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Optimal control problems for hyperbolic equations with damping terms involving p-Laplacian

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Abstract

In this paper we study optimal control problems for the hyperbolic equations with a damping term involving p-Laplacian. We prove the existence of an optimal control and the Gâteaux differentiability of a solution mapping on control variables. And then we characterize the optimal controls by giving necessary conditions for optimality.

AMS Subject Classification: 49K20; 93C20

Keywords: optimal control; hyperbolic equation; p-Laplacian; conditions for optimality; Gâteaux differentiability

1 Introduction

In this paper, we are concerned with optimal control problems for the hyperbolic equation with a damping term involving p-Laplacian:

$$\begin{cases} y'' - \triangle y' - \operatorname{div}(|\nabla y|^{p-2}\nabla y) = f & \text{in } \Omega \times (0,\infty), \\ y = 0 & \text{on } \partial\Omega \times (0,\infty), \\ y(x,0) = y_0(x), \qquad y'(x,0) = y_1(x) & \text{in } x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, f is a forcing function, and $y' = \frac{\partial y}{\partial t}$, $y'' = \frac{\partial^2 y}{\partial t^2}$. The background of these variational problems are in physics, especially in solid mechanics. The precise hypotheses on the above system will be given in the next section.

Recently, much research has been devoted to the study of hemivariational inequalities [1–8]. The research works have mainly considered the existence of weak solutions for differential inclusions of various forms [2, 3, 5, 6, 9]. In particular, the case where the nonlinear wave equation includes a nonlinear damping term in a bounded domain was proved by many authors for both existence and nonexistence of global solutions [10–14]. Especially, [15] showed the existence of global weak solutions for (1.1) and the asymptotic stability of the solution by using the Nakao lemma [9].

On the other hand, it is interesting to mention that optimal control problems for the equation of Kirchhoff type with a damping term have been studied by Hwang and Nakagiri [16]. We can find some articles about the studies on some kinds of semilinear partial

Ju and Jeong *Journal of Inequalities and Applications* 2013, 2013:92
http://www.journalofinequalitiesandapplications.com/content/2013/1/92

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differential equations and on quasilinear partial differential equations; see Ha and Naka-
giri [17], Hwang and Nakagiri [18]. Based on these methods, we intend to study optimal
control problems for the hyperbolic hemivariational inequality (1.1) due to the theory of
Lions [19] in which the optimal control problems are surveyed on many types of linear
partial differential equations.

Our goal in this paper is to extend the optimal control theory in the framework of Lions
[19] to the hyperbolic equation (1.1) involving p-Laplacian with a damping term. Let H be a
Hilbert space and let $\mathcal{U}$ be another Hilbert space of control variables, and B be a bounded
linear operator from $\mathcal{U}$ into $L^2(0, T; H)$, which is called a controller. We formulate our
optimal control problem as follows:

$$y''(v) - \triangle y'(v) - \mathrm{div}\big(\big|\nabla y(v)\big|^{p-2}\nabla y(v)\big) = f + Bv \quad \text{in } \Omega \times (0, \infty). \tag{1.2}$$

The plan of this paper is as follows. In Section 2, the main results besides notations and
assumptions are stated. In Section 3, we show the existence of an optimal control $u \in \mathcal{U}$
which minimizes the quadratic cost function. In Section 4, we characterize the optimal
controls by giving necessary conditions for optimality. For this we prove the Gâteaux dif-
ferentiability of the nonlinear mapping $v \to y(v)$, which is used to define the associated
adjoint system.

2 Preliminaries
Throughout this paper we denote

$$V = \big\{y \in W^{1,2}(\Omega) : y = 0 \text{ on } \partial\Omega\big\}, \qquad (y, z) = \int_\Omega y(x)z(x)\,dx, \quad \forall y, z \in V.$$

For every $q \in (1, \infty)$, we denote $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$. For brevity, we denote $\|\cdot\|_2$ by $\|\cdot\|$. For
a Banach space X, we denote by $\|\cdot\|_X$ the norm of X.

