Infinitely many solutions for p-harmonic equation with singular term

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Abstract

In this paper, we study the following p-harmonic problem involving the Hardy term:

$$\Delta(|\Delta u|^{p-2}\Delta u) - \lambda \frac{|u|^{p-2}u}{|x|^{2p}} = f(x, u), \quad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where Ω is an open bounded domain containing the origin in $\mathbb{R}^N$, $1 < p < \frac{N}{2}$ and $0 \le \lambda < [N(p-1)(N-2p)/p^2]^p$. By using the variational method, we prove that the above problem has infinitely many solutions with positive energy levels.

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1 Introduction

The main purpose of this paper is to show the existence of infinitely many solutions for the following p-harmonic equation:

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \lambda \frac{|u|^{p-2}u}{|x|^{2p}} = f(x, u), & x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain containing the origin in $\mathbb{R}^N$, the boundary $\partial\Omega$ is smooth. $1 < p < \frac{N}{2}$, $0 \le \lambda < \overline{\lambda} = [N(p-1)(N-2p)/p^2]^p$. $\frac{\partial}{\partial n}$ is the outer normal derivative. Nonlinearity $f(x, u)$ satisfies the following conditions:

(f_1) $f(x, u)$ is continuous on $\Omega \times \mathbb{R}$ and limits subcritical growing at infinity; that is,

$$\lim_{u \to \infty} \frac{f(x, u)}{|u|^{p^*-1}} = 0, \quad \text{uniformly for } x \in \Omega, \tag{1.2}$$

where $p^* = Np/(N - 2p)$ is the critical exponent of Sobolev's embedding $W_0^{2,p}(\Omega) \hookrightarrow L^s(\Omega)$.

(f_2) For any $x \in \Omega$, $f(x, u)$ satisfies

$$\lim_{u \to 0} \frac{f(x, u)u}{|u|^p} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{f(x, u)u}{|u|^p} = +\infty. \tag{1.3}$$

(f_3) Denote $G(x,t) = f(x,t)t - p\int_0^t f(x,s)\,ds$. For all $x \in \Omega$, there exists a constant $M \geq 0$ such that

$$G(x,t_1) \leq G(x,t_2) \quad \text{for any } M \leq |t_1| \leq |t_2|. \tag{1.4}$$

(f_4) $f(x,u)$ is odd with respect to u.

By the Hardy-Rellich inequality (see [1, 2]), we know that

$$\int_\Omega \frac{|u|^p}{|x|^{2p}}\,dx \leq \frac{1}{\bar{\lambda}}\int_\Omega |\Delta u|^p\,dx. \tag{1.5}$$

Obviously, for any $\lambda \in [0, \bar{\lambda})$,

$$\left(1 - \frac{\lambda}{\bar{\lambda}}\right)\int_\Omega |\Delta u|^p\,dx \leq \int_\Omega \left(|\Delta u|^p - \lambda\frac{|u|^p}{|x|^{2p}}\right)dx \leq \int_\Omega |\Delta u|^p\,dx.$$

In $W_0^{2,p}(\Omega)$, for $\lambda \in [0, \bar{\lambda})$, we define

$$\|u\| = \|u\|_{W_0^{2,p}(\Omega)} := \left(\int_\Omega \left(|\Delta u|^p - \lambda\frac{|u|^p}{|x|^{2p}}\right)dx\right)^{\frac{1}{p}},$$

this norm is equivalent to $(\int_\Omega |\Delta u|^p\,dx)^{1/p}$.

A weak solution of the problem (1.1) is a critical point of the energy functional

$$I(u) = \frac{1}{p}\int_\Omega \left(|\Delta u|^p - \lambda\frac{|u|^p}{|x|^{2p}}\right)dx - \int_\Omega F(x,u)\,dx, \quad u \in W_0^{2,p}(\Omega), \tag{1.6}$$

where $F(x,u) = \int_0^u f(x,s)\,ds$. It is easy to check that $I(u)$ is a continuous even functional.

Biharmonic equations can describe the static form change of a beam or the sport of a rigid body. For example, this type of equation furnishes a model for studying traveling wave in suspension bridges (see [3]). By using variational arguments, many authors investigated nonlinear biharmonic equations under Dirichlet boundary conditions or Navier boundary conditions and got interesting results (see [4–10]).

Li and Squassina in [11] considered the superlinear p-harmonic equation with Navier boundary conditions

$$\Delta\left(|\Delta u|^{p-2}\Delta u\right) = f(x,u), \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. $p > 1$, $N \geq 2p+1$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for some positive constant C,

$$\left|f(x,u)\right| \leq C\left(1 + |u|^{q-1}\right) \quad \text{for all } q \in [1, p^*). \tag{1.8}$$

By means of the Morse theory, they proved the existence of two nontrivial solutions to (1.7). After that, Li and Tang [12] considered a more general problem than (1.7). They got three solutions by the three critical points theorem which was obtained in [13]. If $f(x,u) = \lambda g(x,u)$, Candito and Bisci [14] established a well-determined interval of values of the parameter λ for which the problem (1.7) admits at least two distinct weak solutions.

