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Strong convergence of modified Halpern's iterations for a k-strictly pseudocontractive mapping

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Abstract

In this paper, we discuss three modified Halpern iterations as follows:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\big((1-\delta)x_n + \delta Tx_n\big), \tag{I}$$

$$x_{n+1} = \alpha_n\big((1-\delta)u + \delta x_n\big) + (1-\alpha_n)Tx_n, \tag{II}$$

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Tx_n, \quad n \geq 0, \tag{III}$$

and obtained the strong convergence results of the iterations (I)-(III) for a k-strictly pseudocontractive mapping, where $\{\alpha_n\}$ satisfies the conditions: (C1) $\lim_{n\to\infty}\alpha_n = 0$ and (C2) $\sum_{n=1}^{\infty}\alpha_n = +\infty$, respectively. The results presented in this work improve the corresponding ones announced by many other authors.

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H.

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in the Hilbert space H is called strongly pseudo-contractive if, for all $x, y \in D(T)$, there exists $k \in (0,1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq k\|x-y\|^2, \tag{1.1}$$

while T is said to be pseudo-contractive if (1.1) holds for $k = 1$. A mapping T is said to be Lipschitzian if, for all $x, y \in D(T)$, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|. \tag{1.2}$$

A mapping T is called nonexpansive if (1.2) holds for $L = 1$ and, further, T is said to be contractive if $L < 1$. T is said to be firmly nonexpansive if for all $x, y \in D(T)$,

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle.$$

Firmly nonexpansive mappings could be looked upon as an important subclass of nonexpansive mappings. A mapping T is called k-strictly pseudocontractive, if for all $x, y \in D(T)$, there exists $\lambda > 0$ such that

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \lambda \bigl\|x - y - (Tx - Ty)\bigr\|^2. \tag{1.3}$$

Without loss of generality, we may assume that $\lambda \in (0,1)$. In Hilbert spaces H, (1.3) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\bigl\|(I - T)x - (I - T)y\bigr\|^2, \quad k = (1 - 2\lambda) < 1,$$

and we can assume also that $k \geq 0$ so that $k \in [0,1)$.

It is obvious that a k-strictly pseudocontractive mapping is Lipschitzian with $L = \frac{k+1}{k}$. The class of nonexpansive mappings is a subclass of strictly pseudocontractive mappings in a Hilbert space, but the converse implication may be false. We remark that the class of strongly pseudo-contractive mappings is independent from the class of k-strict pseudo-contractions.

In 1967, Halpern [1] was the first who introduced the following iteration scheme for a nonexpansive mapping T which was referred to as Halpern iteration: For any initialization $x_0 \in C$ and any anchor $u \in C$, $\alpha_n \in [0,1]$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0. \tag{1.4}$$

He proved that the sequence (1.4) converges weakly to a fixed point of T, where $\alpha_n = n^{-a}$, $a \in (0,1)$. In 1977, Lions [2] further proved that the sequence (1.4) converges strongly to a fixed point of T in a Hilbert space, where $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(C3) $\lim_{n\to\infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_{n+1}^2} = 0$.

But, in [1, 2], the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = \frac{1}{n+1}$. In 1992, Wittmann [3] proved, still in Hilbert spaces, the strong convergence of the sequence (1.4) to a fixed point of T, where $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$.

The strong convergence of Halpern's iteration to a fixed point of T has also been proved in Banach spaces; see, *e.g.*, [4–10]. Reich [4, 5] has showed the strong convergence of the sequence (1.4), where $\{\alpha_n\}$ satisfies the conditions (C1), (C2) and (C5), $\{\alpha_n\}$ is decreasing (noting that the condition (C5) is a special case of condition(C4)). In 1997, Shioji and

Takahashi [6] extended Wittmann's result to Banach spaces. In 2002, Xu [9] obtained a strong convergence theorem, where $\{\alpha_n\}$ satisfies the following conditions: (C1), (C2) and (C6) $\lim_{n\to\infty}\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}} = 0$. In particular, the canonical choice of $\alpha_n = \frac{1}{n+1}$ satisfies the conditions (C1), (C2) and (C6).

However, is a real sequence $\{\alpha_n\}$ satisfying the conditions (C1) and (C2) sufficient to guarantee the strong convergence of Halpern's iteration (1.4) for nonexpansive mappings? It remains an open question, see [1].