Define $A : V \to V^*$ by

$$\langle Ay, z\rangle = \big(|\nabla y|^{p-2}\nabla y, \nabla z\big) \quad \text{for all } y, z \in V,$$

where V^* denotes the dual space of V and $\langle\cdot, \cdot\rangle$ the dual pairing between V and V^*. Then
the operator A is bounded, monotone, hemicontinuous (see, *e.g.*, [20]) and

$$\langle Ay, y\rangle = \|\nabla y\|_p^p, \qquad \big\langle Ay, y'\big\rangle = \frac{1}{p}\frac{d}{dt}\|\nabla y\|_p^p \quad \text{for } y \in V.$$

Now, we formulate the following assumption:

(H) We assume that p is an even natural number satisfying

$$2 \le p < \frac{(N-1)p}{2(N-p)} + 1 \quad (2 \le p < \infty \text{ if } p = N).$$

Definition 2.1 A function $y(x, t)$ is a weak solution to problem (1.1) if for every $T > 0$,
y satisfies $y \in L^\infty(0, T; D(\triangle))$, $y' \in L^2(0, T; D(\triangle)) \cap L^\infty(0, T; V)$, $y'' \in L^2(0, T; H)$. And for

any $z \in V$ and $f \in L^2(0, T; H)$, the following relations hold:

$$\begin{cases} \int_0^T \{\langle y''(t), z \rangle + (\nabla y'(t), \nabla z) + (|\nabla y(t)|^{p-2} \nabla y(t), \nabla z)\}\, dt = \int_0^T (f, z)\, dt, \\ y(0) = y_0, \qquad y'(0) = y_1. \end{cases}$$

The following theorem is from Jeong *et al.* [15].

Theorem 2.1 *Let assumptions* (H) *be satisfied. Then, for* $y_0 \in D(\triangle)$, $y_1 \in V$ *and* $f \in L^2(0, T; H)$, *problem* (1.1) *has a weak solution.*

For any initial data $(y_0, y_1) \in D(\triangle) \times V$, we define the solution space $\mathcal{W} = L^\infty(0, T; V) \cap W^{1,2}(0, T; V) \cap W^{2,2}(0, T; H) \cap W^{1,\infty}(0, T; V)$.

Here we remark that $\mathcal{W}$ is continuously imbedded in $C([0, T]; V) \cap C^1([0, T]; V)$, so that we assume that there exists

$$|\nabla y(t)| < C, \qquad \left| \frac{d}{dt} \nabla y(t) \right| < C. \tag{2.1}$$

Theorem 2.2 *Assume that* $y_0 \in D(\triangle)$, $y_1 \in V$ *and* $f \in L^2(0, T; H)$. *The solution mapping* $p = (y_0, y_1, f) \to y(p)$ *of* $P \equiv D(\triangle) \times V \times L^2(0, T; H)$ *into* $\mathcal{W}$ *is strongly continuous. Further, for each* $p_1 = (y_0^1, y_1^1, f_1) \in P$ *and* $p_2 = (y_0^2, y_1^2, f_2) \in P$, *we have the inequality*

$$\left| y'(p_1; t) - y'(p_2; t) \right|^2 + \left| \nabla \big(y(p_1; t) - y(p_2; t) \big) \right|^2 + \int_0^t \left| \nabla \big(y'(p_1) - y'(p_2) \big) \right|^2 ds$$

$$\leq C \big(\left| \nabla \big(y_0^1 - y_0^2 \big) \right|^2 + \left| y_1^1 - y_1^2 \right|^2 + \| f_1 - f_2 \|_{L^2(0,T;H)}^2 \big). \tag{2.2}$$

Proof Due to [15], we can infer that a weak solution $y \in \mathcal{W}$ under the data condition $p = (y_0, y_1, f) \in D(\triangle) \times V \times L^2(0, T : H)$. Based on the above results, we prove the inequality (2.2). For that purpose, we denote $y_1 - y_2 \equiv y(p_1) - y(p_2)$ by ψ. Then we can observe from (1.1) that

$$\begin{cases} \psi'' - \triangle \psi' - \operatorname{div}(|\nabla y_1|^{p-2} \nabla \psi) = f_1 - f_2 + \operatorname{div}(\varepsilon(\psi)) & \text{in } (0, \infty), \\ \psi = 0 & \text{on } (0, \infty), \\ \psi(0) = y_0^1 - y_0^2, \qquad \psi'(0) = y_1^1 - y_1^2 & \text{in } \Omega, \end{cases} \tag{2.3}$$

where

$$\varepsilon(\psi) = \big(|\nabla y_1|^{p-2} - |\nabla y_2|^{p-2} \big) \nabla y_2$$

$$= \big(\nabla \psi, |\nabla y_1|^{p-3} + \cdots + |\nabla y_2|^{p-3} \big) \nabla y_2. \tag{2.4}$$

Multiplying both sides of (2.3) by ψ', we have

$$\frac{1}{2} \frac{d}{dt} |\psi'(t)|^2 + |\nabla \psi'(t)|^2 + |\nabla y_1(t)|^{p-2} \frac{1}{2} \frac{d}{dt} |\nabla \psi(t)|^2$$

$$= -\big(\varepsilon(\psi), \nabla \psi' \big) + \big(f_1 - f_2, \psi' \big). \tag{2.5}$$

