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OD-characterization of the automorphism groups of simple K_3-groups

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Abstract

The degree pattern of a finite group G associated to its prime graph has been introduced in (Moghaddamfar *et al.* in Algebra Colloq. 12(3):431-442, 2005) and denoted by $D(G)$. The group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups H satisfying conditions (1) $|G| = |H|$ and (2) $D(G) = D(H)$. Moreover, a one-fold OD-characterizable group is simply called OD-characterizable group. In this problem, those groups with connected prime graphs are somewhat much difficult to be solved. In the present paper, we continue this investigation and show that the automorphism groups of simple K_3-groups are characterized by their orders and degree patterns. In fact, the automorphism groups of simple K_3-groups except A_6 and $U_4(2)$ are OD-characterizable. Moreover, $\mathrm{Aut}(A_6)$ is fourfold OD-characterizable and $\mathrm{Aut}(U_4(2))$ is at least fourfold OD-characterizable.

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1 Introduction

Let G be a finite group, $\pi_e(G)$ denote the set of element orders of G and $\pi(G)$ the set of all prime divisors of $|G|$. The prime graph of G was defined by Gruenberg and Kegel (ref. to [1]), which was denoted by $\Gamma(G)$ and constructed as follows: The vertex set of this graph is $\pi(G)$ and two distinct vertices p and q are jointed by an edge if and only if $pq \in \pi_e(G)$. In this case, we say vertices p and q are adjacent and denote this fact by $p \sim q$. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$ and the sets of vertices of connected components of $\Gamma(G)$ are denoted as $\pi_i = \pi_i(G)$ $(i = 1, 2, \ldots, t(G))$. If $|G|$ is even, we always assume that $2 \in \pi_1(G)$. Set $T(G) = \{\pi_i(G) | i = 1, 2, \ldots, t(G)\}$.

Let n be a positive integer, we use $\pi(n)$ to denote the set of all prime divisors of n. If the prime graph of G is known, then $|G|$ can be expressed as a product of $m_1, m_2, \ldots, m_{t(G)}$, where m_is are positive integers such that $\pi(m_i) = \pi_i$. These m_is were called the *order components* of G by the second author, who proved a lot of finite simple groups can be uniquely determined by their order components (ref. to [2]). The set of order components of G is denoted as $OC(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$. We also use the following notations. Given a finite group G, denote by $\mathrm{Soc}(G)$ the socle of G which is the subgroup generated by the set of all minimal normal subgroups of G. $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$. And P_r denotes a Sylow r-subgroup of G for $r \in \pi(G)$. All further unexplained notations are referred to [3].

Definition 1.1 [4] Let G be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_is are primes and α_i are integers. For $p \in \pi(G)$, let $\deg(p) := |\{q \in \pi(G) | p \sim q\}|$, called the *degree* of p. We also define $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$. We call $D(G)$ the *degree pattern* of G.

Definition 1.2 [4] A group M is called *k-fold OD-characterizable* if there exist exactly k non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$. Moreover, a one-fold OD-characterizable group is simply called an *OD-characterizable group*.

Definition 1.3 A group G is said to be almost simple related to S if and only if $S \trianglelefteq G \leq \mathrm{Aut}(S)$ for some non-abelian simple group S.

In a series of articles such as [4–7], many finite non-abelian simple groups or almost simple groups were shown to be OD-characterizable. For convenience, we recall some of them in the following proposition.

Proposition 1.4 *A finite group G is OD-characterizable if G is one of the following groups*:

(1) *All sporadic simple groups and their automorphism groups except* $\mathrm{Aut}(J_2)$ *and* $\mathrm{Aut}(M^cL)$;
(2) *The alternating groups A_p, A_{p+1}, A_{p+2} and the symmetric groups S_p and S_{p+1}, where p is a prime*;
(3) *All finite simple K_4-groups except A_{10}*;
(4) *The simple groups of Lie type $L_2(q)$, $L_3(q)$, $U_3(q)$, ${}^2B_2(q)$ and ${}^2G_2(q)$ for certain prime power q*;
(5) *All finite simple $C_{2,2}$-groups*;
(6) *The alternating groups A_{p+3}, where $p + 2$ is a composite number, $p + 4$ is a prime and $7 \neq p \in \pi(1{,}000!)$*;
(7) *The almost simple groups of* $\mathrm{Aut}(O_{10}^+(2))$, $\mathrm{Aut}(O_{10}^-(2))$ *and* $\mathrm{Aut}(F_4(2))$.