Some mathematicians considered the open question. In [11], Song proved that for a firmly nonexpansive mapping T, an important subclass of nonexpansive mappings, the answer of the Halpern open problem is affirmative. A partial answer to this question was given independently by Chidume and Chidume [12] and Suzuki [7]. They defined the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\big((1-\delta)x_n + \delta Tx_n\big), \tag{1.5}$$

where $\delta \in [0,1]$, I is the identity, and obtained the strong convergence of the iteration (1.5), where $\{\alpha_n\}$ satisfies the conditions (C1) and (C2). Recently, Xu [10] gave another partial answer to this question. He obtained the strong convergence of the iterative sequence

$$x_{n+1} = \alpha_n\big((1-\delta)u + \delta x_n\big) + (1-\alpha_n)Tx_n, \tag{1.6}$$

where $\delta \in [0,1]$ and $\{\alpha_n\}$ satisfies the conditions (C1) and (C2).

In [13], Liang-Gen Hu introduced the modified Halpern's iteration: For any $u, x_0 \in C$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Tx_n, \quad n \geq 0, \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three real sequences in $[0,1]$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$. He showed that the sequence $\{\alpha_n\}$ satisfying the conditions (C1) and (C2) is sufficient to guarantee the strong convergence of the modified Halpern's iterative sequence (1.7) for nonexpansive mappings.

The purpose of this paper is to present a significant answer to the above open question. We will show that the sequence $\{\alpha_n\}$ satisfying the conditions (C1) and (C2) is sufficient to guarantee the strong convergence of the modified Halpern's iterative sequences (1.5)-(1.7) for k-strictly pseudocontractive mappings, respectively. The results present in this paper improve and develop the corresponding results of [7, 10, 12, 13].

2 Preliminaries

In what follows we will need the following:

Lemma 2.1 [14] *Let H be a real Hilbert space, then the following well-known results hold*:

(i) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $x, y \in H$ *and for all* $t \in [0,1]$.

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, $\forall x, y \in H$.

Let C be a nonempty closed convex subset of a real Hilbert space H. The nearest point projection $P_C : H \to C$ defined from H onto C is the function which assigns to each $x \in H$

its nearest point denoted by $P_C x$ in C. Thus $P_C x$ is the unique point in C such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

It is known that for each $x \in H$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

Lemma 2.2 [15] *Let C be a nonempty closed convex subset of a real Hilbert space H, T : $C \to C$ be a k-strictly pseudocontractive mapping. Then $(I - T)$ is demiclosed at zero.*

Lemma 2.3 [9] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \delta_n)a_n + \delta_n \xi_n$, $\forall n \geq 0$, where $\{\delta_n\}$ is a sequence in $[0,1]$ and $\{\xi_n\}$ is a sequence in R satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \delta_n = +\infty$;

(ii) $\limsup_{n\to\infty} \xi_n \leq 0$ *or* $\sum_{n=1}^{\infty} \delta_n |\xi_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, proving the following theorems, we show that the conjunction of (C1) and (C2) is a sufficient condition on our iteration (I)-(III), respectively.

Theorem 3.1 *Let C be a closed and convex subset of a real Hilbert space H, $T : C \to C$ be a k-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. For an arbitrary initial value $x_0 \in C$ and fixed anchor $u \in C$, define iteratively a sequence $\{x_n\}$ as follows*:

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \tag{I}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three real sequences in $(0,1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < k < \frac{\beta_n}{\beta_n+\gamma_n}$. Suppose that $\{\alpha_n\}$ satisfies the conditions: (C1) $\lim_{n\to\infty} \alpha_n = 0$; (C2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$. *Then as $n \to \infty$, $\{x_n\}$ converges strongly to some fixed point x^* of T and $x^* = P_{F(T)}u$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.*

Proof Firstly, we show that $\{x_n\}$ is bounded. Rewrite the iterative process (I) as follows:

$$\begin{aligned} x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n T x_n \\ &= \alpha_n u + (1 - \alpha_n)\frac{\beta_n x_n + \gamma_n T x_n}{1 - \alpha_n} \\ &= \alpha_n u + (1 - \alpha_n) y_n, \end{aligned} \tag{3.1}$$

where $y_n = \frac{\beta_n x_n + \gamma_n T x_n}{1-\alpha_n}$. Take any $p \in F(T)$, then, from Lemma 2.1 and (3.1), we estimate as follows:

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\quad = \alpha_n \|u - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 - \alpha_n(1 - \alpha_n)\|u - y_n\|^2 \\ &\quad = \alpha_n \|u - p\|^2 + (1 - \alpha_n)\frac{\beta_n}{1 - \alpha_n}\|x_n - p\|^2 + (1 - \alpha_n)\frac{\gamma_n}{1 - \alpha_n}\|T x_n - p\|^2 \end{aligned}$$

$$\begin{aligned}
&\quad - (1-\alpha_n)\frac{\beta_n}{1-\alpha_n}\frac{\gamma_n}{1-\alpha_n}\|x_n - Tx_n\|^2 - \alpha_n(1-\alpha_n)\|u - y_n\|^2 \\
&= \alpha_n\|u-p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|Tx_n - p\|^2 - \alpha_n(1-\alpha_n)\|u - y_n\|^2 \\
&\quad - \frac{\beta_n\gamma_n}{1-\alpha_n}\|x_n - Tx_n\|^2 \\
&\le \alpha_n\|u-p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\bigl(\|x_n - p\|^2 + k\|x_n - Tx_n\|^2\bigr) \\
&\quad - \frac{\beta_n\gamma_n}{1-\alpha_n}\|x_n - Tx_n\|^2 - \alpha_n(1-\alpha_n)\|u - y_n\|^2 \\
&= \alpha_n\|u-p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|u - y_n\|^2 \\
&\le \alpha_n\|u-p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\le \max\bigl\{\|u-p\|^2, \|x_n - p\|^2\bigr\}.
\end{aligned} \tag{3.2}$$

By induction,

$$\|x_{n+1} - p\|^2 \le \max\bigl\{\|u-p\|^2, \|x_0 - p\|^2\bigr\}.$$

This proves the boundedness of the sequence $\{x_n\}$, which leads to the boundedness of $\{Tx_n\}$.

Next, we claim that

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

In fact, we have from (3.2) (for some appropriate constant $M > 0$) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|u-p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2 \\
&= \alpha_n\bigl(\|u-p\|^2 - \|x_n - p\|^2\bigr) + \|x_n - p\|^2 - \left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2 \\
&\le \alpha_n M + \|x_n - p\|^2 - \left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2,
\end{aligned}$$

which implies that

$$\left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2 - \alpha_n M \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \tag{3.3}$$

If $(\frac{\beta_n}{1-\alpha_n} - k)\gamma_n\|x_n - Tx_n\|^2 - \alpha_n M \le 0$, then

$$\begin{aligned}
&\left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2 \le \alpha_n M, \\
&\|x_n - Tx_n\|^2 \le \frac{\alpha_n}{(\frac{\beta_n}{1-\alpha_n} - k)\gamma_n}M,
\end{aligned}$$

and hence the desired result is obtained by the condition (C1) and $0 < k < \frac{\beta_n}{1-\alpha_n}$.

If $(\frac{\beta_n}{1-\alpha_n} - k)\gamma_n\|x_n - Tx_n\|^2 - \alpha_n M > 0$, then following (3.3), we have

$$\sum_{n=0}^{m}\left[\left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2 - \alpha_n M\right] \le \|x_0 - p\|^2 - \|x_m - p\|^2 \le \|x_0 - p\|^2.$$

Then

$$\sum_{n=0}^{\infty}\left[\left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2 - \alpha_n M\right] < +\infty.$$

Thus

$$\lim_{n\to\infty}\left[\left(\frac{\beta_n}{1-\alpha_n} - k\right)\gamma_n\|x_n - Tx_n\|^2 - \alpha_n M\right] = 0,$$

and hence

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.4}$$

In order to prove $x_n \to x^* = P_{F(T)}u$, we next show that

$$\limsup_{n\to\infty}\langle P_{F(T)}u - u, P_{F(T)}u - x_{n+1}\rangle \le 0.$$

Indeed, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle P_{F(T)}u - u, P_{F(T)}u - x_{n+1}\rangle = \lim_{k\to\infty}\langle P_{F(T)}u - u, P_{F(T)}u - x_{n_k+1}\rangle.$$

We may assume that $x_{n_k} \rightharpoonup z$ since H is reflexive and $\{x_n\}$ is bounded. From (3.4), it follows from Lemma 2.2 that $z \in F(T)$.