And we integrate (2.5) over [0,t] to have

$$
\left|\psi'(t)\right|^2 + \left|\nabla\psi(t)\right|^2 \left|\nabla y_1(t)\right|^{p-2} + 2\int_0^t \left|\nabla\psi'(s)\right|^2 ds
$$

$$
= -2\int_0^t \left(\varepsilon(\psi), \nabla\psi'\right) ds + \int_0^t |\nabla\psi|^2 \left(\frac{d}{ds}|\nabla y_1|^{p-2}\right) ds
$$

$$
+ \left|\psi'(0)\right|^2 + \left|\nabla\psi(0)\right|^2 \left|\nabla y_0^1\right|^{p-2} + 2\int_0^t \left(f_1 - f_2, \psi'\right) ds. \tag{2.6}
$$

The right member of (2.6) can be estimated as follows:

$$
\left|2\int_0^t \left(f_1 - f_2, \psi'\right) ds\right| \le 2\int_0^t |f_1 - f_2|\left|\psi'\right| ds \le \int_0^t |f_1 - f_2|^2 ds + \int_0^t \left|\psi'\right|^2 ds, \tag{2.7}
$$

$$
\left|-2\int_0^t \left(\varepsilon(\psi), \nabla\psi'\right) ds\right| \le 2\int_0^t |\varepsilon(\psi)|\left|\nabla\psi'\right| ds \le \int_0^t |\varepsilon(\psi)|^2 ds + \int_0^t \left|\nabla\psi'\right|^2 ds
$$

$$
\le c_1\int_0^t |\nabla\psi|^2 ds + \int_0^t \left|\nabla\psi'\right|^2 ds, \tag{2.8}
$$

$$
\left|\int_0^t |\nabla\psi|^2 \left(\frac{d}{ds}|\nabla y_1|^{p-2}\right) ds\right| \le \left|(p-2)\int_0^t \left(|\nabla y_1|^{p-3}, \nabla y_1'\right)|\nabla\psi|^2 ds\right|
$$

$$
\le c_0\int_0^t |\nabla\psi|^2 ds, \tag{2.9}
$$

where c_0, c_1 are constants. By using the boundedness of (2.1), we obtain (2.8) and (2.9). Finally, we replace the right-hand side of (2.6) by the right members of (2.7)-(2.9) and apply Gronwall's inequality to the replaced inequality to obtain

$$
\left|\psi'(t)\right|^2 + \left|\nabla\psi(t)\right|^2 + \int_0^t \left|\nabla\psi'(s)\right|^2 ds
$$

$$
\le C\left(\left|\psi'(0)\right|^2 + \left|\nabla\psi(0)\right|^2 + \|f_1 - f_2\|_{L^2(0,T;H)}^2\right), \tag{2.10}
$$

where C is a constant. This completes the proof. □

3 Existence of an optimal control

Let $\mathcal{U}$ be a Hilbert space of control variables, and let B be a bounded linear operator from $\mathcal{U}$ into $L^2(0,T;H)$, which is denoted by

$$
B \in \mathcal{L}\left(\mathcal{U}, L^2(0,T;H)\right). \tag{3.1}
$$

We consider the following nonlinear control system:

$$
\begin{cases}
y''(v) - \triangle y'(v) - \operatorname{div}(|\nabla y(v)|^{p-2}\nabla y(v)) = f + Bv & \text{in } \Omega \times (0,\infty), \\
y(v) = 0 & \text{on } \partial\Omega \times (0,\infty), \\
y(v; x, 0) = y_0(x), \qquad y'(v; x, 0) = y_1(x) & \text{in } \Omega,
\end{cases} \tag{3.2}
$$

where $y_0 \in D(\triangle)$, $y_1 \in V$, and $v \in \mathcal{U}$ is a control. By virtue of Theorem 2.2 and (3.1), we can define uniquely the solution map $v \to y(v)$ of $\mathcal{U}$ into $\mathcal{W}$. We will call the solution $y(v)$

of (3.2) the state of the control system (3.2). The observation of the state is assumed to be given by

$$z(v) = Cy(v), \quad C \in \mathcal{L}(\mathcal{W}, M), \tag{3.3}$$

where C is an operator called the observer and M is a Hilbert space of observation variables. The quadratic cost function associated with the control system (3.2) is given by

$$J(v) = \left\| Cy(v) - z_d \right\|_M^2 + (Rv, v)_{\mathcal{U}} \quad \text{for } v \in \mathcal{U}, \tag{3.4}$$

where $z_d \in M$ is a desired value of $y(v)$ and $R \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ is symmetric and positive, *i.e.*,

$$(Rv, v)_{\mathcal{U}} = (v, Rv)_{\mathcal{U}} \geq d\|v\|_{\mathcal{U}}^2 \tag{3.5}$$

for some $d > 0$. Let U_{ad} be a closed convex subset of $\mathcal{U}$, which is called an admissible set. An element $u \in U_{ad}$ which attains the minimum of $J(v)$ over U_{ad} is called an optimal control for the cost function (3.4).