Till now a lot of finite simple groups have been shown to be OD-characterizable, and also some finite groups, especially the automorphism groups of some finite simple groups, have been shown not to be OD-characterizable but k-fold OD-characterizable for some $k > 1$. In this paper, we continue this topic and get the following Main Theorem.

Main Theorem *Let M be a simple K_3-group and G be a finite group such that $|G| = |\mathrm{Aut}(M)|$ and $D(G) = D(\mathrm{Aut}(M))$.*

(1) *If M is one of the following simple K_3-groups*: A_5, A_6, $L_2(7)$, $L_2(8)$, $U_3(3)$, $L_3(3)$ *and* $L_2(17)$, *then $G \cong \mathrm{Aut}(M)$. In other words*, $\mathrm{Aut}(M)$ *is OD-characterizable.*
(2) *If $M = A_6$, then G is isomorphic to one of the following groups*: $\mathrm{Aut}(A_6)$, $Z_2 \times Z_2 \times A_6$, $Z_2 \times (Z_2 \cdot A_6)$ *and* $Z_4 \times A_6$. *In other words*, $\mathrm{Aut}(A_6)$ *is fourfold OD-characterizable.*
(3) *If $M = U_4(2)$, then G is isomorphic to one of the following groups*: $Z_2 \cdot U_4(2)$, $Z_2 \times U_4(2)$, $\mathrm{Aut}(U_4(2))$ *and* $(P_3 \rtimes P_5)P_2$, *where $P_r \in \mathrm{Syl}_r(G)$ for each $r \in \pi(G)$. In other words*, $\mathrm{Aut}(U_4(2))$ *is at least fourfold OD-characterizable.*

2 Preliminaries

In this section, we give some results which will be applied to our further investigations.

Lemma 2.1 [1, Theorem A] *Let G be a finite group with $t(G) \geq 2$, then G is one of the following groups*:

(a) *G is a Frobenius or 2-Frobenius group*;

(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group and $|G/H| \mid |\mathrm{Aut}(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.*

Remark 2.2 A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.3 [3] *Let S be a finite non-abelian simple group with order having prime divisors at most 17. Then S is isomorphic to one of the simple groups listed in Table 1. In particular, if $|\mathrm{Out}(G)| \neq 1$, then $\pi(\mathrm{Out}(G)) \subseteq \{2, 3\}$.*

Table 1 Finite non-abelian simple groups with $\pi(S) \subseteq \{2, 3, 5, 7, 11, 13, 17\}$

S	$\|S\|$	$\|\mathrm{Out}(S)\|$	S	$\|S\|$	$\|\mathrm{Out}(S)\|$
A_5	$2^2 \cdot 3 \cdot 5$	2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2	$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 7$	3
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4	$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7$	6
$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4	A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2
$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2
$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1	HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2	$L_2(5^2)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	$L_3(3^3)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	6
$L_2(2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6	$L_3(3)$	$2^4 \cdot 3^3 \cdot 7 \cdot 13$	2
$L_3(3^2)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	4	$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4
$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	2	$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	4
$U_3(2^2)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4	$U_4(5)$	$2^5 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 13$	4
A_{18}	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2	$U_4(5)$	$2^5 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 13$	4
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2	$S_4(2^3)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	6
$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2	$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	2
$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	24	$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	2
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	3	$G_2(2^2)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	2
A_{13}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2	A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	2
A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2	A_{16}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	2
$Sz(2^3)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3	${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2	F_{i22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2
$L_2(13^2)$	$2^3 \cdot 3 \cdot 5 \cdot 13^2 \cdot 17$	4	$L_3(2^4)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	24
$U_3(17)$	$2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$	6	$U_4(2^2)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
$S_4(13)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$	2	$S_6(2^2)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2
$O_7(2^2)$	$2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$	2	$O_8^+(2^2)$	$2^{24} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 13 \cdot 17^2$	2
$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2	$F_4(2)$	$2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$	2
A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	2			

Lemma 2.4 [8] *Let G be a Frobenius group with kernel F and complement C. Then the following assertions hold*:

(a) *F is a nilpotent group.*
(b) $|F| \equiv 1 (\mathrm{mod}\, |C|)$.