The authors in [15] considered the p-harmonic equation with Dirichlet conditions and obtained the existence of a nontrivial solution. Under the condition of (1.8), [16] proved the existence and multiplicity of weak solutions for the nonuniformly nonlinear problem

$$\Delta\big(a(x,\Delta u)\big) = f(x,u), \quad \text{in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{1.9}$$

where $|a(x,t)| \le c_0(h_0(x) + h_1(x)|t|^{p-1})$ with $h_0(x) \ge 0$, $h_1(x) \ge 1$. $a(x,t)$ and $f(x,t)$ are odd with respect to the second variable. Furthermore, $f(x,t)$ is subcritical and satisfies the Ambrosetti-Rabinowitz condition, that is, there exists a constant $\theta > p$ such that

$$0 < \theta \int_0^t f(x,s)\,ds \le f(x,t)t \quad \text{for any } x \in \Omega. \tag{1.10}$$

The aim of this paper is to obtain infinitely many solutions for the problem (1.1) when $2p < N$. The main difficulty lies in the fact that the embeddings $W_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-2p}\,dx)$ are not compact. Furthermore, the assumption (f_1) is not the usual subcritical growth (1.8). The condition (f_3) is weaker than the A-R condition (1.10). We use the concentration compactness principle (see [17, 18]) to overcome those difficulties. The following theorem is our main result.

Theorem 1.1 *Assume $f(x,u)$ satisfies* (f_1)-(f_4), *then the problem* (1.1) *possesses infinitely many weak solutions and the corresponding critical values are positive.*

This paper proceeds as follows. In the next section, we prove the energy functional $I(u)$ satisfies the Palais-Smale condition. In Section 3, by using the symmetric mountain pass theorem (see [19]), we get the main result of this paper. Throughout the paper, denote

$$S_{p,\lambda} := \inf_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_{p^*}^p}.$$

We use $\|u\|_q = (\int_\Omega |u|^q\,dx)^{1/q}$ to denote the norm of $L^q(\Omega)$, C, C_i stand for universal constants. We omit dx and Ω in the integrals if there is no other indication.

2 Palais-Smale condition

To show the (PS) sequence $\{u_n\}$ of the variational functional $I(u)$ is compact in $W_0^{2,p}(\Omega)$, we first prove the boundedness of $\{u_n\}$ by the analytic argument which has been used in [20]. Then, using the concentration compactness principle, we get the compactness of $\{u_n\}$.

Lemma 2.1 *The condition* (f_1) *implies that for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$F(x,u) \le \varepsilon |u|^{p^*} + C_\varepsilon |u|, \quad (x,u) \in \Omega \times \mathbb{R}. \tag{2.1}$$

(f_2) *implies that $F(x,u) = o(u^p)$ as $u \to 0$. Furthermore, for any $|u| \ge M > 0$, there exists a small positive constant θ*

$$F(x,u) \ge C|u|^{p+\theta} \quad and \quad f(x,u)u > 0. \tag{2.2}$$

Lemma 2.2 *Under the conditions* (f_1), (f_2) *and* (f_3), *if the sequence* $\{u_n\} \in W_0^{2,p}(\Omega)$ *satisfies*

$$I(u_n) \to c, \qquad I'(u_n) \to 0 \quad \text{as } n \to +\infty, \tag{2.3}$$

then the sequence $\{u_n\}$ *is bounded in* $W_0^{2,p}(\Omega)$.

Proof We prove this lemma by contradiction. Without loss of generality, we assume that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty. \tag{2.4}$$

Set $w_n = u_n / \|u_n\|$, then $\|w_n\| = 1$ for any $n \in \mathbb{N}$. As $n \to +\infty$, there exists $w \in W_0^{2,p}(\Omega)$ such that

$$w_n \rightharpoonup w \quad \text{weakly in } W_0^{2,p}(\Omega),$$

$$w_n \to w \quad \text{strongly in } L^s(\Omega) \text{ for any } 1 \le s < p^*,$$

$$w_n \to w \quad \text{a.e. } x \in \Omega.$$

If $w \not\equiv 0$, we denote $\Omega_0 = \{x \in \Omega : w = 0\}$, set $\Omega \setminus \Omega_0$ is nonempty. (2.3) implies that

$$\|u_n\|^p - \int_\Omega f(x, u_n) u_n \, dx = o(1).$$

Therefore,

$$1 - o(1) = \int_\Omega \frac{f(x, u_n) u_n}{\|u_n\|^p} \, dx = \int_{\Omega_0} \frac{f(x, u_n) u_n}{\|u_n\|^p} \, dx + \int_{\Omega \setminus \Omega_0} \frac{f(x, u_n) u_n}{\|u_n\|^p} \, dx. \tag{2.5}$$

