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Best proximity points for generalized proximal C -contraction mappings in metric spaces with partial orders

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Abstract

In this paper we extend the notion of weakly C -contraction mappings to the case of non-self mappings and establish the best proximity point theorems for this class. Our results generalize the result due to Harjani *et al.* (Comput. Math. Appl. 61:790-796, 2011) and some other authors.

1 Introduction and preliminaries

In 1922, Banach proved that every contractive mapping in a complete metric space has a unique fixed point, which is called Banach's fixed point theorem or Banach's contraction principle. Since Banach's fixed point theorem, many authors have extended, improved and generalized this theorem in several ways and, further, some applications of Banach's fixed point theorem can be found in [1–6] and many others.

In 1972, Chatterjea [7] introduced the following definition.

Definition 1.1 Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is called a C -contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx)).$$

In 2009, Choudhury [8] introduced a generalization of C -contraction given by the following definition.

Definition 1.2 Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is called a *weakly C -contraction* if, for all $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx)), \quad (1.1)$$

where $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

In 2011, Harjani *et al.* [9] presented some fixed point results for weakly C -contraction mappings in a complete metric space endowed with a partial order as follows.

Theorem 1.3 *Let (X, \preceq) be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \rightarrow X$ be a continuous and nondecreasing mapping such that*

$$d(Tx, Ty) \leq \frac{1}{2} [d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx))$$

for $x \preceq y$, where $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

On the other hand, most of the results on Banach's fixed point theorem dilate upon the existence of a fixed point for self-mappings. Nevertheless, if T is a non-self mapping, then it is probable that the equation $Tx = x$ has no solution, in which case best approximation theorems explore the existence of an approximate solution, whereas best proximity point theorems analyze the existence of an approximate solution that is optimal.

A classical best approximation theorem was introduced by Fan [10], that is, if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \rightarrow B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors including Prolla [11], Reich [12], Sehgal and Singh [13, 14] have derived the extensions of Fan's theorem in many directions. Other works on the existence of a best proximity point for some contractions can be seen in [15–19]. In 2005, Eldred, Kirk and Veeramani [20] obtained best proximity point theorems for relatively nonexpansive mappings, and some authors have proved best proximity point theorems for several types of contractions (see, for example, [21–26]).

Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Let A and B be nonempty subsets of a metric space (X, d) . Now, we recall the following notions:

$$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\},$$

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

If $A \cap B \neq \emptyset$, then A_0 and B_0 are nonempty. Further, it is interesting to notice that A_0 and B_0 are contained in the boundaries of A and B , respectively, provided A and B are closed subsets of a normed linear space such that $d(A, B) > 0$ (see [27]).

Definition 1.4 A mapping $T : A \rightarrow B$ is said to be *increasing* if

$$x \preceq y \implies Tx \preceq Ty$$

for all $x, y \in A$.

Definition 1.5 [28] A mapping $T : A \rightarrow B$ is said to be *proximally order-preserving* if and only if it satisfies the condition that

$$\left. \begin{array}{l} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \implies u \preceq v \tag{1.2}$$

for all $u, v, x, y \in A$.

It is easy to observe that for a self-mapping, the notion of a proximally order-preserving mapping reduces to that of an increasing mapping.

Definition 1.6 A point $x \in A$ is called a *best proximity point* of the mapping $T : A \rightarrow B$ if

$$d(x, Tx) = d(A, B).$$

In view of the fact that $d(x, Tx) \geq d(A, B)$ for all x in A , it can be observed that the global minimum of the mapping $x \mapsto d(x, Tx)$ is attained from a best proximity point. Moreover, it is easy to see that the best proximity point reduces to a fixed point if the underlying mapping T is a self-mapping.

In this paper, we introduce a new class of proximal contractions, which extends the class of weakly C -contractive mappings to the class of non-self mappings, and also give some examples to illustrate our main results. Our results extend and generalize the corresponding results given by Harjani *et al.* [9] and some authors in the literature.