Lemma 2.5 [9] *Let G be a Frobenius group of even order with H and K being its Frobenius kernel and Frobenius complement, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(K), \pi(H)\}$.*

Lemma 2.6 [10] *Let G be a simple $C_{p,p}$-group, where p is a prime. Then*

(a) *If $p = 5$, G is isomorphic to one of the following simple groups*: A_5, A_6, A_7, M_{11}, M_{22}, $L_3(4)$, $S_4(3)$, $S_4(7)$, $U_4(3)$, $Sz(8)$, $Sz(32)$, $L_2(49)$, $L_2(5^m)$, $L_2(2 \cdot 5^m \pm 1)$, *where* $m \in \mathbb{N}$ *and* $2 \cdot 5^m \pm 1 \in \mathbb{P}$.
(b) *If $p = 7$, G is isomorphic to one of the following simple groups*: A_7, A_8, A_9, M_{22}, J_1, J_2, HS, $L_3(4)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $G_2(13)$, $U_3(3)$, $U_3(5)$, $U_3(19)$, $U_4(3)$, $U_6(2)$, $Sz(8)$, $L_2(8)$, $L_2(7^m)$, $L_2(2 \cdot 7^m - 1)$, *where* $m \in \mathbb{N}$ *and* $2 \cdot 7^m - 1 \in \mathbb{P}$.
(c) *If $p = 13$, G is isomorphic to one of the following simple groups*: A_{13}, A_{14}, A_{15}, Suz, Fi_{22}, $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $U_3(4)$, $U_3(23)$, $G_2(4)$, $G_2(3)$, $F_4(2)$, $Sz(8)$, ${}^2E_6(2)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, $L_2(27)$, $L_2(25)$, $L_2(13^m)$, $L_2(2 \cdot 13^m - 1)$, *where* $m \in \mathbb{N}$ *and* $2 \cdot 13^m - 1 \in \mathbb{P}$.
(d) *If $p = 17$, G is isomorphic to one of the following simple groups*: A_{17}, A_{18}, A_{19}, J_3, He, F_{i23}, F'_{i24}, $L_2(q)$ ($q = 2^4, 17^m, 2 \cdot 17^m \pm 1$ *which is a prime*, $m \geq 1$), $S_4(4)$, $S_8(2)$, $O_8^-(2)$, $F_4(2)$, ${}^2E_6(2)$.

Remark 2.7 Let p be a prime. A group G is called a $C_{p,p}$-group if and only if $p \in \pi(G)$ and the centralizers of its elements of order p in G are p-groups.

3 Proof of Main Theorem

In this section, we give the proof of Main Theorem.

Remark 3.1 Let n be a positive integer and $n > 1$. We say that a finite group G is a K_n-group if and only if $|\pi(G)| = n$.

Proof of Main Theorem Let M be a simple K_3-group, then M is isomorphic to one of the following simple K_3-groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $U_3(3)$, $L_3(3)$, $L_2(17)$ and $U_4(2)$. For convenience, using [3], we have tabulated $|\mathrm{Aut}(M)|$, $D(\mathrm{Aut}(M))$ and $|\mathrm{Out}(M)|$ in Table 2.

Let G be a finite group satisfying (1) $|G| = |\mathrm{Aut}(M)|$ and (2) $D(G) = D(\mathrm{Aut}(M))$. We prove the theorem up to choice of M one by one. The proof is written in four cases.