As $n \to +\infty$, $\int_\Omega |u_n|^p \, dx \le \|u_n\| \to 0$. So, $|u_n| \to +\infty$ for $x \in \Omega \setminus \Omega_0$. It follows from (f_2) that

$$\lim_{u_n \to \infty} \frac{f(x, u_n) u_n}{\|u_n\|^p} = \lim_{u_n \to \infty} \frac{f(x, u_n) u_n}{|u_n|^p} |w_n|^p = +\infty.$$

Since $|\Omega \setminus \Omega_0| > 0$, then

$$\int_{\Omega \setminus \Omega_0} \frac{f(x, u_n) u_n}{\|u_n\|^p} \, dx \to +\infty \quad \text{as } n \to +\infty. \tag{2.6}$$

Using (2.1) and (2.2), we derive that

$$\begin{aligned}
\int_{\Omega_0} \frac{f(x, u_n) u_n}{\|u_n\|^p} \, dx &= \frac{1}{\|u_n\|^p} \left(\int_{\Omega_0(|u_n| > M)} \cdots + \int_{\Omega_0(|u_n| \le M)} \cdots \right) \\
&\ge -\frac{1}{\|u_n\|^p} \int_{\Omega_0(|u_n| \le M)} |f(x, u_n) u_n| \, dx \\
&\ge -\frac{C}{\|u_n\|^p} \left(M + M^{p^*} \right) |\Omega_0| \\
&\ge -C_1 \quad \text{as } n \to +\infty.
\end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we get

$$\int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^p}\,dx \to +\infty \quad \text{as } n \to +\infty. \tag{2.8}$$

This contradicts (2.5).

If $w \equiv 0$, then $w_n \to 0$ strongly in $L^1(\Omega)$. Since $I(u_n) \to c$, we know that

$$c \leftarrow \frac{1}{p}\|u_n\|^p - \int F(x,u_n)\,dx \geq \frac{1}{p}\|u_n\|^p - \varepsilon\int |u_n|^{p^*}\,dx - C_\varepsilon\int |u_n|\,dx. \tag{2.9}$$

In (2.9), choose $\varepsilon = 1$, then

$$\frac{1}{p}\|u_n\|^p \leq \|u_n\|_{p^*}^{p^*} + C_2\|u_n\|_1 + c \leq C_3\|u_n\|_{p^*}^{p^*} + c.$$

From (2.4), it follows that $\|u_n\|_{p^*} \to +\infty$ as $n \to +\infty$. Denote

$$A_n = M\frac{\|u_n\|}{\|u_n\|_{p^*}} \geq MS_{p,\lambda}^{1/p}, \tag{2.10}$$

where M is any fixed positive constant. It is clear that

$$\frac{A_n}{\|u_n\|} = \frac{M}{\|u_n\|_{p^*}} \to 0 \quad (\text{as } n \to +\infty).$$

Therefore, for n large enough, $A_n/\|u_n\| \in (0,1)$.

For every $n \in \mathbb{N}$, we define a sequence of t_n as follows:

$$I(t_n u_n) = \max_{t \in [0,1]} I(t u_n).$$

We claim that

$$I(t_n u_n) \to +\infty \quad (\text{as } n \to +\infty). \tag{2.11}$$

In fact, by (2.1)-(2.10) and the definition of t_n, we have

$$I(t_n u_n) \geq I\left(\frac{A_n}{\|u_n\|}u_n\right) \geq \frac{1}{p}A_n^p - \varepsilon\int\left|\frac{A_n}{\|u_n\|}u_n\right|^{p^*}\,dx - C_\varepsilon A_n\int |w_n|\,dx$$

$$= \frac{1}{p}A_n^p - \varepsilon M^{p^*} - C_\varepsilon A_n\int |w_n|\,dx.$$

If $A_n \to +\infty$ as $n \to +\infty$, then (2.11) follows from the above estimate. If A_n is bounded for any n, that is, there exists a constant $K > 0$ such that $A_n \leq K$, then

$$I(t_n u_n) \geq \frac{1}{p}A_n^p - \varepsilon M^{p^*} - C_\varepsilon A_n\int |w_n|\,dx \geq \frac{1}{p}S_{p,\lambda}M^p - \varepsilon M^{p^*} - C_\varepsilon K\int |w_n|\,dx$$

$$\to \frac{1}{p}S_{p,\lambda}M^p - \varepsilon M^{p^*} \quad \text{as } n \to +\infty,$$

where we use the fact that $\|w_n\|_1 \to 0$ as $n \to +\infty$. The positive constant ε is arbitrary, thus

$$\lim_{n \to +\infty} I(t_n u_n) \geq \frac{1}{p} S_{p,\lambda} M^p \quad \text{for any positive constant } M.$$

(2.11) follows from the above inequality.