As indicated in Introduction, we need to show the existence of an optimal control and to give the characterizations of it. The existence of an optimal control u for the cost function (3.4) can be stated by the following theorem.

Theorem 3.1 *Assume that the hypotheses of Theorem* 2.2 *are satisfied. Then there exists at least one optimal control u for the control problem* (3.2) *with* (3.4).

Proof Set $J = \inf_{v \in U_{ad}} J(v)$. Since U_{ad} is non-empty, there is a sequence $\{v_n\}$ in $\mathcal{U}$ such that

$$\inf_{v \in U_{ad}} J(v) = \lim_{n \to \infty} J(v_n) = J.$$

Obviously, $\{J(v_n)\}$ is bounded in $\mathbb{R}^+$. Then by (3.5) there exists a constant $K_0 > 0$ such that

$$d\|v_n\|_{\mathcal{U}}^2 \leq (Rv_n, v_n)_{\mathcal{U}} \leq J(v_n) \leq K_0. \tag{3.6}$$

This shows that $\{v_n\}$ is bounded in $\mathcal{U}$. Since U_{ad} is closed and convex, we can choose a subsequence (denoted again by $\{v_n\}$) of $\{v_n\}$ and find a $u \in U_{ad}$ such that

$$v_n \to u \quad \text{weakly in } \mathcal{U} \tag{3.7}$$

as $n \to \infty$. From now on, each state $y_n = y(v_n) \in \mathcal{W}$ corresponding to v_n is the solution of

$$\begin{cases} y_n''(t) - \triangle y_n'(t) - \operatorname{div}(|\nabla y_n(t)|^{p-2} \nabla y_n(t)) = f + Bv_n & \text{a.e. } t \in [0, T], \\ y_n(0) = y_0, \qquad y_n'(0) = y_1. \end{cases} \tag{3.8}$$

By (3.6) the term Bv_n is estimated as

$$\|Bv_n\|_{L^2(0,T;H)} \leq \|B\|_{\mathcal{L}(\mathcal{U},L^2(0,T;H))} \|v_n\|_{\mathcal{U}} \leq \|B\|_{\mathcal{L}(\mathcal{U},L^2(0,T;H))} \sqrt{K_0 d^{-1}} \equiv K_1. \tag{3.9}$$

Hence it follows from the inequality (2.2), nothing $y((0,0,0);t) \equiv 0$, that

$$\left|y_n'(t)\right|^2 + \left|\nabla y_n(t)\right|^2 + \int_0^t \left|\nabla y_n'(s)\right|^2 ds \leq C\left(|\nabla y_0|^2 + |y_1|^2 + \|f\|_{L^2(0,T;H)}^2 + K_1^2\right) \qquad (3.10)$$

for some $C > 0$. Combining (3.10) and (3.8), we deduce that

$$\{y_n\} \quad \text{is bounded in } \mathcal{W}. \qquad (3.11)$$

Therefore, by the extraction theorem of Rellich, we can find a subsequence of $\{y_n\}$, say again $\{y_n\}$, and find a $y \in \mathcal{W}$ such that

$$y_n \to y \quad \text{weakly in } \mathcal{W}. \qquad (3.12)$$

By using the fact that $D(\triangle) \hookrightarrow V$ is compact and by virtue of (3.11), we can refer to the result of Aubin-Lions Teman's compact imbedding theorem (*cf.* Teman [21]) to verify that $\{y_n\}$ is precompact in $L^2(0,T;V)$. Hence there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that

$$y_{n_k} \to y \quad \text{strongly in } L^2(0,T;V) \text{ as } k \to \infty. \qquad (3.13)$$

Then we can choose a subsequence of $\{y_{n_k}\}$, denoted again by $\{y_{n_k}\}$, such that

$$y_{n_k}(t) \to y(t) \quad \text{in } V \text{ for a.e. } t \in [0,T]. \qquad (3.14)$$

Now we will show that

$$\text{div}\left(|\nabla y_{n_k}|^{p-2}\nabla y_{n_k}\right) \to \text{div}\left(|\nabla y|^{p-2}\nabla y\right) \quad \text{weakly in } L^2\left(0,T;V^*\right) \text{ as } k \to \infty. \qquad (3.15)$$