From (2.1), we conclude

$$\begin{aligned}\limsup_{n\to\infty}\langle P_{F(T)}u - u, P_{F(T)}u - x_{n+1}\rangle &= \lim_{k\to\infty}\langle P_{F(T)}u - u, P_{F(T)}u - x_{n_k+1}\rangle\\ &= \langle P_{F(T)}u - u, P_{F(T)}u - z\rangle \le 0.\end{aligned} \tag{3.5}$$

Finally, we show that $x_n \to P_{F(T)}u$. As a matter of fact, from Lemma 2.1 and (3.1), we obtain

$$\begin{aligned}&\|x_{n+1} - P_{F(T)}u\|^2\\ &\quad\le (1-\alpha_n)\|y_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle\\ &\quad= (1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n}\|x_n - P_{F(T)}u\|^2 + \frac{\gamma_n}{1-\alpha_n}\|Tx_n - P_{F(T)}u\|^2 - \frac{\beta_n\gamma_n}{(1-\alpha_n)^2}\|x_n - Tx_n\|^2\right]\\ &\qquad+ 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle\\ &\quad\le (1-\alpha_n)\left[\frac{\beta_n}{1-\alpha_n}\|x_n - P_{F(T)}u\|^2 + \frac{\gamma_n}{1-\alpha_n}\|x_n - P_{F(T)}u\|^2 + \frac{\gamma_n k}{1-\alpha_n}\|x_n - Tx_n\|^2\right.\\ &\qquad\left.- \frac{\beta_n\gamma_n}{1-\alpha_n}\|x_n - Tx_n\|^2\right] + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle\end{aligned}$$

$$= (1-\alpha_n)\|x_n - P_{F(T)}u\|^2 - \gamma_n(\beta_n - k)\|x_n - Tx_n\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle$$
$$\leq (1-\alpha_n)\|x_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle.$$

It follows from the conditions (C1), (C2) and (3.5), using Lemma 2.3, that

$$\lim_{n\to\infty}\|x_n - P_{F(T)}u\| = 0.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2 *Let C be a closed and convex subset of a real Hilbert space H, $T: C \to C$ be a k-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. For an arbitrary initial value $x_0 \in C$ and fixed anchor $u \in C$, define iteratively a sequence $\{x_n\}$ as follows*:

$$x_{n+1} = \alpha_n\big((1-\delta)u + \delta x_n\big) + (1-\alpha_n)Tx_n, \tag{II}$$

where $\{\alpha_n\} \subset (0,1)$, $0 < k < \alpha_n\delta$. Suppose that $\{\alpha_n\}$ satisfies the conditions: (C1) $\lim_{n\to\infty}\alpha_n = 0$; (C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$. *Then as $n \to \infty$, $\{x_n\}$ converges strongly to some fixed point x^* of T and $x^* = P_{F(T)}u$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.*

Proof Firstly, we show that $\{x_n\}$ is bounded. Rewrite the iterative process (II) as follows:

$$\begin{aligned} y_n &= (1-\delta)u + \delta x_n, \\ x_{n+1} &= \alpha_n y_n + (1-\alpha_n)Tx_n. \end{aligned} \tag{3.6}$$

Take any $p \in F(T)$, then, from Lemma 2.1 and (3.6), we estimate as follows:

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\quad= \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|Tx_n - p\|^2 - \alpha_n(1-\alpha_n)\|y_n - Tx_n\|^2 \\ &\quad= \alpha_n(1-\delta)\|u-p\|^2 + \alpha_n\delta\|x_n - p\|^2 - \alpha_n(1-\delta)\delta\|u - x_n\|^2 \\ &\qquad+ (1-\alpha_n)\|Tx_n - p\|^2 - \alpha_n(1-\alpha_n)(1-\delta)\|u - Tx_n\|^2 \\ &\qquad- \alpha_n(1-\alpha_n)\delta\|x_n - Tx_n\|^2 + \alpha_n(1-\alpha_n)\delta(1-\delta)\|u - x_n\|^2 \\ &\quad\leq \alpha_n(1-\delta)\|u-p\|^2 + \alpha_n\delta\|x_n - p\|^2 - \alpha_n(1-\delta)\delta\|x_n - u\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\qquad+ (1-\alpha_n)k\|x_n - Tx_n\|^2 - \alpha_n(1-\alpha_n)(1-\delta)\|Tx_n - u\|^2 \\ &\qquad- \alpha_n(1-\alpha_n)\delta\|x_n - Tx_n\|^2 + \alpha_n(1-\alpha_n)\delta(1-\delta)\|x_n - u\|^2 \\ &\quad\leq \big[1 - \alpha_n(1-\delta)\big]\|x_n - p\|^2 + \alpha_n(1-\delta)\|u-p\|^2 - (1-\alpha_n)(\alpha_n\delta - k)\|x_n - Tx_n\|^2 \\ &\quad\leq \big[1 - \alpha_n(1-\delta)\big]\|x_n - p\|^2 + \alpha_n(1-\delta)\|u-p\|^2 \\ &\quad\leq \max\big\{\|u-p\|^2, \|x_n - p\|^2\big\}. \end{aligned}$$