From (2.3), we know that

$$pc + o(1) = pI(u_n) - \langle I'(u_n), u_n \rangle = \int (f(x, u_n)u_n - pF(x, u_n)) \, dx. \tag{2.12}$$

By the definition of t_n, we obtain $\frac{d}{dt} I(tu_n)|_{t=t_n} = 0$ and $t_n \in [0,1]$, then $|t_n u_n| \leq |u_n|$. The condition (f$_3$) and (2.11) derive that

$$
\begin{aligned}
\int (f(x, u_n)u_n - pF(x, u_n)) \, dx &\geq \int (f(x, t_n u_n)t_n u_n - pF(x, t_n u_n)) \, dx \\
&= pI(t_n u_n) - \langle I'(t_n u_n), t_n u_n \rangle \\
&= pI(t_n u_n) - t_n \frac{d}{dt} I(tu_n)\Big|_{t=t_n} \\
&\to +\infty.
\end{aligned}
$$

Which is a contradiction to (2.12). Therefore, the sequence $\{u_n\}$ is bounded in $W_0^{2,p}(\Omega)$.
□

Under the conditions of Lemma 2.2, we know $\|u_n\| \leq C$. Therefore, there exists a subsequence, still denoted by $\{u_n\}$, and some $u \in W_0^{2,p}(\Omega)$ such that

$u_n \rightharpoonup u$ weakly in $W_0^{2,p}(\Omega)$,

$u_n \rightharpoonup u$ weakly in $L^p(\Omega, |x|^{-2p} \, dx)$,

$u_n \to u$ strongly in $L^s(\Omega)$ for any $1 \leq s < p^*$,

$u_n \to u$ a.e. $x \in \Omega$.

Obviously, u is a weak solution of (1.1). Now we prove that $u \not\equiv 0$. According to the concentration compactness principle (see [17, 18]), there exists a subsequence, still denoted by $\{u_n\}$, at most countable set J, a set of different points $\{x_j\}_{j \in J} \subset \overline{\Omega} \setminus \{0\}$ and two positive number sequences $\{\mu_j\}_{j \in J \cup \{0\}}$, $\{\nu_j\}_{j \in J \cup \{0\}}$ such that

$$|\Delta u_n|^p \rightharpoonup d\mu \geq |\Delta u|^p + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0 \quad \text{weakly in the sense of measure,}$$

$$|u_n|^{p^*} \rightharpoonup d\nu = |u|^{p^*} + \sum_{j \in J} \nu_j \delta_{x_j} + \nu_0 \delta_0 \quad \text{weakly in the sense of measure,}$$

$$\frac{|u_n|^p}{|x|^{2p}} \rightharpoonup d\gamma = \frac{|u|^p}{|x|^{2p}} + \gamma_0 \delta_0 \quad \text{weakly in the sense of measure,}$$

$$S_{p,\lambda} \nu_0^{\frac{p}{p^*}} \leq \mu_0 - \lambda \gamma_0, \quad \text{and} \quad S_{p,0} \nu_j^{\frac{p}{p^*}} \leq \mu_j \quad \text{for } j \in J,$$

where δ_{x_j} is the unit Dirac measure at x_j.

Lemma 2.3 *Assume $f(x, u)$ satisfies* (f_1), *the sequence* $\{u_n\}$ *is bounded in* $W_0^{2,p}(\Omega)$ *and satisfies* (2.3). *Then the set* $J = \{x_1, x_2, \ldots, x_j\} \subset \overline{\Omega}$ *is finite,* $\mu_0 - \lambda \gamma_0 = 0$.

Proof We first prove that $\mu_j = 0$ for any $x_j \in J$.

Let $\varepsilon > 0$ be small enough such that $0 \notin B_\varepsilon(x_j)$ and $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \emptyset$ for $i \neq j$. $\phi_j(x) \in [0, 1]$ is a cutting-off function in $C_0^\infty(\Omega)$. $\phi_j(x) \equiv 1$ for $|x - x_j| < \varepsilon/2$, $\phi_j(x) \equiv 0$ for $|x - x_j| \geq \varepsilon$, and $|\Delta \phi_j| \leq 4/\varepsilon^2$. Since

$$\langle I'(u_n), u_n \phi_j \rangle = \int |\Delta u_n|^p \phi_j \, dx + 2 \int |\Delta u_n|^{p-2} \Delta u_n \nabla u_n \nabla \phi_j \, dx$$

$$+ \int u_n |\Delta u_n|^{p-2} \Delta u_n \Delta \phi_j \, dx$$

$$- \int \lambda \frac{|u_n|^p}{|x|^{2p}} \phi_j \, dx - \int f(x, u_n) u_n \phi_j \, dx, \tag{2.13}$$

we deduce that

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int |\Delta u_n|^p \phi_j \, dx = \lim_{\varepsilon \to 0} \int \phi_j \, d\mu \geq \lim_{\varepsilon \to 0} \int |\Delta u|^p \phi_j \, dx + \mu_j = \mu_j,$$