Table 2 K_3**-groups**

M	$\|\mathrm{Aut}(M)\|$	$D(\mathrm{Aut}(M))$	$\|\mathrm{Out}(M)\|$
A_5	$2^3 \cdot 3 \cdot 5$	$(1,1,0)$	2
$L_2(7)$	$2^4 \cdot 3 \cdot 7$	$(1,1,0)$	2
A_6	$2^5 \cdot 3^2 \cdot 5$	$(2,1,1)$	4
$L_2(8)$	$2^3 \cdot 3^3 \cdot 7$	$(1,1,0)$	3
$L_2(17)$	$2^5 \cdot 3^2 \cdot 17$	$(1,1,0)$	2
$L_3(3)$	$2^5 \cdot 3^3 \cdot 13$	$(1,1,0)$	2
$U_4(2)$	$2^7 \cdot 3^4 \cdot 5$	$(2,1,1)$	2
$U_3(3)$	$2^6 \cdot 3^3 \cdot 7$	$(1,1,0)$	2

Case 1. To prove the theorem if $M = A_5$.

Evidently, $t(G) = 2$. In fact, we have $\pi_1(G) = \{2,3\}$ and $\pi_2(G) = \{5\}$. We first show that G is neither Frobenius nor 2-Frobenius group. Suppose $G = NH$ is a Frobenius group with kernel N and complement H, and hence $T(G) = \{\pi(N), \pi(H)\}$ by Lemma 2.5. Since $|H|$ divides $|N| - 1$, it follows that $|N| = 2^3 \cdot 3$ and $|H| = 5$. Clearly, this is impossible because P_5 cannot act fixed-point-freely for instance on P_3 as $5 \nmid (3 - 1)$.

Now assume that G is a 2-Frobenius group with kernels H and K/H, respectively. Since $T(G) = \{\pi(H) \cup \pi(G/K), \pi(K/H)\}$ and $2 \in \pi(H) \cup \pi(G/K)$, it follows that $|K/H| = 5$. On the other hand, $G/K \lesssim \mathrm{Aut}(K/H) \cong Z_4$. Hence $|G/K| \mid 4$, which implies $\{3,5\} \subseteq \pi(K)$. In this case, we have $3 \in \pi(H)$, so an element of order 5 must act fixed-point-freely on a subgroup of order 3 in H, which is clearly a contradiction by Table 2.

By Lemma 2.1, G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N and G/G_1 are π_1-groups and G_1/N is a non-abelian simple group, N is a nilpotent group. Note that one of the components of the prime graph of G_1/N must be $\{5\}$ and G_1/N is a simple $C_{5,5}$ group. By Lemma 2.6, G_1/N can only be isomorphic to one of the following simple groups: A_5, $A_6, A_7, M_{11}, M_{22}, L_3(4), S_4(3), S_4(7), U_4(3), Sz(8), Sz(32), L_2(49), L_2(5^m)$ and $L_2(2 \cdot 5^m \pm 1)$, where $m \in \mathbb{N}$ and $2 \cdot 5^m \pm 1 \in \mathbb{P}$.

Considering the orders of the simple groups, G_1/N can only be isomorphic to A_5, that is, $G_1/N \cong A_5$. Since $G/N \lesssim \mathrm{Aut}(G_1/N)$, we get $A_5 \lesssim G/N \lesssim \mathrm{Aut}(A_5)$.

If $G/N \cong \mathrm{Aut}(A_5)$, and since $|G| = |\mathrm{Aut}(A_5)|$, we deduce $N = 1$ and $G \cong \mathrm{Aut}(A_5)$.

If $G/N \cong A_5$, then $|N| = 2$ and so $N \leq Z(G)$. Therefore G is a central extension of Z_2 by A_5 and G is isomorphic to one of the following groups:

$2 \cdot A_5$ (a non-split extension of Z_2 by A_5);

$2 : A_5 \cong Z_2 \times A_5$ (a split extension of Z_2 by A_5).

But whether G is isomorphic to $2 \cdot A_5$ or $2 : A_5 \cong Z_2 \times A_5$, it always follows that $15 \in \pi_e(G)$ by [3] (see ATLAS), a contradiction.