Let $\phi(\cdot) \in L^2(0,T;V)$ be given. Then we have

$$\left| \int_0^T \left\{ \left\langle \text{div}\left(|\nabla y_{n_k}(t)|^{p-2}\nabla y_{n_k}(t)\right), \phi(t) \right\rangle - \left\langle \text{div}\left(|\nabla y(t)|^{p-2}\nabla y(t)\right), \phi(t) \right\rangle \right\} dt \right|$$

$$\leq \int_0^T \left| |\nabla y_{n_k}(t)|^{p-2} - |\nabla y(t)|^{p-2} \right| \left| \left(\nabla y_{n_k}(t), \nabla \phi(t)\right) \right| dt$$

$$+ \left| \int_0^T \left(\nabla\left(y_{n_k}(t) - y(t)\right), |\nabla y(t)|^{p-2}\nabla \phi(t)\right) dt \right|. \qquad (3.16)$$

Since $|\nabla y(t)|^{p-2}\nabla\phi \in L^2(0,T;H)$, it follows from (3.13) and the Hölder inequality that the last term of (3.16) converges to 0 as $k \to \infty$. By the continuous imbedding $\mathcal{W} \hookrightarrow C([0,T];V)$ and (3.12), we see that the set $\{|\nabla y_{n_k}(t)| : t \in [0,T], k = 1,2,\ldots\}$ is bounded with a bound $M > 0$. Then, by the Schwartz inequality, the middle term of (3.16) is estimated by

$$\int_0^T \left| |\nabla y_{n_k}(t)|^{p-2} - |\nabla y(t)|^{p-2} \right| \left| \left(\nabla y_{n_k}(t), \nabla \phi(t)\right) \right| dt$$

$$\leq M\left(\int_0^T \left| |\nabla y_{n_k}(t)|^{p-2} - |\nabla y(t)|^{p-2} \right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T |\nabla\phi(t)|^2 dt\right)^{\frac{1}{2}}. \qquad (3.17)$$

We use (3.14) and apply the Lebesgue dominated convergence theorem to have that the term of (3.17) converges to 0 as $k \to \infty$. This proves (3.15). We replace y_n by y_{n_k} and take $k \to \infty$ in (3.8). Since $y_{n_k} \in \mathcal{W}$ are also the weak solutions, as in the standard argument in Dautray and Lions [22] and Lions and Magenes [23], we conclude that the limit y is a weak solution of

$$\begin{cases} y'' + \triangle y' - \operatorname{div}(|\nabla y|^{p-2} \nabla y) = f + Bu & \text{in } \Omega \times (0, \infty), \\ y = 0 & \text{on } \partial\Omega \times (0, \infty), \\ y(x, 0) = y_0(x) y'(x, 0) = y_1(x) & \text{in } \Omega. \end{cases} \tag{3.18}$$

Also, since equation (3.18) has a unique strong solution $y \in \mathcal{W}$ by Theorem 2.2, we conclude that $y = y(v)$ in $\mathcal{W}$ by the uniqueness of solutions, which implies $y(v_n) \to y(u)$ weakly in $\mathcal{W}$. Since C is continuous on $\mathcal{W}$ and $\|\cdot\|_M$ is lower semicontinuous, it follows that

$$\|Cy(u) - z_d\|_M \le \liminf_{n \to \infty} \|Cy(v_n) - z_d\|_M.$$

It is also clear from $\liminf_{n \to \infty} \|R^{\frac{1}{2}} v_n\|_{\mathcal{U}} \ge \|R^{\frac{1}{2}} u\|_{\mathcal{U}}$ that $\liminf_{n \to \infty} (Rv_n, v_n)_{\mathcal{U}} \ge (Ru, u)_{\mathcal{U}}$. Hence

$$J = \liminf_{n \to \infty} J(v_n) \ge J(u).$$

But since $J(u) \ge J$ by definition, we conclude that $J(u) = \inf_{v \in U_{ad}} J(v)$. This completes the proof. □

4 Necessary condition for optimality

In this section we will characterize the optimal controls by giving necessary conditions for optimality. For this it is necessary to write down the necessary optimality condition

$$DJ(u)(v - u) \ge 0 \quad \text{for all } v \in U_{ad} \tag{4.1}$$

and to analyze (4.1) in view of the proper adjoint state system, where $DJ(u)$ denotes the Gâteaux derivative of $J(v)$ at $v = u$. That is, we have to prove that the mapping $v \to y(v)$ of $\mathcal{U} \to \mathcal{W}$ is Gâteaux differentiable at $v = u$. At first we can see the continuity of the mapping. The following lemma follows immediately from Theorem 2.2.