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left| \int \frac{|u_n|^p}{|x|^{2p}} \phi_j \, dx \right| \leq \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int_{B_\varepsilon(x_j)} \frac{|u_n|^p}{(|x_j| - \varepsilon)^{2p}} |\phi_j| \, dx = 0,$$

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int |u_n|^{p^*} \phi_j \, dx = \lim_{\varepsilon \to 0} \int |u|^{p^*} \phi_j \, dx + \nu_j = \nu_j, \tag{2.14}$$

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int |u_n| \phi_j \, dx = \lim_{\varepsilon \to 0} \int |u| \phi_j \, dx = 0. \tag{2.15}$$

By Hölder's inequality,

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left| \int |\Delta u_n|^{p-2} \Delta u_n \nabla u_n \nabla \phi_j \, dx \right|$$

$$\leq \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \|\Delta u_n\|_{L^p(B_\varepsilon(x_j))}^{p-1} \|\nabla u_n\|_{L^{Np/(N-p)}(B_\varepsilon(x_j))} \|\nabla \phi_j\|_{L^N(B_\varepsilon(x_j))}$$

$$\leq C \lim_{\varepsilon \to 0} \|\Delta u_n\|_{L^p(B_\varepsilon(x_j))}^p = 0,$$

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left| \int |\Delta u_n|^{p-2} \Delta u_n u_n \Delta \phi_j \, dx \right|$$

$$\leq \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \|\Delta u_n\|_{L^p(B_\varepsilon(x_j))}^{p-1} \|u_n\|_{L^{p^*}(B_\varepsilon(x_j))} \|\Delta \phi_j\|_{L^{N/2}(B_\varepsilon(x_j))}$$

$$\leq C \lim_{\varepsilon \to 0} \|u_n\|_{L^{p^*}(B_\varepsilon(x_j))} = 0.$$

Using (f_1), (2.14) and (2.15), we have

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int \left| f(x, u_n) u_n \phi_j \right| dx \leq \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left(\varepsilon \int |u_n|^{p^*} \phi_j \, dx + C_\varepsilon \int |u_n| \phi_j \, dx \right) = 0.$$

By the above estimates, (2.13) becomes

$$0 = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \langle I'(u_n), u_n \phi_j \rangle \geq \mu_j. \tag{2.16}$$

Therefore, for any $j \in J$, $\mu_j = 0$, which implies that J is finite.

If the concentration is at the origin, let $\phi_0(x) \in [0,1]$ be a cutting-off function in $C_0^\infty(\Omega)$. $\phi_0(x) \equiv 1$ for $|x| < \varepsilon/2$, $\phi_0(x) \equiv 0$ for $|x| \geq \varepsilon$, and $|\Delta \phi_0| \leq 4/\varepsilon^2$. Choose $\varepsilon > 0$ small enough such that $x_j \notin B_\varepsilon(0)$ for all $j \in J$. Then

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int |\Delta u_n|^p \phi_0 \, dx = \lim_{\varepsilon \to 0} \int \phi_0 \, d\mu \geq \lim_{\varepsilon \to 0} \int |\Delta u_n|^p \phi_0 \, dx + \mu_0 = \mu_0, \qquad (2.17)$$

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left| \int \frac{|u_n|^p}{|x|^{2p}} \phi_0 \, dx \right| = \lim_{\varepsilon \to 0} \int_{B_\varepsilon(x_j)} \frac{|u|^p}{|x|^{2p}} |\phi_0| \, dx + \gamma_0 = \gamma_0. \qquad (2.18)$$

Similarly, we can get

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int \left(2|\Delta u_n|^{p-2} \Delta u_n \nabla u_n \nabla \phi_0 + |\Delta u_n|^{p-2} \Delta u_n u_n \Delta \phi_j \right) dx = 0, \qquad (2.19)$$

$$\lim_{\varepsilon \to 0} \lim_{n \to +\infty} \int \left| f(x, u_n) u_n \phi_0 \right| dx = 0. \qquad (2.20)$$

From equalities (2.17)-(2.20), we get

$$0 = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \left\langle I'(u_n), u_n \phi_0 \right\rangle \geq \mu_0 - \lambda \gamma_0. \qquad (2.21)$$

On the other hand, $\mu_0 - \lambda \gamma_0 \geq S_{p,\lambda} \nu_0^{\frac{p}{p^*}} \geq 0$, thus $\mu_0 - \lambda \gamma_0 = 0$. The proof is complete. $\quad\square$