2 Main results

In this section, we first introduce the notion of a generalized proximal C -contraction mapping and establish the best proximity point theorems.

Definition 2.1 A mapping $T : A \rightarrow B$ is said to be a *generalized proximal C -contraction* if, for all $u, v, x, y \in A$, it satisfies

$$\left. \begin{array}{l} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \implies d(u, v) \leq \frac{1}{2}(d(x, v) + d(y, u)) - \psi(d(x, v), d(y, u)), \tag{2.1}$$

where $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

For a self-mapping, it is easy to see that equation (2.1) reduces to (1.1).

Theorem 2.2 Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \rightarrow B$ satisfy the following conditions:

- (a) T is a continuous, proximally order-preserving and generalized proximal C -contraction such that $T(A_0) \subseteq B_0$;

(b) there exist elements x_0 and x_1 in A_0 such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Then there exists a point $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Tx_n) = d(A, B)$$

converges to the point x .

Proof By the hypothesis (b), there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Since $T(A_0) \subseteq B_0$, there exists a point $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

By the proximally order-preserving property of T , we get $x_1 \preceq x_2$. Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that $x_{n-1} \preceq x_n$ and

$$d(x_n, Tx_{n-1}) = d(A, B).$$

Having found the point x_n , one can choose a point $x_{n+1} \in A_0$ such that $x_n \preceq x_{n+1}$ and

$$d(x_{n+1}, Tx_n) = d(A, B). \tag{2.2}$$

Since T is a generalized proximal C -contraction, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)) - \psi(d(x_{n-1}, x_{n+1}), d(x_n, x_n)) \\ &= \frac{1}{2}d(x_{n-1}, x_{n+1}) - \psi(d(x_{n-1}, x_{n+1}), 0) \\ &\leq \frac{1}{2}d(x_{n-1}, x_{n+1}) \\ &\leq \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \end{aligned} \tag{2.3}$$

and so it follows that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, that is, the sequence $\{d(x_{n+1}, x_n)\}$ is non-increasing and bounded below. Then there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = r. \tag{2.4}$$

Taking $n \rightarrow \infty$ in (2.3), we have

$$r \leq \lim_{n \rightarrow \infty} \frac{1}{2} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2}(r+r) = r$$

and so

$$\lim_{n \rightarrow \infty} d(x_{n-1}, x_{n+1}) = 2r. \tag{2.5}$$

Again, taking $n \rightarrow \infty$ in (2.3) and using (2.4), (2.5) and the continuity of ψ , we get

$$r \leq \frac{1}{2}(2r) = r - \psi(2r, 0) \leq r$$

and hence $\psi(2r, 0) = 0$. So, by the property of ψ , we have $r = 0$, which implies that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0. \tag{2.6}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}, \{x_{n_k}\}$ of $\{x_n\}$ such that $n_k > m_k \geq k$ with

$$r_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \quad d(x_{m_k}, x_{n_k-1}) < \varepsilon \tag{2.7}$$

for each $k \in \{1, 2, 3, \dots\}$. For each $n \geq 1$, let $\alpha_n := d(x_{n+1}, x_n)$. So, we have

$$\begin{aligned} \varepsilon \leq r_k &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< \varepsilon + \alpha_{n_k-1}. \end{aligned}$$

It follows from (2.6) that

$$\lim_{k \rightarrow \infty} r_k = \varepsilon. \tag{2.8}$$

Notice also that

$$\begin{aligned} r_k &= d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) \\ &= d(x_{n_k}, x_{m_k+1}) + \alpha_{m_k} \\ &\leq d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1}) + \alpha_{m_k} \\ &= r_k + \alpha_{m_k} + \alpha_{m_k}. \end{aligned} \tag{2.9}$$

Taking $k \rightarrow \infty$ in (2.9), by (2.6) and (2.8), we conclude that

$$\lim_{k \rightarrow \infty} d(x_{n_k}, x_{m_k+1}) = \varepsilon. \tag{2.10}$$

Similarly, we can show that

$$\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k+1}) = \varepsilon. \tag{2.11}$$