By induction,

$$\|x_{n+1} - p\|^2 \leq \max\big\{\|u-p\|^2, \|x_0 - p\|^2\big\}.$$

This proves the boundedness of the sequence $\{x_n\}$, which implies that the sequence $\{Tx_n\}$ is bounded also.

Using the same technique as in Theorem 3.1, we can prove

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0$$

and

$$\limsup_{n\to\infty}\langle P_{F(T)}u - u, P_{F(T)}u - x_{n+1}\rangle \le 0.$$

Finally, we show that $x_n \to P_{F(T)}u$. Writing $z_n = \frac{\alpha_n\delta x_n+(1-\alpha_n)Tx_n}{1-(1-\delta)\alpha_n}$, then

$$x_{n+1} = (1-\delta)\alpha_n u + \bigl[1-(1-\delta)\alpha_n\bigr]z_n,$$

from Lemma 2.1 and (3.1), we obtain

$$\begin{aligned}
&\|x_{n+1} - P_{F(T)}u\|^2 \\
&\quad\le \bigl[1-(1-\delta)\alpha_n\bigr]\|z_n - P_{F(T)}u\|^2 + 2(1-\delta)\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\quad= \bigl[1-(1-\delta)\alpha_n\bigr]\biggl[\frac{\alpha_n\delta}{1-(1-\delta)\alpha_n}\|x_n - P_{F(T)}u\|^2 + \frac{1-\alpha_n}{1-(1-\delta)\alpha_n}\|Tx_n - P_{F(T)}u\|^2 \\
&\qquad - \frac{\alpha_n(1-\alpha_n)\delta}{[1-(1-\delta)\alpha_n]^2}\|x_n - Tx_n\|^2\biggr] + 2(1-\delta)\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\quad\le \bigl[1-(1-\delta)\alpha_n\bigr]\biggl[\frac{\alpha_n\delta}{1-(1-\delta)\alpha_n}\|x_n - P_{F(T)}u\|^2 + \frac{1-\alpha_n}{1-(1-\delta)\alpha_n}\|x_n - P_{F(T)}u\|^2 \\
&\qquad + \frac{1-\alpha_n}{1-(1-\delta)\alpha_n}k\|x_n - Tx_n\|^2 - \frac{\alpha_n(1-\alpha_n)\delta}{1-(1-\delta)\alpha_n}\|x_n - Tx_n\|^2\biggr] \\
&\qquad + 2(1-\delta)\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\quad= \bigl[1-(1-\delta)\alpha_n\bigr]\|x_n - P_{F(T)}u\|^2 - \frac{(1-\alpha_n)(\alpha_n\delta - k)}{1-(1-\delta)\alpha_n}\|x_n - Tx_n\|^2 \\
&\qquad + 2(1-\delta)\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\quad\le \bigl[1-(1-\delta)\alpha_n\bigr]\|x_n - P_{F(T)}u\|^2 + 2(1-\delta)\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle.
\end{aligned}$$

It follows from the conditions (C1), (C2) and (3.5), using Lemma 2.3, that

$$\lim_{n\to\infty}\|x_n - P_{F(T)}u\| = 0.$$

This completes the proof of Theorem 3.2. □

Theorem 3.3 *Let C be a closed and convex subset of a real Hilbert space H, $T : C \to C$ be a k-strictly pseudocontractive mapping such that $F(T) \neq \emptyset$. For an arbitrary initial value $x_0 \in C$ and fixed anchor $u \in C$, define iteratively a sequence $\{x_n\}$ as follows*:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_\beta x_n, \tag{III}$$

where $T_\beta = \beta I + (1-\beta)T$, $\{\alpha_n\} \subset [0,1]$, $\beta \in (k,1)$. *Suppose that* $\{\alpha_n\}$ *satisfies the conditions*: (C1) $\lim_{n\to\infty} \alpha_n = 0$; (C2) $\sum_{n=1}^\infty \alpha_n = \infty$. *Then as* $n \to \infty$, $\{x_n\}$ *converges strongly to some fixed point* x^* *of* T, *and* $x^* = P_{F(T)}u$, *where* $P_{F(T)}$ *is the metric projection from* H *onto* $F(T)$.

