_\varepsilon}\|$ and $b = (2^{-k})_{k\in\mathbb{N}}$ belong to the Banach space ℓ_p. Calculating ℓ_p-norms of these elements and applying the previous inequality, we obtain

$$\|x\|_{\mathcal{E}_p^t} \le \|a + b\|_{\ell_p} \le \|a\|_{\ell_p} + \|b\|_{\ell_p} = \|x_{n_\varepsilon}\|_{\mathcal{E}_p^t} + 4\varepsilon.$$

Hence, $x \in \mathcal{E}_p^t(A)$. Moreover, $\|(A/t)^k(x_n - x)\| \le \|(A/t)^k(x_{m_{\varepsilon,k}} - x)\| + \|(A/t)^k(x_n - x_{m_{\varepsilon,k}})\|$, where in this inequality all sequences by k belong to ℓ_p. We obtain $\|x_n - x\|_{\mathcal{E}_p^t} \le 4\varepsilon$, $n \ge n_\varepsilon$. So, $\mathcal{E}_p^t(A)$ is complete. $\square$

On the subspace $\mathcal{E}_p(A) := \bigcup_{t>0} \mathcal{E}_p^t(A)$ we define the function

$$|x|_p := \|x\| + \inf\{t > 0 : x \in \mathcal{E}_p^t(A)\}. \tag{2}$$

Theorem 2 (i) *For every p ($1 \le p \le \infty$) the embedding $\mathcal{E}_\infty^t(A) \subset \mathcal{E}_p^\tau(A)$ with $\tau > t$ and the equality $\mathcal{E}(A) = \mathcal{E}_p(A)$ hold.*

(ii) *The function* (2) *is a quasi-norm satisfying the inequality $|x+y|_p \le |x|_p + |y|_p$ for all $x, y \in \mathcal{E}(A)$. Moreover, the contractive embedding $\mathcal{E}_p(A) \hookrightarrow \mathfrak{X}$ is true.*

Proof (i) Let $x \in \mathcal{E}_\infty^t(A)$. We reason similarly to the above. For every $k \in \mathbb{Z}_+$ we have $\|A^k x\|^p \le t^{pk} \|x\|_{\mathcal{E}_\infty^t}^p$. So $\|A^k x\|^{p/k} \le t^p \|x\|_{\mathcal{E}_\infty^t}^{p/k}$. It follows that $\limsup_{k\to\infty} \|A^k x\|^{p/k} \le t^p$. Therefore, for every $\tau > t$ the series $\|x\|_{\mathcal{E}_p^\tau}^p = \sum_k \|(A/\tau)^k x\|^p$ is convergent, *i.e.* $x \in \mathcal{E}_p^\tau(A)$. Hence, $\mathcal{E}_\infty^t(A) \subset \mathcal{E}_p^\tau(A)$.

The constant c in the definition $\mathcal{E}(A)$ is independent on the index $k \in \mathbb{Z}_+$. It yields the equality $\mathcal{E}_\infty^t(A) = \bigcap_{k\in\mathbb{Z}_+} \{x \in \mathcal{C}^\infty(A) : \|A^k x\| \le ct^k\}$. Hence, $\mathcal{E}(A) = \bigcup_{t>0} \mathcal{E}_\infty^t(A)$. Therefore, the embedding $\mathcal{E}_\infty^t(A) \subset \mathcal{E}_p^\tau(A)$ from Theorem 2(i) yields the embedding $\mathcal{E}(A) \subset \mathcal{E}_p(A)$ for any index p. The inverse embedding $\mathcal{E}_p(A) \subset \mathcal{E}(A)$ follows from Theorem 1(i).

(ii) Use that $\mathcal{E}(A) = \mathcal{E}_p(A)$ and set $r(x) = \inf\{t > 0 : x \in \mathcal{E}_p^t(A)\}$. For each $x, y \in \mathcal{E}(A)$ and $\varepsilon > 0$ the values $\|x\|_{\mathcal{E}_p^{r(x)+\varepsilon}}$, $\|y\|_{\mathcal{E}_p^{r(y)+\varepsilon}}$ are finite and the inequalities

$$\|x+y\|_{\mathcal{E}_p^{r+\varepsilon}} \le \|x\|_{\mathcal{E}_p^{r+\varepsilon}} + \|y\|_{\mathcal{E}_p^{r+\varepsilon}} \le \|x\|_{\mathcal{E}_p^{r(x)+\varepsilon}} + \|y\|_{\mathcal{E}_p^{r(y)+\varepsilon}}$$

with $r = \max\{r(x), r(y)\}$ hold. It follows that $r(x+y) \le r + \varepsilon \le r(x) + r(y) + \varepsilon$. Since ε is arbitrary, $r(x+y) \le r(x) + r(y)$ for all $x, y \in \mathcal{E}(A)$. Evidently, $r(x) = r(-x)$ for all $x \in \mathcal{E}(A)$. So $|\cdot|_p$ is a quasi-norm. The contractility of $\mathcal{E}_p(A) \subset \mathfrak{X}$ is a direct consequence of (2). □

3 Besov-type scales of approximation spaces

Let $1 \le p \le \infty$. In what follows we denote by $\mathcal{E}_p(A)$ the subspace $\mathcal{E}(A)$ endowed with the quasi-norm $|\cdot|_p$. Consider the auxiliary functional

$$E_p(t, x) = \inf\{\|x - x^0\| : x^0 \in \mathcal{E}_p(A), |x^0|_p < t\}, \quad x \in \mathfrak{X}.$$

Given a pair of numbers $\{0 < \alpha < \infty, 0 < \tau \le \infty\}$ and $\{0 \le \alpha < \infty, \tau = \infty\}$ we consider the scale of spaces $\mathcal{B}_{p,\tau}^\alpha(A) = \{x \in \mathfrak{X} : |x|_{\mathcal{B}_{p,\tau}^\alpha} < \infty\}$,

$$|x|_{\mathcal{B}_{p,\tau}^\alpha} = \begin{cases} (\int_0^\infty [t^\alpha E_p(t,x)]^\tau \frac{dt}{t})^{1/\tau}, & 0 < \tau < \infty, \\ \sup_{t>0} t^\alpha E_p(t,x), & \tau = \infty, \end{cases}$$

where by [1, Lemma 7.1.6] the function $|x|_{\mathcal{B}_{p,\tau}^\alpha}$ is a quasi-norm on $\mathcal{B}_{p,\tau}^\alpha(A)$.