On the other hand, by the construction of $\{x_n\}$, we may assume that $x_{m_k} \preceq x_{n_k}$ such that

$$d(x_{n_k+1}, Tx_{n_k}) = d(A, B) \tag{2.12}$$

and

$$d(x_{m_k+1}, Tx_{m_k}) = d(A, B). \tag{2.13}$$

By the triangle inequality, (2.12), (2.13) and the generalized proximal C -contraction of T , we have

$$\begin{aligned} \varepsilon &\leq r_k \leq d(x_{m_k}, x_{m_k+1}) + d(x_{n_k+1}, x_{n_k}) + d(x_{m_k+1}, x_{n_k+1}) \\ &= \alpha_{m_k} + \alpha_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &\leq \alpha_{m_k} + \alpha_{n_k} + \frac{1}{2} [d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})] \\ &\quad - \psi(d(x_{n_k}, x_{m_k+1}), d(x_{m_k}, x_{n_k+1})). \end{aligned}$$

Taking $k \rightarrow \infty$ in the above inequality, by (2.6), (2.10), (2.11) and the continuity of ψ , we get

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \psi(\varepsilon, \varepsilon) \leq \varepsilon.$$

Therefore, $\psi(\varepsilon, \varepsilon) = 0$. By the property of ψ , we have that $\varepsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since A is a closed subset of the complete metric space X , there exists $x \in A$ such that

$$\lim_{n \rightarrow \infty} x_n = x. \tag{2.14}$$

Letting $n \rightarrow \infty$ in (2.2), by (2.14) and the continuity of T , it follows that

$$d(x, Tx) = d(A, B). \tag{2.15}$$

Corollary 2.3 *Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \rightarrow B$ satisfy the following conditions:*

- (a) T is continuous, increasing such that $T(A_0) \subseteq B_0$ and

$$\left. \begin{aligned} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{aligned} \right\} \implies d(u, v) \leq \alpha(d(x, v) + d(y, u)), \tag{2.15}$$

where $\alpha \in (0, \frac{1}{2})$;

- (b) there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Then there exists a point $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Tx_n) = d(A, B)$$

converges to the point x .

Proof Let $\alpha \in (0, \frac{1}{2})$ and the function ψ in Theorem 2.2 be defined by

$$\psi(a, b) = \left(\frac{1}{2} - \alpha\right)(a + b).$$

Obviously, it follows that $\psi(a, b) = 0$ if and only if $a = b = 0$ and (2.1) become (2.15). Hence we obtain Corollary 2.3. \square

For a self-mapping, the condition (b) implies that $x_0 \preceq Tx_0$ and so Theorem 2.2 includes the results of Harjani *et al.* [9] as follows.

Corollary 2.4 [9] *Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Let $T : X \rightarrow X$ be a continuous and nondecreasing mapping such that, for all $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx))$$

for $x \preceq y$, where $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Now, we give an example to illustrate Theorem 2.2.

Example 2.5 Consider the complete metric space \mathbb{R}^2 with an Euclidean metric. Let

$$A = \{(x, 0) : x \in \mathbb{R}\}, \quad B = \{(0, y) : y \in \mathbb{R}, y \geq 1\}.$$

Then $d(A, B) = 1$, $A_0 = \{(0, 0)\}$ and $B_0 = \{(0, 1)\}$. Define a mapping $T : A \rightarrow B$ as follows:

$$T((x, 0)) = (0, 1 + |x|)$$

for all $(x, 0) \in A$. Clearly, T is continuous and $T(A_0) \subseteq B_0$. If $x_1 \preceq x_2$ and

$$d(u_1, Tx_1) = d(A, B) = 1, \quad d(u_2, Tx_2) = d(A, B) = 1$$

for some $u_1, u_2, x_1, x_2 \in A$, then we have

$$u_1 = u_2 = (0, 0), \quad x_1 = x_2 = (0, 0).$$

Therefore, T is a generalized proximal C -contraction with $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ defined by

$$\psi(a, b) = \frac{1}{4}(a + b).$$

Further, observe that $(0, 0) \in A$ such that

$$d((0, 0), T(0, 0)) = d(A, B) = 1.$$

In Theorem 2.6, we do not need the condition that T is continuous. Now, we improve the condition in Theorem 2.2 to prove the new best proximity point theorem as follows.