Till now we have proved that $G \cong \mathrm{Aut}(A_5)$ if $|G| = |\mathrm{Aut}(A_5)|$ and $D(G) = D(\mathrm{Aut}(A_5))$, that is, $\mathrm{Aut}(A_5)$ is *OD*-characterizable.

Case 2. To prove the theorem holds for M, one of the following simple groups: $L_2(7)$, $L_2(8)$, $U_3(3)$, $L_3(3)$ and $L_2(17)$.

Since $|G| = |\mathrm{Aut}(M)|$ and $D(G) = D(\mathrm{Aut}(M))$, we have to discuss the following five cases. The method used below is the same as Case 1, so the detailed processes are omitted.

(a) If $M = L_2(7)$, then $G \cong \mathrm{Aut}(L_2(7))$;
(b) If $M = L_2(8)$, then $G \cong \mathrm{Aut}(L_2(8))$;
(c) If $M = U_3(3)$, then $G \cong \mathrm{Aut}(U_3(3))$;
(d) If $M = L_3(3)$, then $G \cong \mathrm{Aut}(L_3(3))$;
(e) If $M = L_2(17)$, then $G \cong \mathrm{Aut}(L_2(17))$.

Hence all the almost simple groups $\mathrm{Aut}(L_2(7))$, $\mathrm{Aut}(L_2(8))$, $\mathrm{Aut}(U_3(3))$, $\mathrm{Aut}(L_3(3))$ and $\mathrm{Aut}(L_2(17))$ are *OD*-characterizable.

Case 3. To prove the theorem holds for $M = A_6$.

By Table 2, $|G| = |\mathrm{Aut}(A_6)| = 2^5 \cdot 3^2 \cdot 5$ and $D(G) = D(\mathrm{Aut}(A_6)) = (2,1,1)$. By these facts, we immediately conclude that $\{2,3,5,6,10\} \in \pi_e(G)$ and $15 \notin \pi_e(G)$. It is evident that the prime graph of G is connected since $\deg(2) = 2$ and $|\pi(G)| = 3$. Moreover, it easy to see that $\Gamma(G) = \Gamma(\mathrm{Aut}(A_6))$. We break up the proof into a sequence of subcases.

Subcase 3.1. Let K be a maximal normal solvable subgroup of G. Then K is a 2-group. In particular, G is nonsolvable.

We first prove that K is a $5'$-group. Assume the contrary, then K possesses an element x of order 5. Set $C = C_G(x)$ and $N = N_G(\langle x\rangle)$. By the structure of $D(G)$, C is a $\{2,5\}$-group. By N-C Theorem, $N/C \lesssim \mathrm{Aut}(\langle x\rangle) \cong Z_4$. Hence, N is a $\{2,5\}$-group. By the Frattini argument, $G = KN$. This implies that $\{3,5\} \subseteq \pi(K)$. Since K is solvable, it possesses a Hall $\{3,5\}$-subgroup L of order $3^2 \cdot 5$. Clearly, L is nilpotent, and hence $15 \in \pi_e(G)$, a contradiction.

Next, we show that K is a $3'$-group. Otherwise, let $P_3 \in \mathrm{Syl}_3(K)$. Again, by the Frattini argument $G = KN_G(P_3)$. Hence 5 divides the order of $N_G(P_3)$. Then $N_G(P_3)$ contains a subgroup of order $3^2 \cdot 5$, which leads to a contradiction as before. Therefore K is a 2-group. Since $K \neq G$, it follows that G is nonsolvable. This completes the proof of Subcase 3.1.

Subcase 3.2. The quotient group G/K is an almost simple group. In fact, $S \lesssim G/K \lesssim \mathrm{Aut}(S)$, where S is a non-abelian simple group.

Let $\overline{G} := G/K$ and $S := \mathrm{Soc}(\overline{G})$. Then $S = B_1 \times B_2 \times \cdots \times B_m$, where B_is are non-abelian simple groups and $S \lesssim \overline{G} \lesssim \mathrm{Aut}(S)$. In what follows, we will prove that $m = 1$.