We can call the space $\mathcal{B}_{p,\tau}^\alpha(A)$ endowed with the quasi-norm $|\cdot|_{\mathcal{B}_{p,\tau}^\alpha}$ an abstract Besov-type space, determined by an operator A. The following properties of $\mathcal{B}_{p,\tau}^\alpha(A)$ are deduced from well-known interpolation theorems.

Theorem 3 (i) *If $[\mathcal{B}_{p,\tau}^\alpha(A)]^\vartheta$ is the space $\mathcal{B}_{p,\tau}^\alpha(A)$ endowed with the quasi-norm $|x|_{\mathcal{B}_{p,\tau}^\alpha}^\vartheta$ with $x \in \mathcal{B}_{p,\tau}^\alpha(A)$ then the equality*

$$[\mathcal{B}_{p,\tau}^\alpha(A)]^\vartheta = (\mathcal{E}_p(A), \mathfrak{X})_{\vartheta,g}, \quad \vartheta = 1/(\alpha+1), \tau = g\vartheta \tag{3}$$

(*up to a quasi-norm equivalence*) *holds.*

(ii) *The spaces* $\mathcal{B}^{\alpha}_{p,\tau}(A)$ *are complete.*

(iii) *If* $0<\tau<\infty$, $0<\vartheta<1$, $\alpha=(1-\vartheta)\alpha_0+\vartheta\alpha_1$ *with* $\alpha_0\neq\alpha_1$ *then*

$$\left(\mathcal{B}^{\alpha_0}_{p,\tau_0}(A),\mathcal{B}^{\alpha_1}_{p,\tau_1}(A)\right)_{\vartheta,\tau}=\mathcal{B}^{\alpha}_{p,\tau}(A) \tag{4}$$

and there exist constants c_1, c_2 *such that*

$$|x|_{\mathcal{B}^{\alpha}_{p,\tau}}\leq c_1|x|^{1-\vartheta}_{\mathcal{B}^{\alpha_0}_{p,\tau_0}}|x|^{\vartheta}_{\mathcal{B}^{\alpha_1}_{p,\tau_1}},\quad x\in\mathcal{B}^{\alpha_0}_{p,\tau_0}(A)\cap\mathcal{B}^{\alpha_1}_{p,\tau_1}(A), \tag{5}$$

$$K\left(t,x;\mathcal{B}^{\alpha_0}_{p,\tau_0},\mathcal{B}^{\alpha_1}_{p,\tau_1}\right)\leq c_2t^{\vartheta}|x|_{\mathcal{B}^{\alpha}_{p,\tau}},\quad x\in\mathcal{B}^{\alpha}_{p,\tau}(A),t>0. \tag{6}$$

(iv) *If* $0<\tau\leq\varrho<\infty$ *then the following continuous embedding holds*:

$$\mathcal{B}^{\alpha}_{p,\tau}(A)\hookrightarrow\mathcal{B}^{\alpha}_{p,\varrho}(A). \tag{7}$$

Proof (i) The equality (3) is a direct consequence of the definition and [1, Theorem 7.1.7].

(ii) To prove the completeness of $\mathcal{B}^{\alpha}_{p,\tau}(A)$, we equip the sum $\mathcal{E}_p+\mathfrak{X}$ (which is equal to $\mathfrak{X}$, because $\mathcal{E}_p\subset\mathfrak{X}$) with the norm $\|x\|_{\mathcal{E}_p+\mathfrak{X}}=\inf_{x=x^0+x^1}(|x^0|_p+\|x^1\|)$ with $x^0\in\mathcal{E}_p$ and $x^1\in\mathfrak{X}$. Since $|x|_p\geq\|x\|$, we have $\|x\|_{\mathcal{E}_p+\mathfrak{X}}=\|x\|$. Hence, the space $\mathfrak{X}$ with the norm $\|\cdot\|_{\mathcal{E}_p+\mathfrak{X}}$ is complete. Consequently, every series $\sum_{n\in\mathbb{N}}x_n$ with $x_n\in(\mathcal{E}_p(A),\mathfrak{X})_{\vartheta,g}$ such that $\sum_{n\in\mathbb{N}}\|x_n\|_{(\mathcal{E}_p(A),\mathfrak{X})_{\vartheta,g}}<\infty$ is convergent to an element $x\in\mathcal{E}_p+\mathfrak{X}=\mathfrak{X}$. Using the inequality $\|\sum_{n\in\mathbb{N}}x_n\|_{(\mathcal{E}_p(A),\mathfrak{X})_{\vartheta,g}}\leq\sum_{n\in\mathbb{N}}\|x_n\|_{(\mathcal{E}_p(A),\mathfrak{X})_{\vartheta,g}}$, we obtain $x\in(\mathcal{E}_p(A),\mathfrak{X})_{\vartheta,g}$. So $(\mathcal{E}_p(A),\mathfrak{X})_{\vartheta,g}$ is complete. The isomorphism (3) implies that the space $[\mathcal{B}^{\alpha}_{p,\tau}(A)]^{\vartheta}$ is complete. Thus, $\mathcal{B}^{\alpha}_{p,\tau}(A)$ is complete as well.