Theorem 2.6 *Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \rightarrow B$ satisfy the following conditions:*

- (a) *T is a proximally order-preserving and generalized proximal C -contraction such that $T(A_0) \subseteq B_0$;*
- (b) *there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

- (c) *if $\{x_n\}$ is an increasing sequence in A converging to x , then $x_n \preceq x$ for all $n \in \mathbb{N}$. Then there exists a point $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

Proof As in the proof of Theorem 2.2, we have

$$d(x_{n+1}, Tx_n) = d(A, B) \tag{2.16}$$

for all $n \geq 0$. Moreover, $\{x_n\}$ is a Cauchy sequence and converges to some point $x \in A$. Observe that for each $n \in \mathbb{N}$,

$$\begin{aligned} d(A, B) &= d(x_{n+1}, Tx_n) \leq d(x_{n+1}, x) + d(x, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x, x_{n+1}) + d(A, B). \end{aligned}$$

Taking $n \rightarrow \infty$ in the above inequality, we obtain $\lim_{n \rightarrow \infty} d(x, Tx_n) = d(A, B)$ and hence $x \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $v \in A$ such that

$$d(v, Tx) = d(A, B). \tag{2.17}$$

Next, we prove that $x = v$. By the condition (c), we have $x_n \preceq x$ for all $n \in \mathbb{N}$. Using (2.16), (2.17) and the generalized proximal C -contraction of T , we have

$$d(x_{n+1}, v) \leq \frac{1}{2} [d(x_n, v) + d(x, x_{n+1})] - \psi(d(x_n, v), d(x, x_{n+1})). \tag{2.18}$$

Letting $n \rightarrow \infty$ in (2.18), we get

$$d(x, v) \leq \frac{1}{2}d(x, v) - \psi(d(x, v), 0),$$

which implies that $d(x, v) = 0$, that is, $x = v$. If we replace v by x in (2.17), we have

$$d(x, Tx) = d(A, B). \quad \square$$

Corollary 2.7 *Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X such that A_0 and B_0 are nonempty. Let $T : A \rightarrow B$ satisfy the following conditions:*

(a) *T is an increasing mapping such that $T(A_0) \subseteq B_0$ and*

$$\left. \begin{array}{l} x \preceq y \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array} \right\} \implies d(u, v) \leq \alpha(d(x, v) + d(y, u)), \quad (2.19)$$

where $\alpha \in (0, \frac{1}{2})$;

(b) *there exist $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

(c) *if $\{x_n\}$ is an increasing sequence in A converging to a point $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists a point $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Corollary 2.8 [9] *Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Assume that if $\{x_n\} \subseteq X$ is a nondecreasing sequence such that $x_n \rightarrow x$ in X , then $x_n \preceq x$ for all $n \in \mathbb{N}$. Let $T : X \rightarrow X$ be a nondecreasing mapping such that*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx))$$

for $x \preceq y$, where $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\psi(x, y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Now, we recall the condition defined by Nieto and Rodríguez-López [3] for the uniqueness of the best proximity point in Theorems 2.2 and 2.6.

$$\text{For } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \quad (2.20)$$

Theorem 2.9 *Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X and let A_0*

and B_0 be nonempty such that A_0 satisfies the condition (2.20). Let $T : A \rightarrow B$ satisfy the following conditions:

- (a) T is a continuous, proximally order-preserving and generalized proximal C -contraction such that $T(A_0) \subseteq B_0$;
- (b) there exist elements x_0 and x_1 in A_0 such that $x_0 \preceq x_1$ and

$$d(x_1, Tx_0) = d(A, B).$$

Then there exists a unique point $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Proof We will only prove the uniqueness of the point $x \in A$ such that $d(x, Tx) = d(A, B)$. Suppose that there exist x and x^* in A which are best proximity points, that is,

$$d(x, Tx) = d(A, B) \quad \text{and} \quad d(x^*, Tx^*) = d(A, B).$$

Case I: x is comparable to x^* , that is, $x \preceq x^*$ (or $x^* \preceq x$). By the generalized proximal C -contraction of T , we have

$$d(x, x^*) \leq \frac{1}{2} [d(x, x^*) + d(x^*, x)] - \psi(d(x, x^*), d(x^*, x)) \leq d(x^*, x),$$

which implies that $\psi(d(x, x^*), d(x^*, x)) = 0$. Using the property of ψ , we get $d(x^*, x) = 0$ and hence $x = x^*$.

Case II: x is not comparable to x^* . Since A_0 satisfies the condition (2.20), then there exists $z \in A_0$ such that z is comparable to x and x^* , that is, $x \preceq z$ (or $z \preceq x$) and $x^* \preceq z$ (or $z \preceq x^*$). Suppose that $x \preceq z$ and $x^* \preceq z$. Since $T(A_0) \subseteq B_0$, there exists a point $v_0 \in A_0$ such that

$$d(v_0, Tv_0) = d(A, B).$$

By the proximally order-preserving property of T , we get $x \preceq v_0$ and $x^* \preceq v_0$. Since $T(A_0) \subseteq B_0$, there exists a point $v_1 \in A_0$ such that

$$d(v_1, Tv_1) = d(A, B).$$

Again, by the proximally order-preserving property of T , we get $x \preceq v_1$ and $x^* \preceq v_1$. One can proceed further in a similar fashion to find v_n in A_0 with $v_{n+1} \in A_0$ such that

$$d(v_{n+1}, Tv_{n+1}) = d(A, B).$$

Hence $x \preceq v_n$ and $x^* \preceq v_n$ for all $n \in \mathbb{N}$. By the generalized proximal C -contraction of T , we have

$$d(v_{n+1}, x) \leq \frac{1}{2} [d(v_n, x) + d(x, v_{n+1})] - \psi(d(v_n, x), d(x, v_{n+1})), \tag{2.21}$$

$$d(v_{n+1}, x^*) \leq \frac{1}{2} [d(v_n, x^*) + d(x^*, v_{n+1})] - \psi(d(v_n, x^*), d(x^*, v_{n+1})). \tag{2.22}$$

It follows from (2.21), (2.22) and the property of ψ that

$$v_n \rightarrow x \quad \text{and} \quad v_n \rightarrow x^* \quad \text{as } n \rightarrow \infty.$$

By the uniqueness of limit, we conclude that $x = x^*$. Other cases can we proved similarly and this completes the proof. \square

Theorem 2.10 *Let X be a nonempty set such that (X, \preceq) is a partially ordered set and let (X, d) be a complete metric space. Let A and B be nonempty closed subsets of X and let A_0 and B_0 be nonempty such that A_0 satisfies the condition (2.20). Let $T : A \rightarrow B$ satisfy the following conditions:*

- (a) *T is a proximally order-preserving and generalized proximal C -contraction such that $T(A_0) \subseteq B_0$;*
- (b) *there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and*

$$d(x_1, Tx_0) = d(A, B);$$

(c) *if $\{x_n\}$ is an increasing sequence in A converging to x , then $x_n \preceq x$ for all $n \in \mathbb{N}$.
Then there exists a unique point $x \in A$ such that*

$$d(x, Tx) = d(A, B).$$

Proof For the proof, combine the proofs of Theorems 2.6 and 2.9. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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