Lemma 2.4 *Assume $f(x, u)$ satisfies* (f$_1$), *the sequence $\{u_n\}$ satisfying* (2.3) *is bounded in $W_0^{2,p}(\Omega)$. Then there exist some $u \in W_0^{2,p}(\Omega)$ and a subsequence, still denoted by $\{u_n\}$, such that*

$$u_n \to u \quad \text{strongly in } L^{p^*}(\Omega_\epsilon),$$

$$\nabla u_n \to \nabla u \quad \text{strongly in } L^{\frac{Np}{N-p}}(\Omega_\epsilon),$$

where $\Omega_\epsilon = \Omega \setminus \bigcup_{i=0}^{j} \overline{B}_\epsilon(x_i)$, $x_i \in J = \{x_1, x_2, \ldots, x_j\} \cup \{0\}$.

Proof Lemma 2.3 implies that J is finite. Choose $\varphi \in C_0^\infty(\Omega)$ with $\varphi(x_i) = 0$ for any $x_i \in J \cup \{0\}$. Then

$$\int |\varphi u_n|^{p^*} dx = \int |\varphi u|^{p^*} dx + \sum_{i=1}^{j} \nu_i \varphi^{p^*}(x_i) + \nu_0 \varphi^{p^*}(0)$$

$$= \int |\varphi u|^{p^*} dx. \qquad (2.22)$$

Since $\varphi u_n \to \varphi u$ a.e. in Ω, we get that

$$u_n \to u \quad \text{strongly in } L^{p^*}(\Omega_\epsilon).$$

Similarly,

$$\nabla u_n \to \nabla u \quad \text{strongly in } L^{\frac{Np}{N-p}}(\Omega_\epsilon). \qquad\qquad\qquad \square$$

By using the Lebesgue decomposition theorem, we have

$$d\mu - \lambda \, d\gamma = |\Delta u|^p - \lambda \frac{|u|^p}{|x|^{2p}} + d\sigma, \tag{2.23}$$

where $(|\Delta u|^p - \lambda |u|^p |x|^{-2p}) \perp d\sigma$. Then $d\sigma \geq \sum_{j \in J} \mu_j \delta_{x_j} + (\mu_0 - \lambda\gamma_0)\delta_0$.

Lemma 2.5 *Assume $f(x, u)$ satisfies* (f$_1$)-(f$_3$), *the sequence $\{u_n\} \in W_0^{2,p}(\Omega)$ satisfies* (2.3). *Then there exist $u \in W_0^{2,p}(\Omega)$ and a subsequence, still denoted by $\{u_n\}$, such that u_n converges to u strongly in $W_0^{2,p}(\Omega)$.*

Proof We first prove that

$$\left(\sigma - \sum_{j \in J} \mu_j \delta_{x_j} - (\mu_0 - \lambda\gamma_0)\delta_0 \right)(\overline{\Omega}) = 0. \tag{2.24}$$

Claim for any closed set $F \subset \overline{\Omega} \setminus \overline{J}$, $\overline{J} = J \cup \{0\}$, $\sigma(F) = 0$. In fact, denote $r = \mathrm{dist}(F, \overline{J}) > 0$. Then, by using a finite covering theorem, there exist finite open balls $B_{r/4}(z_i)$, with $z_i \in F$, $i = 1, \ldots, m$, such that $F \subset \bigcup_{i=1}^{m} B_{r/4}(z_i) \subset \overline{\Omega} \setminus \overline{J}$. Let $\varrho(t)$ be a smooth cutting-off function. $0 \leq \varrho(t) \leq 1$ for any $t \in [0, \infty)$. $\varrho(x) \equiv 1$ for $0 \leq t \leq 1/2$, and $\varrho \equiv 0$ for $t \geq 1$. Denote $\eta_{\varepsilon,y} = \varrho(|x - y|/\varepsilon)$, then

$$\int_{\mathbb{R}^N} |\nabla \eta_{\varepsilon,y}|^N \, dx = \int_{1/2}^{1} |\varrho'(t)|^N \, dt = C_1, \tag{2.25}$$

$$\int_{\mathbb{R}^N} |\Delta \eta_{\varepsilon,y}|^{\frac{N}{2}} \, dx := \int_{1/2}^{1} |\varrho''(t)|^{\frac{N}{2}} \, dt = C_2. \tag{2.26}$$