Suppose that $m \geq 2$. It is easy to see that 5 does not divide the order of S, since otherwise $15 \in \pi_e(G)$, a contradiction. On the other hand, by the order of G, we obtain that $\pi(S) \subseteq \{2,3\}$, which is impossible. Therefore $m = 1$ and $S = B_1$.

Subcase 3.3. $S \cong A_6$ and G is isomorphic to one of the following groups: $\mathrm{Aut}(A_6)$, $Z_2 \times Z_2 \times A_6$, $Z_2 \times (Z_2 \cdot A_6)$ and $\mathbb{Z}_4 \times A_6$.

By Lemma 2.3 and Subcase 3.1, we may assume that $|S| = 2^a \cdot 3^2 \cdot 5$, where $2 \leq a \leq 5$. Using Table 1, we see that S can only be isomorphic to the simple group A_6. Thus $A_6 \lesssim G/K \lesssim \mathrm{Aut}(A_6)$.

If $G/K \cong \mathrm{Aut}(A_6)$, then by the order comparison, we obtain that $K = 1$ and $G \cong \mathrm{Aut}(A_6)$.

If $G/K \cong A_6$, then $|K| = 4$ and so $K \cong Z_2 \times Z_2$ or $K \cong Z_4$. Now, we divide the proof into two subcases.

Subsubcase 3.3.1. $G/K \cong A_6$ and $K \cong Z_2 \times Z_2$. By N-C Theorem, we know that the factor group $G/C_G(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$. Thus $|G/C_G(K)| \mid (2^2 - 1)(2^2 - 2)$, that is, $|G/C_G(K)| \mid 6$, which implies that $5 \mid |C_G(K)|$. In particular, $K < C_G(K)$. On the other hand, we have $C_G(K)/K \trianglelefteq G/K \cong A_6$ and hence we obtain $G = C_G(K)$. So $K \leq Z(G)$. Therefore G is a central extension of K by A_6. If G is a non-split extension of K by A_6, then $G \cong Z_2 \times (Z_2 \cdot A_6)$ (see [3]). If G is a split extension over K, we have $G \cong Z_2 \times Z_2 \times A_6$.

Subsubcase 3.3.2. $G/K \cong A_6$ and $K \cong Z_4$. In this case, we have $G/C_G(K) \lesssim \mathrm{Aut}(Z_4) \cong Z_2$, and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 2$, then $K < C_G(K)$. Since $C_G(K)/K \trianglelefteq G/K \cong A_6$, we obtain $G = C_G(K)$, a contradiction. Therefore $|G/C_G(K)| = 1$ and $K \leq Z(G)$. Furthermore, G is a central extension of Z_4 by A_6. Obviously, G cannot be a non-split extension central extension of Z_4 by A_6 since the order of Schur multiplier of A_6 is 6. If G is a split extension over K, we obtain $G \cong Z_4 \times A_6$. This completes the proof of Subcase 3.3 and the case.

Case 4. To prove the theorem if $M = U_4(2)$.

In this case, we have $|G| = |\mathrm{Aut}(U_4(2))| = 2^7 \cdot 3^4 \cdot 5$ and $D(G) = D(\mathrm{Aut}(U_4(2))) = (2,1,1)$ by Table 2. By these hypotheses, we immediately conclude that $\{2,3,5,6,10\} \in \pi_e(G)$ and $15 \notin \pi_e(G)$. Clearly, the prime graph of G is connected, because the vertex 2 is adjacent to all other vertices. Moreover, it is easy to see that $\Gamma(G) = \Gamma(\mathrm{Aut}(U_4(2)))$. We separate the proof into a sequence of subcases.

Subcase 4.1. To prove the theorem holds while G is nonsolvable.

Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-group by the same approach as that in Subcase 3.1. We assert that G/K is an almost simple group. And in fact, $S \lesssim G/K \lesssim \mathrm{Aut}(S)$.

Let $\overline{G} := G/K$ and $S := \mathrm{Soc}(\overline{G})$. Then $S = B_1 \times B_2 \times \cdots \times B_m$, where B_is are non-abelian simple groups and $S \lesssim \overline{G} \lesssim \mathrm{Aut}(S)$. It is easy to see that $m = 1$ by Table 2. Therefore $S = B_1$.