(iii) Applying the reiteration property of the real interpolation [1, Theorem 3.11.5] for the indices $\vartheta=(1-\eta)\vartheta_0+\eta\vartheta_1$ with $\vartheta_i=1/(\alpha_i+1)$ $(i=0,1)$, $\vartheta=1/(\alpha+1)$, $\tau=g\vartheta$ and $0<\eta<1$, we obtain

$$\left(\left[\mathcal{B}^{\alpha_0}_{p,\tau_0}(A)\right]^{\vartheta_0},\left[\mathcal{B}^{\alpha_1}_{p,\tau_1}(A)\right]^{\vartheta_1}\right)_{\eta,g}=\left[\mathcal{B}^{\alpha}_{p,\tau}(A)\right]^{\vartheta}. \tag{8}$$

Applying the interpolation degree property [1, Theorem 3.11.6], we obtain

$$\left(\left[\mathcal{B}^{\alpha_0}_{p,\tau_0}(A)\right]^{\vartheta_0},\left[\mathcal{B}^{\alpha_1}_{p,\tau_1}(A)\right]^{\vartheta_1}\right)_{\eta,g}=\left(\mathcal{B}^{\alpha_0}_{p,\tau_0}(A),\mathcal{B}^{\alpha_1}_{p,\tau_1}(A)\right)^{\vartheta}_{\varrho,\tau},\qquad\varrho=\eta\vartheta_1/\vartheta. \tag{9}$$

The equalities (8) and (9) for $\alpha=(1-\varrho)\alpha_0+\varrho\alpha_1$ yield (4) with $\varrho=\vartheta$. The inequalities (5), (6) are a consequence of (4) and the well-known interpolation properties [1, Theorem 3.11.2].

(iv) For every $x\in(\mathcal{B}^{\alpha_0}_{p,\tau_0}(A),\mathcal{B}^{\alpha_1}_{p,\tau_1}(A))_{\vartheta,\tau}$ there exists $c>0$ such that

$$\begin{aligned}|x|_{(\mathcal{B}^{\alpha_0}_{p,\tau_0},\mathcal{B}^{\alpha_1}_{p,\tau_1})_{\vartheta,\varrho}}&\leq\left(\sup_{t>0}t^{-\vartheta}K(t,x;\cdot)\right)^{1-\tau/\varrho}\left(\int_0^{\infty}\left[t^{-\vartheta}K(t,x;\cdot)\right]^{\tau}\frac{dt}{t}\right)^{1/\varrho}\\&\leq c|x|_{(\mathcal{B}^{\alpha_0}_{p,\tau_0},\mathcal{B}^{\alpha_1}_{p,\tau_1})_{\vartheta,\tau}}.\end{aligned}$$

Hence, the embedding $(\mathcal{B}^{\alpha_0}_{p,\tau_0}(A),\mathcal{B}^{\alpha_1}_{p,\tau_1}(A))_{\vartheta,\tau}\hookrightarrow(\mathcal{B}^{\alpha_0}_{p,\tau_0}(A),\mathcal{B}^{\alpha_1}_{p,\tau_1}(A))_{\vartheta,\varrho}$ is continuous. Finally using (4), we obtain (7). □

Corollary 4 *If* $\mathcal{E}(A)$ *is norm dense in* $\mathfrak{X}$ *then* $\mathcal{B}^{\alpha}_{p,\tau}(A)$ *is as well.*

4 Bernstein-Jackson-type inequalities

Let $1 \le p \le \infty$ and let the space $\mathcal{E}_p(A)$ be endowed with the quasi-norm $|\cdot|_p$. Consider the problem of the approximation of a given element in a Banach space $\mathfrak{X}$ by elements of an A-invariant subspace $\mathcal{E}_p^t(A)$ with a fixed index p. The distance between $x \in \mathfrak{X}$ and $\mathcal{E}_p^t(A)$ we denote by

$$d_p(t,x) = \inf\{\|x - x^0\| : x^0 \in \mathcal{E}_p^t(A)\}, \quad t > 0.$$

To investigate this problem, we will use spaces $\mathcal{B}_{p,\tau}^{\alpha}(A)$ defined for pair indices $\{0 < \alpha < \infty, 0 < \tau \le \infty\}$ or $\{0 \le \alpha < \infty, \tau = \infty\}$.

Theorem 5 *There are constants c_1 and c_2 such that the following inequalities hold*:

$$|x|_{\mathcal{B}_{p,\tau}^{\alpha}} \le c_1 |x|_p^{\alpha} \|x\|, \quad x \in \mathcal{E}_p(A), \tag{10}$$

$$d_p(t,x) \le c_2 t^{-\alpha} |x|_{\mathcal{B}_{p,\tau}^{\alpha}}, \quad x \in \mathcal{B}_{p,\tau}^{\alpha}(A). \tag{11}$$

Proof Via Theorem 3(i) the space $[\mathcal{B}_{p,\tau}^{\alpha}(A)]^{\vartheta}$ is interpolating between $\mathcal{E}_p(A)$ and $\mathfrak{X}$ for any $\vartheta = 1/(\alpha+1)$ and $\tau = g\vartheta$. As a consequence, $\mathcal{E}_p(A) \subset [\mathcal{B}_{p,\tau}^{\alpha}(A)]^{\vartheta} = (\mathcal{E}_p(A), \mathfrak{X})_{\vartheta,g} \subset \mathfrak{X}$. Hence, by [1, Theorem 3.11.4(b)] for some constant $c(\vartheta, g)$ we obtain

$$|x|_{(\mathcal{E}_p,\mathfrak{X})_{\vartheta,g}} \le c|x|_p^{1-\vartheta} \|x\|^{\vartheta}, \quad x \in \mathcal{E}_p(A).$$