The sequence $\{\eta_{\varepsilon,y} u_n\}$ is still bounded in $W_0^{2,p}(\Omega)$. Define

$$\eta(x) = \sum_{i=1}^{m} \eta_{r/4, z_i}(x), \tag{2.27}$$

then $\eta(x) \in C_0^\infty(\overline{\Omega} \setminus \overline{J})$ and $0 \leq \eta(x) \leq m$. Furthermore,

$$F \subset \bigcup_{i=1}^{m} B_{\frac{r}{4}}(z_i) \subset \mathrm{spt}\,\eta(x) \subset \Omega_{\frac{r}{4}}, \tag{2.28}$$

where $\mathrm{spt}\,\eta(x)$ denotes the support of $\eta(x)$. As $n \to +\infty$, (2.3) implies

$$\begin{aligned}
0 \leftarrow & \langle I'(u_n) - I'(u), (u_n - u)\eta \rangle \\
= & \int_\Omega \eta \big(|\Delta u_n|^p - |\Delta u_n|^{p-2}\Delta u_n \Delta u - |\Delta u|^{p-2}\Delta u \Delta u_n + |\Delta u|^p \big) \, dx \\
& - \lambda \int_\Omega \frac{\eta}{|x|^{2p}} \big(|u_n|^p - |u_n|^{p-2}u_n u - |u|^{p-2}u u_n + |u|^p \big) \, dx \\
& + 2 \int_\Omega \nabla\eta \nabla(u_n - u) \big(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u \big) \, dx \\
& + \int_\Omega \Delta\eta(u_n - u) \big(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u \big) \, dx
\end{aligned}$$

$$- \int_{\Omega} \eta (u_n - u)\big(f(x, u_n) - f(x, u)\big)\, dx$$

$$= \ I + II + III + IV + V.$$

As $n \to +\infty$,

$$I + II = \int_{\Omega} \eta \big(|\Delta u_n|^p - |\Delta u_n|^{p-2} \Delta u_n \Delta u - |\Delta u|^{p-2} \Delta u \Delta u_n + |\Delta u|^p \big)\, dx$$

$$- \lambda \int_{\Omega} \frac{\eta}{|x|^{2p}} \big(|u_n|^p - |u_n|^{p-2} u_n u - |u|^{p-2} u u_n + |u|^p \big)\, dx$$

$$= \int_{\Omega} \eta \left(|\Delta u_n|^p - |\Delta u|^p - \lambda \frac{|u_n|^p - |u|^p}{|x|^{2p}}\right) dx + o(1)$$

$$= \int_{\Omega} \eta\, d\sigma + o(1),$$

where we have used (2.23). Lemma 2.2 implies that $\|u_n\| \le M$. Hölder's inequality, (2.25)-(2.28) and Lemma 2.4 deduce that

$$|III| = 2 \left| \int_{\Omega} \nabla \eta \nabla (u_n - u) \big(|\Delta u_n|^{p-2} \Delta u_n - |\Delta u|^{p-2} \Delta u\big)\, dx \right|$$

$$\le C \int_{\Omega} \big(|\Delta u_n|^p + |\Delta u|^p\big)\, dx \|\nabla \eta\|_N \left(\int_{\mathrm{spt}(\eta)} \big|\nabla (u_n - u)\big|^{\frac{Np}{N-p}}\, dx \right)^{\frac{N-p}{Np}}$$

$$\le C(m, M, p) C_1 \left(\int_{\Omega_{\frac{r}{4}}} \big|\nabla (u_n - u)\big|^{\frac{Np}{N-p}}\, dx \right)^{\frac{N-p}{Np}}$$

$$\to 0 \quad \text{as } n \to +\infty,$$

$$|IV| = \left| \int_{\Omega} (u_n - u) \Delta \eta \big(|\Delta u_n|^{p-2} \Delta u_n - |\Delta u|^{p-2} \Delta u\big)\, dx \right|$$

$$\le C(m, M, p) C_2 \left(\int_{\Omega_{\frac{r}{4}}} |u_n - u|^{p^*}\, dx \right)^{\frac{1}{p^*}}$$

$$\to 0 \quad \text{as } n \to +\infty.$$

By (2.1), for any $\varepsilon > 0$, we have

$$|V| \ = \ \left| \int_{\Omega} \big(f(x, u_n) - f(x, u)\big)(u_n - u)\eta\, dx \right|$$

$$\le \int_{\Omega} \big(|f(x, u_n)| + |f(x, u)|\big)|u_n - u|\eta\, dx$$

$$\le \int_{\Omega} \big(C_\varepsilon + \varepsilon\big(|u_n|^{p^*-1} + |u|^{p^*-1}\big)\big)|u_n - u|\eta\, dx$$

$$\le C_\varepsilon \int_{\Omega} |u_n - u|\eta\, dx + \varepsilon \big(\|u_n\|_{p^*}^{p^*-1} + \|u\|_{p^*}^{p^*-1}\big) \left(\int_{\mathrm{spt}(\eta)} |u_n - u|^{p^*}\, dx \right)^{\frac{1}{p^*}}$$

$$\le C_\varepsilon \int_{\Omega} |u_n - u|\, dx + \varepsilon C(M)$$

$$\to 0 \quad \text{as } n \to +\infty, \varepsilon \to 0.$$

Therefore, as $n \to +\infty$,

$$0 \leq \sigma(F) \leq \sigma\left(\mathrm{spt}(\eta)\right) \leq \int_{\Omega} \eta \, d\sigma \leq |III| + |IV| + |V| = 0. \tag{2.29}$$

Since the set F is arbitrary, (2.24) is obtained.