By Lemma 2.3, we can suppose that $|S| = 2^a \cdot 3^2 \cdot 5$, where $2 \leq a \leq 7$, $1 \leq b \leq 4$. Using Table 1, we see that S can only be isomorphic to one of the following simple groups: A_5, A_6 and $U_4(2)$.

If $S \cong A_5$, then $A_5 \lesssim G/K \lesssim \mathrm{Aut}(A_5) \cong S_5$. Hence $2 \cdot 5 \in \pi_e(G) \setminus \pi_e(S_5)$, a contradiction.

If $S \cong A_6$, then $A_6 \lesssim G/K \lesssim \mathrm{Aut}(A_6)$ and $2^2 \cdot 3^2$ divides the order of K.

Let $P_r \in \mathrm{Syl}_r(K)$ for each $r \in \pi(G)$. By the Frattini argument $G = KN_G(P_3)$, 5 divides the order of $N_G(P_3)$. Let T be a subgroup of $N_G(P_3)$ of order 5. By N-C Theorem, the factor group $N_G(P_3)/C_G(P_3)$ is isomorphic to a subgroup of $\mathrm{Aut}(P_3)$. Thus $|G/C_G(K)| \mid (3^2-1)(3^2-3)$, which implies that $T \leq C_G(K)$. Then $15 \in \pi_e(G)$, a contradiction.

If $S \cong U_4(2)$, then $U_4(2) \lesssim G/K \lesssim \mathrm{Aut}(U_4(2))$. In this case, $G/K \cong U_4(2)$, then $|K| = 2$ and $K \leq Z(G)$. Therefore G is a central extension of K by $U_4(2)$. If G is a non-split extension of K by $U_4(2)$, then $G \cong Z_2 \cdot U_4(2)$. If G is a split extension over K, we have $G \cong Z_2 \times U_4(2)$. In the latter case $G/K \cong \mathrm{Aut}(U_4(2))$, by order comparison, we deduce that $K = 1$ and $G \cong \mathrm{Aut}(U_4(2))$.

Till now we have proved that G is isomorphic to one of the following groups: $Z_2 \cdot U_4(2)$, $Z_2 \times U_4(2)$ and $\mathrm{Aut}(U_4(2))$ if G is nonsolvable. It is easy to see that the groups $Z_2 \cdot U_4(2)$, $Z_2 \times U_4(2)$ and $\mathrm{Aut}(U_4(2))$ satisfy the conditions (1) $|G| = |\mathrm{Aut}(U_4(2))|$ and (2) $D(G) = D(\mathrm{Aut}(U_4(2)))$ (see ATLAS).

Subcase 4.2. To prove the theorem holds while G is solvable.

Since G is solvable, we may take a normal series of G: $1 \unlhd N_1 \unlhd N_2 \unlhd G$ such that N_1 is unity or a 2-group, N_2/N_1 is a 3-group or 5-group. While N_2/N_1 is a 5-group, we consider the action a 3-element xN_1 on N_2/N_1, then we see that G/N_1 has an element of order 15, so does G, a contradiction. While N_2/N_1 is a 3-group, it is enough to consider the action of a 5-element of G/N_1 on N_2/N_1, a contradiction appears too if $|N_2/N_1| \mid 3^3$. Hence, $|N_2/N_1| = 3^4$ and the 5-element of G/N_1 must act fixed-point-freely on N_2/N_1. Moreover, the $\{3,5\}$-Hall subgroup H of G is a Frobenius group with kernel P_3 and complement P_5, since otherwise there exists an automorphism of P_3 of order 5, say ϕ, such that $\phi(x) = x$ for each $x \in P_3$. We first show that P_3 is an elementary abelian 3-group.

Set $H = P_3P_5$. Since $Z(\Omega(P_3))$ char $P_3 \unlhd H$ and $|P_3| = 3^4$, then $Z(\Omega(P_3)) \unlhd H$ and $|Z(\Omega(P_3))| \leq 3^4$. By the structure of $D(G)$, G has no elements of order 15, neither does H. Therefore, $Z(\Omega(P_3))$ is an elementary abelian 3-group of order 3^4, as required.