This inequality and the isomorphism (3) imply that there is a constant $c_1(\alpha,\tau)$ such that the inequality (10) is true. By [1, Theorem 3.11.4(a)] for some constant $c'(\vartheta, g)$ we have

$$K(t,x;\mathcal{E}_p(A),\mathfrak{X}) \le c' t^{\vartheta} |x|_{(\mathcal{E}_p,\mathfrak{X})_{\vartheta,g}}, \quad x \in (\mathcal{E}_p(A),\mathfrak{X})_{\vartheta,g}.$$

Hence, by virtue of the isomorphism (3) there is a constant $c_0(\alpha,\tau)$ such that

$$K(t,x;\mathcal{E}_p(A),\mathfrak{X}) \le c_0 t^{\vartheta} |x|_{\mathcal{B}_{p,\tau}^{\alpha}}^{\vartheta}, \quad x \in \mathcal{B}_{p,\tau}^{\alpha}(A).$$

Following [1, Section 7.1], we introduce the function

$$K_\infty(t,x;\mathcal{E}_p(A),\mathfrak{X}) = \inf_{x=x^0+x^1} \max\{|x^0|_p, t\|x^1\|\}, \quad x^0 \in \mathcal{E}_p(A), x^1 \in \mathfrak{X}.$$

From the inequality $K_\infty(t,x;\mathcal{E}_p(A),\mathfrak{X}) \le K(t,x;\mathcal{E}_p(A),\mathfrak{X})$ it follows that

$$t^{-\vartheta} K_\infty(t,x;\mathcal{E}_p(A),\mathfrak{X}) \le c_0 |x|_{\mathcal{B}_{p,\tau}^{\alpha}}^{\vartheta}, \quad x \in \mathcal{B}_{p,\tau}^{\alpha}(A). \tag{12}$$

By [1, Lemma 7.1.2] for every $t > 0$ there exists $s > 0$ such that

$$K_\infty(t,x;\mathcal{E}_p(A),\mathfrak{X}) = s, \qquad \lim_{\nu \downarrow s} E_p(\nu,x) = E_p(s+0,x) \le s/t.$$

So, for every $s_1 > 0$ there is $t > 0$ such that $s_1 \le K_\infty(t,x;\mathcal{E}_p(A),\mathfrak{X}) = s$. For any fixed x the function $E_p(s,x)$ is decreasing, so $E_p(s,x) \le E_p(s_1+0,x) \le s_1/t$. Hence, we have $[E_p(s,x)]^{\vartheta} \le$

$t^{-\vartheta}s_1{}^{\vartheta} \le t^{-\vartheta}s^{\vartheta-1}s$. As a result,

$$s^{1-\vartheta}\left[E_p(s,x)\right]^{\vartheta} \le t^{-\vartheta}K_\infty\left(t,x;\mathcal{E}_p(A),\mathfrak{X}\right).$$

Using (12), we have $s^{1-\vartheta}[E_p(s,x)]^{\vartheta} \le c_0|x|^{\vartheta}_{\mathcal{B}^\alpha_{p,\tau}}$. Putting $\alpha = (1-\vartheta)/\vartheta$, we obtain

$$s^\alpha E_p(s,x) \le c_0^{1/\vartheta}|x|_{\mathcal{B}^\alpha_{p,\tau}}, \quad x \in \mathcal{B}^\alpha_{p,\tau}(A). \tag{13}$$

If $|x^0|_p = r(x^0) + \|x^0\| < s$, then $r(x^0) < s - \|x^0\|$, where $r(x^0) = \inf\{t > 0: x^0 \in \mathcal{E}^t_p(A)\}$. Therefore, $x^0 \in \mathcal{E}^t_p(A)$ for all numbers $t > 0$ such that $r(x^0) < t < s - \|x^0\|$. By Theorem 1(i), we have $\mathcal{E}^t_p(A) \subset \mathcal{E}^s_p(A)$. Therefore, $x^0 \in \mathcal{E}^s_p(A)$. Hence, the inequality

$$d_p(s,x) \le E_p(s,x), \quad x \in \mathfrak{X}, s > 0 \tag{14}$$

holds. Taking $c_2 = c_0^{1/\vartheta}$ in (13) and using (14), we obtain (11). □

Theorem 6 *Let A be an operator with the discrete spectrum $\sigma(A) = \{\lambda_n \in \mathbb{C}\}$, $n \in \mathbb{N}$ and let $\mathcal{R}^t$ be the complex linear span of all $\{\mathcal{R}(\lambda_n): |\lambda_n| < t\}$, where $\mathcal{R}(\lambda_n)$ is the root subspace of A corresponding to λ_n. Then for every α, τ there is a constant c such that*

$$\inf\left\{\left\|x - x^0\right\|: x^0 \in \mathcal{R}^t\right\} \le ct^{-\alpha}|x|_{\mathcal{B}^\alpha_{1,\tau}}, \quad x \in \mathcal{B}^\alpha_{1,\tau}(A). \tag{15}$$

Proof In [12] it is proven that for operators A, having discrete spectra, the equality $\mathcal{E}^t_1(A) = \mathcal{R}^t$ holds. Hence, the inequality (11) directly implies the estimation (15) for the distance from an element $x \in \mathcal{B}^\alpha_{1,\tau}(A)$ to the spectral subspace $\mathcal{R}^t$. □

5 Connections with classical results

Let us put $A = D_q$, where D_q is the closure in $\mathfrak{X} = L_q(\mathbb{R})$ $(1 < q \le \infty)$ of the operator of differentiation. In the considered case we have $\mathcal{E}^t_\infty(D_q) = \{u \in \mathcal{C}^\infty(D_q): \|u\|_{\mathcal{E}^t_\infty} < \infty\}$, where $\|u\|_{\mathcal{E}^t_\infty} = \sup_{k\in\mathbb{Z}_+}\|(D_q/t)^k u\|_{L_q}$ $(t > 0)$. Thus, $\mathcal{E}_\infty(D_q) = \bigcup_{t>0}\mathcal{E}^t_\infty(D_q)$.