Lemma 2.3 means that $\mu_j = 0$ for any $j \in J$ and $\mu_0 - \lambda \gamma_0 = 0$, so $\sigma(\overline{\Omega}) = 0$, which implies that as $n \to +\infty$,

$$\int_{\Omega} \left(|\Delta u_n|^p - \lambda \frac{|u_n|^p}{|x|^{2p}} \right) dx \to \int_{\Omega} \left(|\Delta u|^p - \lambda \frac{|u|^p}{|x|^{2p}} \right) dx. \tag{2.30}$$

Together with $u_n \to u$ a.e. in Ω, we complete the proof. $\qquad\square$

3 The proof of the main result

In this section, by using the following symmetric mountain pass theorem (see [19]), we give the proof of Theorem 1.1.

Lemma 3.1 (Symmetric mountain pass theorem) *Assume functional I satisfies the following conditions*:

(1) *$I \in \mathcal{C}^1(W_0^{2,p}(\Omega), \mathbb{R})$ is even and satisfies the Palais-Smale condition.*
(2) *There exists a finite dimensional subspace $X \in W_0^{2,p}(\Omega)$ such that $I|_{X \cap \partial B_r} \geq a > 0$.*
(3) *There exists a sequence of the finite dimensional subspace $\{X_j\}$, $\dim(X_j) = j$ and $r_j > 0$ such that*

$$I(u) \leq 0 \quad \text{for any } u \in X_j \setminus B_{r_j}, j = 1, 2, \dots.$$

Then I has infinitely many different critical points, and the corresponding energy values are positive.

Proof of Theorem 1.1 We know the energy functional $I(u)$ is continuous and even. Lemma 2.5 implies that $I(u)$ satisfies the PS condition in $W_0^{2,p}(\Omega)$. In any finite dimensional subspace $X \subset W_0^{2,p}(\Omega)$, all norms are equivalent. For any $u \in X$, since $F(x, u) = o(u^p)$ as $u \to 0$, there exists a constant $q > p$ such that

$$\begin{aligned} I(u) &= \frac{1}{p} \int_{\Omega} \left(|\Delta u|^p - \lambda \frac{|u|^p}{|x|^{2p}} \right) dx - \int_{\Omega} F(x, u) \, dx \\ &\geq \frac{1}{p} \|u\|^p - C \|u\|^q > 0 \quad \text{for } u \text{ small enough.} \end{aligned} \tag{3.1}$$

Thus, there exists $r > 0$ such that $I|_{X \cap \partial B_r} \geq a > 0$. On the other hand, by using (2.2), we have

$$\begin{aligned} I(u) &= \frac{1}{p} \int_{\Omega} \left(|\Delta u|^p - \lambda \frac{|u|^p}{|x|^{2p}} \right) dx - \int_{\Omega} F(x, u) \, dx \\ &\leq \frac{1}{p} \|u\|^p - C \int_{\{x \in \Omega : |u| > M\}} |u|^{p+\theta} \, dx - C' \\ &\leq \frac{1}{p} \|u\|^p - C \|u\|^{p+\theta} - C' \to -\infty \quad \text{as } u \to \infty. \end{aligned}$$

There exists a sequence of the finite dimensional subspace $\{X_j\}$, $\dim(X_j) = j$ and $r_j > 0$ such that

$$I(u) \leq 0 \quad \text{for any } u \in X_j \setminus B_{r_j}, j = 1, 2, \ldots.$$

Finally, all the assumptions of Lemma 3.1 are satisfied. Hence, the problem (1.1) possesses infinitely many weak solutions, and the corresponding critical values are positive. □

Remark 3.1 Under the same conditions of Theorem 1.1, we can also prove that the following p-harmonic type equation with Navier boundary conditions:

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) - \lambda \frac{|u|^{p-2}u}{|x|^{2p}} = f(x, u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

has infinitely many solutions $\{u_n\} \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, where Ω containing the origin is an open bounded domain in $\mathbb{R}^N$, $\partial\Omega$ is smooth. $1 < p < \frac{N}{2}$, $0 \leq \lambda < \overline{\lambda} = [N(p-1)(N - 2p)/p^2]^p$.

Remark 3.2 All the results obtained above obviously hold if we choose $f(x, u) = |u|^{q-2}u + |u|^{r-2}u$ with $p < q < r < p^*$.

Authors' contributions
HX carried out the existence of infinitely many solutions for the *p*-harmonic equation studies and drafted the manuscript. JW participated in its coordination. All authors read and approved the final manuscript.

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