Proof It is easy to see that $F(T_\beta) = F(T) \neq \emptyset$. For any $x, y \in C$, we have

$$\begin{aligned}
&\|T_\beta x - T_\beta y\|^2 \\
&\quad = \big\|\beta(x-y) + (1-\beta)(Tx - Ty)\big\|^2 \\
&\quad = \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\big\|x - Tx - (y - Ty)\big\|^2 \\
&\quad \le \beta\|x-y\|^2 + (1-\beta)\big[\|x-y\|^2 + k\big\|x - Tx - (y-Ty)\big\|^2\big] \\
&\qquad - \beta(1-\beta)\big\|x - Tx - (y-Ty)\big\|^2 \\
&\quad = \|x-y\|^2 - (1-\beta)(\beta-k)\big\|x - Tx - (y-Ty)\big\|^2 \\
&\quad = \|x-y\|^2 - \frac{\beta-k}{1-\beta}\big\|x - T_\beta x - (y - T_\beta y)\big\|^2 \\
&\quad \le \|x-y\|^2 - (\beta-k)\big\|x - T_\beta x - (y - T_\beta y)\big\|^2.
\end{aligned}$$

Thus, for all $x \in C$ and for all $p \in F(T_\beta) = F(T)$, we have

$$\|T_\beta x - p\|^2 \le \|x-p\|^2 - (\beta-k)\|x - T_\beta x\|^2.$$

This implies that T_β is a quasi-firmly type nonexpansive mapping (see, for example, [11]). T_β is also a strongly quasi-nonexpansive mapping (see, for example, [16]). Hence it follows from [11, 16] (see Theorem 3.1 and Remark 1 of [11] or Corollary 8 of [16]) that $\{x_n\}$ converges strongly to a point $x^* \in F(T_\beta) = F(T)$.

Finally, we show $x^* = P_{F(T)}u$. From Lemma 2.1 and the iterative process (III), we estimate as follows:

$$\begin{aligned}
&\|x_{n+1} - P_{F(T)}u\|^2 \\
&\quad \le (1-\alpha_n)\big\|\beta(x_n - P_{F(T)}u) + (1-\beta)(Tx_n - P_{F(T)}u)\big\|^2 \\
&\qquad + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\quad \le (1-\alpha_n)\big[\beta\|x_n - P_{F(T)}u\|^2 + (1-\beta)\|Tx_n - P_{F(T)}u\|^2 - \beta(1-\beta)\|x_n - Tx_n\|^2\big] \\
&\qquad + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\quad \le (1-\alpha_n)\big[\beta\|x_n - P_{F(T)}u\|^2 + (1-\beta)\|x_n - P_{F(T)}u\|^2 + (1-\beta)k\|x_n - Tx_n\|^2 \\
&\qquad - \beta(1-\beta)\|x_n - Tx_n\|^2\big] + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\quad = (1-\alpha_n)\big[\|x_n - P_{F(T)}u\|^2 - (1-\beta)(\beta-k)\|x_n - Tx_n\|^2\big] \\
&\qquad + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\quad \le (1-\alpha_n)\|x_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle.
\end{aligned}$$

It follows from the conditions (C1), (C2) and

$$\lim_{n\to\infty}\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle = \big\langle u - P_{F(T)}u, x^* - P_{F(T)}u\big\rangle \le 0,$$

using Lemma 2.3, that

$$\lim_{n\to\infty} \|x_n - P_{F(T)}u\| = 0.$$

This completes the proof of Theorem 3.3. □

Remark 3.1 Theorems 3.1-3.3 improve the main results of [7, 10, 12, 13] from a nonexpansive mapping to a k-strictly pseudocontractive mapping, respectively. Theorems 3.1-3.3 show that the real sequence $\{\alpha_n\}$ satisfying the two conditions (C1) and (C2) is sufficient for the strong convergence of the iterative sequences (I)-(III) for k-strictly pseudocontractive mappings, respectively. Therefore, our results give a significant partial answer to the open question.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SL and LL carried out the proof of convergence of the theorems. XH and LZ carried out the check of the manuscript. All authors read and approved the final manuscript.

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