Let x be an element of P_3 of order 3, then we have $\phi(g\langle x\rangle) = g\langle x\rangle$ for every $g \in P_3$. Now a direct computation shows that $\phi(g) = g \cdot x^i$, where $i = 0,1,2$. Hence $\phi^3(g) = g \cdot x^{3i} = g$. However, the order of ϕ is 5, a contradiction.

We have $G = (P_3 \rtimes P_5)P_2$, a product of P_2 and $P_3 \rtimes P_5$. It is obvious that there exists such a finite group satisfying the following conditions: (1) $|G| = |\mathrm{Aut}(M)|$ and (2) $D(G) = D(\mathrm{Aut}(M))$. This completes the proof of Main Theorem. □

In 1987, Shi in [11] put forward the following conjecture:

Conjecture 3.2 *Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if* (1) $|G| = |M|$ *and* (2) $\pi_e(G) = \pi_e(M)$.

Corollary 3.3 *Let M be one of the following simple K_3-groups*: A_5, A_6, $L_2(7)$, $L_2(8)$, $U_3(3)$, $L_3(3)$, $L_2(17)$ *and G be a finite group such that* $|G| = |\operatorname{Aut}(M)|$ *and* $\pi_e(G) = \pi_e(\operatorname{Aut}(M))$. *Then* $G \cong \operatorname{Aut}(M)$.

Proof If $\pi_e(G) = \pi_e(\operatorname{Aut}(M))$, then G and $\operatorname{Aut}(M)$ have the same degree pattern. Hence the result follows from Main Theorem. □

4 An example and a question

Example 4.1 According to Main Theorem, let $G = (P_3 \rtimes P_5) \times P_2$ and $M = U_4(2)$, then $|G| = |\operatorname{Aut}(M)|$ and $D(G) = D(\operatorname{Aut}(M))$. However, G is not isomorphic to $\operatorname{Aut}(M)$. Hence, we put forward the following question:

Question 4.2 Is $\operatorname{Aut}(U_4(2))$ exactly fourfold *OD*-characterizable?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YY carried out the study of the alternating group of degree 5. HX carried out the study of the alternating group of degree 6. GC and LH carried out the study of the group $U_4(2)$. All authors read and approved the final manuscript.

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References

1. Williams, JS: Prime graph components of finite groups. J. Algebra **69**(2), 487-513 (1981)
2. Chen, GY: A new characterization of sporadic simple groups. J. Algebra **3**(1), 49-58 (1996)
3. Conway, JH, Curtis, RT, Norton, SP, Parker, RA, Wilson, RA: Atlas of Finite Groups. Clarendon, Oxford (1985)
4. Moghaddamfar, AR, Zokayi, AR, Darafsheh, MR: A characterization of finite simple groups by the degrees of vertices of their prime graphs. Algebra Colloq. **12**(3), 431-442 (2005)
5. Zhang, LC, Shi, WJ: *OD*-characterization of simple K_4-groups. Algebra Colloq. **16**(2), 275-282 (2009)
6. Yan, YX, Chen, GY, Wang, LL: *OD*-characterization of the automorphism groups of $O^{\pm}_{10}(2)$. Indian J. Pure Appl. Math. **3**(43), 183-195 (2012)
7. Yan, YX, Chen, GY: Recognizing finite groups having connected prime graphs through order and degree pattern. Chin. Ann. Math., Ser. B (to appear)
8. Gorenstein, D: Finite Groups. Harper & Row, New York (1980)
9. Chen, GY: On structure of Frobenius and 2-Frobenius group. J. Southwest China Norm. Univ. **20**(5), 485-487 (1995)
10. Chen, ZM, Shi, WJ: On C_{pp}-simple groups. J. Southwest China Norm. Univ. **18**(3), 249-256 (1993)
11. Shi, WJ: A new characterization of some simple groups of Lie type. Contemp. Math. **82**, 171-180 (1989)

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