Consider the space $\mathcal{M}^t_q$ of entire complex functions $U: \mathbb{C} \ni \xi + i\eta \longrightarrow U(\xi + i\eta)$ of exponential type $t > 0$, belonging to $L_q(\mathbb{R})$ for $\eta = 0$. Denote $\mathcal{M}_q = \bigcup_{t>0}\mathcal{M}^t_q$. Following [1, Section 7.2], we can define on $\mathcal{M}_q$ the quasi-norm

$$|u|_{\mathcal{M}_q} = \|u\|_{L_q} + \sup\left\{|\zeta|: \zeta \in \operatorname{supp} Fu\right\}, \quad u \in \mathcal{M}_q,$$

where $\operatorname{supp} Fu$ is a support of the Fourier-image Fu of a function $u \in \mathcal{M}_q$.

For any pair $\{0 < \alpha < \infty, 0 < \tau \le \infty\}$ or $\{0 \le \alpha < \infty, \tau = \infty\}$ and $1 < q \le \infty$ we define the classical Besov space $B^\alpha_{q,\tau}(\mathbb{R})$ with the norm $\|\cdot\|_{B^\alpha_{q,\tau}}$ (see *e.g.* [1, Section 6.2]). Let us show a relationship between the spaces $\mathcal{B}^\alpha_{p,\tau}(D_q)$ and $B^\alpha_{q,\tau}(\mathbb{R})$.

Theorem 7 *The following isomorphism holds*:

$$\mathcal{B}^\alpha_{\infty,\tau}(D_q) = B^\alpha_{q,\tau}(\mathbb{R}).$$

Proof Let us denote $u(\xi) = U(\xi + i0)$ for any $U \in \mathcal{M}_q^t$, where $\xi \in \mathbb{R}$, $t > 0$. For every such entire function u the Bernstein inequality $\|D_q^k u\|_{L_q} \le t^k \|u\|_{L_q}$ for all $k \in \mathbb{Z}_+$ holds (see [1, Section 7.2]). It follows that

$$\|u\|_{\mathcal{E}_\infty^t} = \sup_{k\in\mathbb{Z}_+} \left\|(D_q/t)^k u\right\|_{L_q} \le \|u\|_{L_q}.$$

Hence, if $U \in \mathcal{M}_q^t$ then $u \in \mathcal{E}_\infty^t(D_q)$.

Vice versa, let $u \in \mathcal{E}_\infty^t(D_q)$ with a fixed $t > 0$. The norm definition in $\mathcal{E}_\infty^t(D_q)$ implies that $\|D_q^k u\|_{L_q} \le t^k \|u\|_{\mathcal{E}_\infty^t}$ for all $k \in \mathbb{Z}_+$. It follows that

$$\left\|U(\cdot + i\eta)\right\|_{L_q} \le \sum_{k\in\mathbb{Z}_+} \left\|D_q^k u\right\|_{L_q} \frac{|\eta|^k}{k!} \le \|u\|_{\mathcal{E}_\infty^t} \exp\left(t|\eta|\right), \quad \eta \in \mathbb{R}$$

for any function $U : \mathbb{C} \ni \xi + i\eta \longrightarrow U(\xi + i\eta)$ such that $u(\xi) = U(\xi + i0)$ for all $\xi \in \mathbb{R}$. Hence, $U(\cdot + i\eta) \in L_q(\mathbb{R})$ for all $\eta \in \mathbb{R}$. The above inequality implies that if $q = \infty$ than $U(\cdot + i\eta) \in \mathcal{M}_\infty^t$.

Show that $U(\cdot + i\eta) \in \mathcal{M}_q^t$ for $1 < q < \infty$. From Sobolev's embedding theorem (see [14, Chapter I, Section 8, Theorem 1] or [13, Theorem 2.8.1]) we have

$$\|u\|_{L_\infty} \le c\|u\|_{W_q^1}, \quad u \in W_q^1(\mathbb{R}), 1 < q \le \infty.$$

Consequently, $\|D_q^k u\|_{L_\infty} \le c\|D_q^k u\|_{W_q^1}$ for all $k \in \mathbb{Z}_+$. Now using the inequality $\|D_q^k u\|_{W_q^1}^q = \|D_q^k u\|_{L_q}^q + \|D_q^{k+1} u\|_{L_q}^q \le (1 + t^q)t^{kq}\|u\|_{\mathcal{E}_\infty^t}^q$, we have

$$\left\|U(\cdot + i\eta)\right\|_{L_\infty} \le \sum_{k\in\mathbb{Z}_+} \left\|D_q^k u\right\|_{L_\infty} \frac{|\eta|^k}{k!} \le c\left(1 + t^q\right)^{1/q} \exp\left(t|\eta|\right) \|u\|_{\mathcal{E}_\infty^t}$$

for all $\eta \in \mathbb{R}$. Hence, $U \in \mathcal{M}_q^t$ for all $1 < q \le \infty$. So, up to the restriction $\mathcal{M}_q^t \ni U \longrightarrow u \in \mathcal{E}_\infty^t(D_q)$, we have

$$\mathcal{M}_q^t = \mathcal{E}_\infty^t(D_q), \qquad \mathcal{M}_q = \mathcal{E}_\infty(D_q). \tag{16}$$