Lemma 4.1 *Let $w \in \mathcal{U}$ be arbitrarily fixed. Then*

$$\lim_{\lambda \to 0} y(u + \lambda w) = y(u) \quad \text{strongly in } \mathcal{W}.$$

The solution map $v \to y(v)$ of $\mathcal{U}$ into $\mathcal{W}$ is said to be Gâteaux differentiable at $v = u$ if for any $w \in \mathcal{U}$ there exists a $Dy(u) \in \mathcal{L}(\mathcal{U}, \mathcal{W})$ such that

$$\left\| \frac{1}{\lambda} \big(y(u + \lambda w) - y(u) \big) - Dy(u)w \right\|_{\mathcal{W}} \to 0 \quad \text{as } \lambda \to 0.$$

The operator $Dy(u)$ denotes the Gâteaux derivative of $y(u)$ at $v = u$ and the function $Dy(u)w \in \mathcal{W}$ is called the Gâteaux derivative in the direction $w \in \mathcal{U}$, which plays an important part in the nonlinear optimal control problem.

Theorem 4.1 *The map $v \to y(v)$ of $\mathcal{U}$ into $\mathcal{W}$ is Gâteaux differentiable at $v = u$ and such a Gâteaux derivative of $y(v)$ at $v = u$ in the direction $v - u \in \mathcal{U}$, say $z = Dy(u)(v - u)$, is a unique strong solution of the following problem*:

$$\begin{cases} z'' - \triangle z' - \operatorname{div}(|\nabla y(u)|^{p-2}\nabla z) = (p-2)|\nabla y(u)|^{p-3}\operatorname{div}(\nabla z \nabla y(u)) + Bw \\ \quad in \ \Omega \times (0, \infty), \\ z = 0 \quad on \ \partial\Omega \times (0, \infty), \\ z(x, 0) = 0, \qquad z'(x, 0) = 0 \quad in \ \Omega. \end{cases} \tag{4.2}$$

Proof Let $\lambda \in (-1, 1)$, $\lambda \neq 0$. We set $w = v - u$ and

$$z_\lambda = \lambda^{-1}\big(y(u + \lambda w) - y(u)\big).$$

Then z_λ satisfies

$$\begin{cases} z_\lambda'' - \triangle z_\lambda' = \frac{1}{\lambda}\{\operatorname{div}(|\nabla y(u + \lambda w)|^{p-2}\nabla y(u + \lambda w)) \\ \qquad\qquad - \operatorname{div}(|\nabla y(u)|^{p-2}\nabla y(u))\} + Bw, \\ z_\lambda(0) = 0, \qquad z_\lambda'(0) = 0 \quad in \ \Omega. \end{cases} \tag{4.3}$$

First we recall the simple equality

$$\frac{1}{\lambda}\Big\{\operatorname{div}\big(\big|\nabla y(u + \lambda w)\big|^{p-2}\nabla y(u + \lambda w)\big) - \operatorname{div}\big(\big|\nabla y(u)\big|^{p-2}\nabla y(u)\big)\Big\}$$

$$= \operatorname{div}\big(\big|\nabla y(u + \lambda w)\big|^{p-2}\nabla z_\lambda\big) + \frac{1}{\lambda}\operatorname{div}\big(\big(\big|\nabla y(u + \lambda w)\big|^{p-2} - \big|\nabla y(u)\big|^{p-2}\big)\nabla y(u)\big).$$

Multiplying the weak form of (4.3) by z_λ' and using the above equality, we have

$$\frac{1}{2}\frac{d}{dt}\big|z_\lambda'(t)\big|^2 + \big|\nabla z_\lambda'(t)\big|^2 + \big|\nabla y(u + \lambda w)\big|^{p-2}\frac{1}{2}\frac{d}{dt}\big|\nabla z_\lambda(t)\big|^2$$

$$= -\big(\nabla z_\lambda, \big|\nabla y(u + \lambda w)\big|^{p-3} + \cdots + \big|\nabla y(u)\big|^{p-3}\big)\big(\nabla y(u), \nabla z_\lambda'\big) + \big(Bw, z_\lambda'\big). \tag{4.4}$$

We integrate (4.4) over $[0, t]$ and use the integration by parts to have

$$\big|z_\lambda'(t)\big|^2 + 2\int_0^t \big|\nabla z_\lambda'(s)\big|^2 ds + \big|\nabla y(u + \lambda w; t)\big|^{p-2}\big|\nabla z_\lambda(t)\big|^2$$

$$= -2\int_0^t \big(\nabla z_\lambda(s), \big|\nabla y(u + \lambda w; s)\big|^{p-3} + \cdots + \big|\nabla y(u; s)\big|^{p-3}\big)\big(\nabla y(u), \nabla z_\lambda'(s)\big) ds$$

$$+ \int_0^t \big|\nabla z_\lambda(s)\big|^2\Big(\frac{d}{ds}\big|\nabla y(u + \lambda w; s)\big|^{p-2}\Big) ds + 2\int_0^t \big(Bw, z_\lambda'\big) ds. \tag{4.5}$$