Now applying (3), (16) and the well-known interpolation properties of Besov spaces (see [1, Theorem 7.2.4]), we obtain the required equality:

$$\mathcal{B}_{\infty,\tau}^\alpha(D_q) = \left(\mathcal{E}_\infty(D_q), L_q(\mathbb{R})\right)_{\vartheta,r}^{1/\vartheta} = \left(\mathcal{M}_q, L_q(\mathbb{R})\right)_{\vartheta,r}^{1/\vartheta} = B_{q,\tau}^\alpha(\mathbb{R})$$

with $\vartheta = 1/(\alpha + 1)$ and $\tau = r\vartheta$. □

Theorem 8 *There exist constants c_1 and c_2 such that*

$$\|u\|_{B_{q,\tau}^\alpha} \le c_1|u|_{\mathcal{M}_q}^\alpha \|u\|_{L_q}, \quad u \in \mathcal{M}_q, \tag{17}$$

$$d_\infty(t,u) \le c_2 t^{-\alpha} \|u\|_{B_{q,\tau}^\alpha}, \quad u \in B_{q,\tau}^\alpha(\mathbb{R}), \tag{18}$$

where $d_\infty(t,u) = \inf\{\|u - v\|_{L_q} : v \in \mathcal{E}_\infty^t(D_q)\} = \inf\{\|u - v\|_{L_q} : v \in \mathcal{M}_q^t\}$.

Proof Using the first equality (16) and the Paley-Wiener theorem, we obtain

$$\sup\{|\zeta|: \zeta \in \operatorname{supp} Fu\} = \inf\{t > 0: u \in \mathcal{E}^t_\infty(D_q)\}, \quad u \in \mathcal{M}_q.$$

Hence, the quasi-norms $|u|_\infty$ and $|u|_{\mathcal{M}_q}$ are equal on $\mathcal{M}_q$. Now the above claims is a consequence of Theorems 5, 7. □

Note that the equalities (17) and (18) exactly coincide with the well-known Bernstein and Jackson inequalities in the form given in [1, Section 7.2].

6 An application to regular elliptic operators

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with the infinitely smooth boundary $\partial\Omega$ and the system of operators

$$(Lu)(\xi) = \sum_{|s|\le 2m} a_s D^s u(\xi), \quad a_s \in \mathbb{C},$$

$$(B_j u)(\xi) = \sum_{|s|\le m_j} b_{j,s}(\xi) D^s u(\xi), \quad b_{j,s}(\xi) \in C^\infty(\partial\Omega), j = 1, \dots, m$$

is regular elliptic (see *e.g.* [13, Section 5.2.1]). Denote $D^s u = \frac{\partial^{|s|} u}{\partial\xi_1^{s_1} \dots \partial\xi_n^{s_n}}$, where $\xi = (\xi_1, \dots, \xi_n) \in \Omega$ and $|s| = s_1 + \cdots + s_n$ for all $s = (s_1, \dots, s_n) \in \mathbb{Z}^n_+$, $\bar{\Omega} = \Omega \cup \partial\Omega$.

In the complex space $L_q(\Omega)$ $(1 < q < \infty)$ we consider the closed linear operator

$$Au = Lu \text{ with the domain } \mathcal{C}^1(A) = W^{2m}_{q,\{B_j\}}(\Omega), \tag{19}$$

where $W^{2m}_{q,\{B_j\}}(\Omega) := \{u \in W^{2m}_q(\Omega): B_j u|_{\partial\Omega} = 0, j = 1, \dots, m\}$ and $W^{2m}_q(\Omega)$ is the classical Sobolev space. As is well known [13, Section 5.4.4], A has a discrete spectrum $\sigma(A)$ and the corresponding root subspaces are independent of q. The subspaces of the root vectors belong to the closed subspaces in $C^\infty(\bar{\Omega})$,

$$C^\infty_{A,\{B_j\}}(\bar{\Omega}) = \{u \in C^\infty(\bar{\Omega}): B_j A^k u|_{\partial\Omega} = 0, j = 1, \dots, m,\ k \in \mathbb{Z}_+\},$$

endowed with the seminorms $\sup_{\xi\in\Omega} |D^s u(\xi)|$, $0 \le |s| < \infty$.

Theorem 9 *The following topological isomorphism holds*:

$$\mathcal{B}^\alpha_{q,\tau}(A) = B^\alpha_{q,\tau,\{B_j\}}(\Omega), \tag{20}$$

where $B^\alpha_{q,\tau,\{B_j\}}(\Omega) = \{u \in B^\alpha_{q,\tau}(\Omega): B_j A^k u|_{\partial\Omega} = 0, j = 1, \dots, m, k \in \mathbb{Z}_+\}$ *and* $B^\alpha_{q,\tau}(\Omega)$ *is the Besov space*.

Proof Consider the space $\mathcal{E}^t_q(D) = \{u \in C^\infty(\bar{\Omega}): D^s u \in L_q(\Omega), |s| = k \in \mathbb{Z}_+\}$ endowed with the norm

$$\|u\|_{\mathcal{E}^t_q(D)} = \left(\sum_{k\in\mathbb{Z}_+} \sum_{|s|=k} t^{-qk} \|D^s u\|^q_{L_q(\Omega)}\right)^{1/q}, \quad t > 0.$$

Check that the union $\mathcal{E}_q(D) = \bigcup_{t>0} \mathcal{E}_q^t(D)$ coincides with the space of all entire analytic functions of exponential type, which restrictions to Ω belong to $L_q(\Omega)$. The space $\mathcal{E}_q(D)$ we endow with the quasi-norm

$$|u|_{\mathcal{E}_q(D)} := \|u\|_{L_q(\Omega)} + \inf\{t > 0 : u \in \mathcal{E}_q^t(D)\}.$$

For simplicity we put $0 \in \Omega$. If $l > n/q$ and $u \in \mathcal{E}_q^t(D)$ then the Sobolev embedding theorem yields

$$\sup_{\xi\in\Omega}\left|D^s u(\xi)\right| \le c\max\{1, t, \dots, t^l\} t^k \|u\|_{\mathcal{E}_q^t(D)} \le c_0 t^k, \quad |s| = k \in \mathbb{Z}_+, \tag{21}$$

where the constants c, c_0 are independent of k. It follows that (see [10])

$$\left|u(\xi + i\eta)\right| \le \sum_{k\in\mathbb{Z}_+} \sum_{|s|=k} \left|D^s u(\xi)\right| \frac{|\eta|^k}{k!} \le c_1 e^{t|\eta|} \tag{22}$$

for all $\xi \in \Omega$ and $\eta \in \mathbb{R}^n$, where the constant c_1 is independent of $k \in \mathbb{Z}_+$. Hence, u has an entire analytic extension onto $\mathbb{C}^n$ of exponential type.

Conversely, let an entire function u satisfy (22). Then the inequality $|D^s u(\xi)| \le c_2(2nt)^k e^{t|\xi|}$ for all $\xi \in \mathbb{R}^n$ and $|s| = k \in \mathbb{Z}_+$ holds. Here the constant c_2 is independent of k. By boundedness of Ω we have

$$\sup_{\xi\in\Omega}\left|D^s u(\xi)\right| \le c_3(2nt)^k \quad \text{and} \quad \sum_{|s|=k} \left\|D^s u\right\|_{L_q(\Omega)} \le c_2\left(2n^2 t\right)^k.$$

It follows that $u \in \mathcal{E}_q^{4n^2 t}(D)$ and consequently $u \in \mathcal{E}_q(D)$, because

$$\sum_{k\in\mathbb{Z}_+} \sum_{|s|=k} \left(4n^2 t\right)^{-qk} \left\|D^s u\right\|^q_{L_q(\Omega)} \le \frac{2^q}{2^q - 1} \sup_{k\in\mathbb{Z}_+} \frac{\sum_{|s|=k} \|D^s u\|^q_{L_q(\Omega)}}{(2n^2 t)^{qk}}. \tag{23}$$

Using the inequality (21), (23), and the Paley-Wiener theorem, we obtain the quasi-norm equivalence

$$|u|_{\mathcal{E}_q(D)} \sim \inf_{v|_\Omega = u, v\in L_q(\mathbb{R}^n)} \left\{\|v\|_{L_q(\mathbb{R}^n)} + \sup_{\zeta\in\operatorname{supp} Fv} |\zeta|\right\},$$

where $\operatorname{supp} Fv$ denotes the support of the Fourier-image Fv of a function $v \in L_q(\mathbb{R}^n)$.

Applying [13, Theorem 4.2.2], [1, Theorem 7.1.7] and the Bernstein-Jackson inequalities from [1, Section 7.2], we find that for any $l \in \mathbb{N}$ there exists a constant c_l such that

$$\begin{aligned} &\|u\|^{1/(l+1)}_{W_q^l(\Omega)} \le c_l |u|^{1-1/(l+1)}_{\mathcal{E}_q(D)} \|u\|^{1/(l+1)}_{L_q(\Omega)}, \quad u \in \mathcal{E}_q(D), \\ &K\left(t, u; \mathcal{E}_q(D), L_q(\Omega)\right) \le c_l t^{1/(l+1)} \|u\|^{1/(l+1)}_{W_q^l(\Omega)}, \quad u \in W_q^l(\Omega). \end{aligned} \tag{24}$$

Following Section 3, we define the space

$$\mathcal{B}^\alpha_{q,\tau}(D) := \left\{u \in L_q(\Omega) : |u|_{\mathcal{B}^\alpha_{q,\tau}(D)} := \left(\int_0^\infty \left(t^\alpha E_q(t, u)\right)^\tau \frac{dt}{t}\right)^{1/\tau} < \infty\right\},$$

where $E_q(t,u) = \inf\{\|u-u^0\|_{L_q(\Omega)} : u^0 \in \mathcal{E}_q(D), |u^0|_{\mathcal{E}_q(D)} < t\}$. Using the inequality (24) and well-known theorems [1, Theorems 3.11.5, 3.11.6, 7.1.7], [13, Theorem 2.4.2/2], we obtain

$$\begin{aligned}\mathcal{B}^{\alpha}_{q,\tau}(D) &= \bigl((\mathcal{E}_q(D), L_q(\Omega))_{1/(\alpha+1),\tau(\alpha+1)}\bigr)^{\alpha+1}\\ &= \bigl(L_q(\Omega), W^l_q(\Omega)\bigr)_{\alpha/l,\tau} = B^{\alpha}_{q,\tau}(\Omega).\end{aligned} \tag{25}$$

Now let us prove the equality

$$\mathcal{E}_q(A) = \bigl\{u \in \mathcal{E}_q(D) : B_j A^k u|_{\partial\Omega} = 0, j = 1,\dots,m, k \in \mathbb{Z}_+\bigr\}. \tag{26}$$