By virtue of (2.2) and Young's inequality, we can estimate each of the integrands of (4.5) as follows:

$$\big|-2\big(\nabla z_\lambda, \big|\nabla y(u + \lambda w)\big|^{p-3} + \cdots + \big|\nabla y(u)\big|^{p-3}\big)\big(\nabla y(u), \nabla z_\lambda'(s)\big)\big|$$

$$\leq 4(p-2)C^{p-2}|\nabla z_\lambda|\big|\nabla z_\lambda'\big| \leq 4(p-2)^2 C^{2(p-2)}|\nabla z_\lambda|^2 + \big|\nabla z_\lambda'\big|^2, \tag{4.6}$$

$$\left| \left| \nabla z_\lambda(s) \right|^2 \left(\frac{d}{ds} \left| \nabla y(u + \lambda w; s) \right|^{p-2} \right) \right| \le (p-2) C^{p-2} |\nabla z_\lambda|^2, \tag{4.7}$$

$$\left| 2 \left(Bw, z_\lambda' \right) \right| \le 2|Bw||z_\lambda'| \le \frac{2}{\alpha} |Bw|^2 + \frac{\alpha}{2} |z_\lambda'|^2, \tag{4.8}$$

where C is a positive constant depending only on the data and $\alpha > 0$. Combining (4.5) with (4.6)-(4.8), we have

$$|z_\lambda'(t)|^2 + |\nabla z_\lambda(t)|^2 + \int_0^t |\nabla z_\lambda'(s)|^2 \, ds$$

$$\le \max \left\{ 4(p-2)^2 C^{2(p-2)} + (p-2)C^{p-2}, \frac{\alpha}{2} \right\} \int_0^t \left(|z_\lambda'(s)|^2 + |\nabla z_\lambda(s)|^2 \right) ds$$

$$+ \frac{2}{\alpha} \|Bw\|_{L^2(0,T;H)}^2. \tag{4.9}$$

By applying Grownwall's inequality to (4.9), we obtain

$$|z_\lambda'(t)|^2 + |\nabla z_\lambda(t)|^2 + \int_0^t |\nabla z_\lambda'(s)|^2 \, ds \le C_1 \|Bw\|_{L^2(0,T;H)}^2, \tag{4.10}$$

where C_1 is a constant. The inequality (4.10) provides the boundedness of z_λ and z_λ' in appropriate spaces, and hence via (4.3) we can infer that there exists a $z \in \mathcal{W}$ and a sequence $\{\lambda_k\} \subset (-1, 1)$ tending to 0 such that

$$\begin{cases} z_{\lambda_k} \to z & \text{weakly star in } L^\infty(0, T; V) \text{ as } k \to \infty, \\ z_{\lambda_k}' \to z' & \text{weakly star in } L^\infty(0, T; H) \\ & \text{and weakly in } L^2(0, T; V) \text{ as } k \to \infty, \\ z_{\lambda_k}'' \to z'' & \text{weakly in } L^2(0, T; H) \text{ as } k \to \infty, \\ z(0) = 0, & z'(0) = 0. \end{cases} \tag{4.11}$$

From Lemma 4.1 and (4.12), we can easily show that

$$\left| \nabla y(u + \lambda_k w) \right|^{p-2} \nabla z_{\lambda_k} \to \left| \nabla y(u) \right|^{p-2} \nabla z \quad \text{weakly in } L^2(0, T; H) \text{ as } k \to \infty. \tag{4.12}$$

By Lemma 4.1,

$$\nabla y(u + \lambda w) \to \nabla y(u) \quad \text{strongly in } C([0, T]; H) \tag{4.13}$$

so that by (4.12)

$$\left(\nabla z_{\lambda_k}, \nabla y(u + \lambda_k w) \right) \to \left(\nabla z, \nabla y(u) \right) \quad \text{in } L^\infty(0, T).$$

This implies that

$$\frac{1}{\lambda_k} \left(\left| \nabla y(u + \lambda_k w) \right|^{p-2} - \left| \nabla y(u) \right|^{p-2} \right) \nabla y(u) \to (p-2) \left(\nabla z_\lambda, \left| \nabla y(u) \right|^{p-3} \right) \nabla y(u). \tag{4.14}$$

Hence we can see from (4.11)-(4.13) that $z_\lambda \to z = Dy(u)w$ weakly in $\mathcal{W}$ as $\lambda \to 0$ in which

z is a strong solution of (4.2). This convergency can be improved by showing the strong convergence of $\{z_\lambda\}$ also in the topology of $\mathcal{W}$.