By [13, Theorem 5.4.3] for any $k \in \mathbb{N}$ there exist positive numbers c and C such that

$$c^k \|u\|_{W^{2mk}_q(\Omega)} \le \|A^k u\|_{L_q(\Omega)} \le C^k \|u\|_{W^{2mk}_q(\Omega)}, \quad u \in \mathcal{C}^k(A).$$

It follows that we have the inequalities

$$\begin{aligned}\sum_{k\in\mathbb{Z}_+} \bigl(C(nt)^{2m}\bigr)^{-kq} \|A^k u\|^q_{L_q(\Omega)} &\le C_1 \sum_{k\in\mathbb{Z}_+} \sum_{|s|=2mk} t^{-2mkq} \|D^s u\|^q_{L_q(\Omega)}\\ &\le C_1 \|u\|^q_{\mathcal{E}^t_q(D)}\end{aligned} \tag{27}$$

with a constant C_1. Thus, the embedding $\{u \in \mathcal{E}^t_q(D) : B_j A^k u|_{\partial\Omega} = 0, j = 1,\dots,m, k \in \mathbb{Z}_+\} \subset \mathcal{E}^{\tau}_q(A)$ with $\tau = C(nt)^{2m}$ holds. Conversely, let $u \in \mathcal{E}^t_q(A)$. Then

$$\|u\|^q_{\mathcal{E}^t_q(A)} = \sum_{k\in\mathbb{Z}_+} t^{-kq} \|A^k u\|^q_{L_q(\Omega)} \ge \sum_{k\in\mathbb{Z}_+} \sum_{|s|=k} \bigl(c^{-1}t\bigr)^{-kq} \|D^s u\|^q_{L_q(\Omega)}. \tag{28}$$

It follows that $\mathcal{E}^t_q(A) \subset \{u \in \mathcal{E}^{c^{-1}t}_q(D) : B_j A^k u|_{\partial\Omega} = 0, j = 1,\dots,m, k \in \mathbb{Z}_+\}$. Using (25) and (26), we obtain the required equality (20). □

Corollary 10 *There exist constants c_1 and c_2 such that*

$$\begin{aligned}&\|u\|_{B^{\alpha}_{q,\tau}(\Omega)} \le c_1 |u|^{\alpha}_{\mathcal{E}_q(D)} \|u\|_{L_q(\Omega)}, \quad u \in \mathcal{E}_q(A),\\ &d_q(t,u) \le c_2 t^{-\alpha} \|u\|_{B^{\alpha}_{q,\tau}(\Omega)}, \quad u \in B^{\alpha}_{q,\tau,\{B_j\}}(\Omega),\end{aligned}$$

where $d_q(t,u) = \inf\{\|u-v\|_{L_q(\Omega)} : v \in \mathcal{E}^t_q(A)\}$. In particular, for every α, τ there is a constant c such that

$$\inf\bigl\{\|u-u^0\|_{L_q(\Omega)} : u^0 \in \mathcal{R}^t\bigr\} \le ct^{-\alpha} \|u\|_{B^{\alpha}_{1,\tau}(\Omega)}, \quad u \in B^{\alpha}_{1,\tau,\{B_j\}}(\Omega),$$

where $\mathcal{R}^t$ is the complex linear span of root subspaces $\{\mathcal{R}(\lambda_n) : |\lambda_n| < t\}$ of the operator (19).

Proof From the inequality (27)-(28) and the Paley-Wiener theorem it follows that we have the quasi-norm equivalence $|u|_{\mathcal{E}_q(D)} \sim |u|_q$ on $\mathcal{E}_q(A)$. It remains to apply Theorems 5, 6, and 9. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Bergh, J, Löfström, J: Interpolation Spaces. Springer, Berlin (1976)
2. DeVore, R, Lorentz, G: Constructive Approximation. Springer, Berlin (1992)
3. Garrigós, G, Tabacco, A: Wavelet decompositions of anisotropic Besov spaces. Math. Nachr. **239/240**, 80-102 (2002)
4. Garrigós, G, Hernández, E: Sharp Jackson and Bernstein inequalities for *n*-term approximation in sequence spaces with applications. Indiana Univ. Math. J. **53**(6), 1741-1764 (2004)
5. Carl, B: Inequalities of Bernstein-Jackson-type and degree of compactness of operators in Banach spaces. Ann. Inst. Fourier **35**(3), 79-118 (1985)
6. Giulini, S: Bernstain and Jackson theorems for the Heisenberg group. J. Aust. Math. Soc. **38**, 241-254 (1985)
7. Giulini, S: Approximation and Besov spaces on stratified groups. Proc. Am. Math. Soc. **96**(4), 1741-1764 (1986)
8. Ljubič, JI, Macaev, VI: On operators with separable spectrum. Transl. Am. Math. Soc. **47**, 89-129 (1965)
9. Sakai, S: Operator Algebras in Dynamic Systems. Cambridge University Press, Cambridge (1991)
10. Radyno, YV: Vectors of exponential type in operators calculus and differential equations. Differ. Equ. **21**(9), 1559-1569 (1985)
11. Gorbachuk, MI, Gorbachuk, VI: On approximation of smooth vectors of a closed operator by entire vectors of exponential type. Ukr. Math. J. **47**(5), 713-726 (1995)
12. Dmytryshyn, M, Lopushansky, O: Operator calculus on the exponential type vectors of operators with point spectrum. In: Banakh, T (ed.) Topology in Banach Spaces, pp. 137-145. Nova Publ., New York (2001)
13. Triebel, H: Interpolation Theory. Function Spaces. Differential Operators. Springer, Berlin (1995)
14. Sobolev, SL: Some Applications of Functional Analysis in Mathematical Physics. Transl. Math. Monographs, vol. 90. Am. Math. Soc., New York (2008)

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