Subtracting (4.3) from (4.2) and denoting $z_\lambda - z$ by ϕ_λ, we see that

$$\begin{cases} \phi_\lambda'' - \triangle\phi_\lambda' - \operatorname{div}(|y(u + \lambda w)|^{p-2}\nabla\phi_\lambda) = -\operatorname{div}(\varepsilon_1(y_\lambda)) + \operatorname{div}(\varepsilon_2(y_\lambda, z_\lambda)), \\ \phi_\lambda(0) = 0, \qquad \phi_\lambda'(0) = 0. \end{cases} \tag{4.15}$$

Here in (4.15), for $\lambda \in (-1, 1)$, we set

$$\epsilon_1(y_\lambda) = (|\nabla y(u + \lambda w)|^{p-2} - |\nabla y(u)|^{p-2})\nabla z,$$

$$\epsilon_2(y_\lambda, z_\lambda) = (\nabla z_\lambda, |\nabla y(u + \lambda w)|^{p-3} + \cdots + |\nabla y(u)|^{p-3})\nabla y(u)$$

$$- (p - 2)(\nabla z, |\nabla y(u)|^{p-3})\nabla y(u).$$

From Lemma 4.1 and (4.14), we know that

$$\epsilon_1(y_\lambda), \epsilon_2(y_\lambda, z_\lambda) \to 0 \quad \text{strongly in } L^2(0, T; H) \text{ as } \lambda \to 0. \tag{4.16}$$

To estimate ϕ_λ, we multiply both sides of (4.15) by ϕ_λ' and integrate it over $[0, t]$ and use integration by parts to have

$$|\phi_\lambda'(t)|^2 + |\nabla y(u + \lambda w; t)|^{p-2}|\nabla\phi_\lambda(t)|^2 + 2\int_0^t |\nabla\phi_\lambda'(s)|^2 \, ds$$

$$= \int_0^t |\nabla\phi_\lambda(s)|^2 \left(\frac{d}{ds}|\nabla y(u + \lambda w)|^{p-2}\right) ds$$

$$- 2\int_0^t (\epsilon_1(y_\lambda) + \epsilon_2(y_\lambda, z_\lambda), \nabla\phi_\lambda'(s)) \, ds. \tag{4.17}$$

The integral parts of the right member of (4.17) can be estimated as follows:

$$\left|-2\int_0^t (\epsilon_1(y_\lambda) + \epsilon_2(y_\lambda, z_\lambda), \nabla\phi_\lambda'(s)) \, ds\right|$$

$$\leq 2\int_0^t |\epsilon_1(y_\lambda) + \epsilon_2(y_\lambda, z_\lambda)||\nabla\phi_\lambda'(s)| \, ds$$

$$\leq \int_0^t |\epsilon_1(y_\lambda) + \epsilon_2(y_\lambda, z_\lambda)|^2 \, ds + \int_0^t |\nabla\phi_\lambda'(s)|^2 \, ds, \tag{4.18}$$

$$\left|\int_0^t |\nabla\phi_\lambda(s)|^2 \left(\frac{d}{ds}|\nabla y(u + \lambda w)|^{p-2}\right) ds\right| \leq C\int_0^t |\nabla\phi_\lambda(s)|^2 \, ds, \tag{4.19}$$

where C is a constant. We replace the right-hand side of (4.17) by the right members of (4.18)-(4.19) and we apply Gronwall's inequality to the replaced inequality, then we arrive at

$$|\phi_\lambda'(t)|^2 + |\nabla\phi_\lambda(t)|^2 + \int_0^t |\nabla\phi_\lambda'(s)|^2 \, ds \leq C_2 \|\epsilon_1(y_\lambda) + \epsilon_2(y_\lambda, z_\lambda)\|_{L^2(0,T;H)}^2, \tag{4.20}$$

where C_2 is a constant. By virtue (4.16) and (4.20), we deduce that

$$\phi_\lambda \to 0 \quad \text{in } C([0,T];H) \text{ as } \lambda \to 0, \tag{4.21}$$

$$\phi'_\lambda \to 0 \quad \text{in } \mathcal{W} \text{ as } \lambda \to 0. \tag{4.22}$$

Finally, by means of (4.15), (4.21) and (4.22), it follows that

$$z_\lambda(\cdot) \to z(\cdot) \quad \text{strongly in } \mathcal{W} \text{ as } \lambda \to 0. \tag{4.23}$$

This completes the proof. □

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