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Homomorphisms and derivations in induced fuzzy C^*-algebras

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Abstract

Using fixed point method, we establish the Hyers-Ulam stability of fuzzy $*$-homomorphisms in fuzzy C^*-algebras and fuzzy $*$-derivations on fuzzy C^*-algebras associated to the following (m,n)-Cauchy-Jensen additive functional equation:

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l\,(\ne i_j,\forall j\in\{1,\ldots,m\})\le n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m}+\sum_{l=1}^{n-m} x_{k_l}\right)=\frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^n f(x_i).$$

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1 Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Rassias [3] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1 (T.M. Rassias) *Let $f: E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality $\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p)$ for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \le p < 1$. Then the limit $L(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ exists for all $x \in E$, and $L: E \to E'$ is the unique additive mapping which satisfies*

$$\|f(x)-L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$, the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [4] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain

X is replaced by an Abelian group. Czerwik [6] proved the Hyers-Ulam stability of the quadratic functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [7–20]).

Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [22–24]). In particular, Bag and Samanta [25], following Cheng and Mordeson [26], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [27]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [28].

Now, we consider a mapping $f : X \to Y$ satisfying the following functional equation, which is introduced by the first author:

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n \\ 1\le k_l\,(\neq i_j,\forall j\in\{1,\dots,m\})\le n}} f\left(\frac{\sum_{j=1}^{m} x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^{n} f(x_i) \tag{1}$$

for all $x_1,\dots,x_n \in X$, where $m,n \in \mathbb{N}$ are fixed integers with $n \ge 2$, $1 \le m \le n$. Especially, we observe that in the case $m = 1$, equation (1) yields the Cauchy-type additive equation $f(\sum_{l=1}^{n} x_{k_l}) = \sum_{l=1}^{n} f(x_i)$. We observe that in the case $m = n$, equation (1) yields the Jensen-type additive equation $f(\frac{\sum_{j=1}^{n} x_j}{n}) = \frac{1}{n}\sum_{l=1}^{n} f(x_i)$. Therefore, equation (1) is a generalized form of the Cauchy-Jensen additive equation and thus every solution of equation (1) may be analogously called a general (m,n)-Cauchy-Jensen additive. For the case $m = 2$, we have established new theorems about the Hyers-Ulam stability in quasi β-normed spaces [29]. Let X and Y be linear spaces. For each m with $1 \le m \le n$, a mapping $f : X \to Y$ satisfies equation (1) for all $n \ge 2$ if and only if $f(x) - f(0) = A(x)$ is a Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f((n-m+1)x) = (n-m+1)f(x)$ and $f(mx) = mf(x)$ for all $x \in X$.

2 Preliminaries

Definition 2.1 Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,t) = 0$ for $t \le 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, c+t) \ge \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

Example 2.1 Let $(X, \|\cdot\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta\|x\|}, & t > 0, x \in X, \\ 0, & t \le 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 2.2 Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N\text{-}\lim_{t\to\infty} x_n = x$.

Definition 2.3 Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [28]).

Definition 2.4 Let X be a $*$-algebra and (X, N) be a fuzzy normed space.

(1) The fuzzy normed space (X, N) is called a fuzzy normed $*$-algebra if

$$N(xy, st) \geq N(x, s) \cdot N(y, t), \qquad N\bigl(x^*, t\bigr) = N(x, t)$$

for all $x, y \in X$ and all positive real numbers s and t.

(2) A complete fuzzy normed $*$-algebra is called a fuzzy Banach $*$-algebra.

Example 2.2 Let $(X, \|\cdot\|)$ be a normed $*$-algebra. Let

$$N(x, t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0, x \in X, \\ 0, & t \leq 0, x \in X. \end{cases}$$

Then $N(x, t)$ is a fuzzy norm on X and (X, N) is a fuzzy normed $*$-algebra.

Definition 2.5 Let $(X, \|\cdot\|)$ be a normed C^*-algebra and N be a fuzzy norm on X.

(1) The fuzzy normed $*$-algebra (X, N) is called an induced fuzzy normed $*$-algebra.

(2) The fuzzy Banach $*$-algebra (X, N) is called an induced fuzzy C^*-algebra.

Definition 2.6 Let (X, N) and (Y, N) be induced fuzzy normed $*$-algebras.

(1) A multiplicative $\mathbb{C}$-linear mapping $H : (X, N) \to (Y, N)$ is called a fuzzy $*$-homomorphism if $H(x^*) = H(x)^*$ for all $x \in X$.

(2) A $\mathbb{C}$-linear mapping $D : (X, N) \to (X, N)$ is called a fuzzy $*$-derivation if $D(xy) = D(x)y + xD(y)$ and $D(x^*) = D(x)^*$ for all $x, y \in X$.

Definition 2.7 Let X be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 2.1 *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, assume that X, Y are unital fuzzy Banach $*$-algebras.

3 Approximate homomorphisms in fuzzy Banach ∗-algebras

In this section, using fixed point method, we prove the Hyers-Ulam stability of homomorphisms in fuzzy Banach $*$-algebras related to functional equation (1).

Theorem 3.1 *Let $\varphi : X^n \to [0,\infty)$ be a function such that there exists an $L < \frac{1}{(n-m+1)^{n-2}}$ with*

$$\varphi\left(\frac{x_1}{n-m+1}, \ldots, \frac{x_n}{n-m+1}\right) \leq \frac{L\varphi(x_1, x_2, \ldots, x_n)}{n-m+1}$$

for all $x_1, \ldots, x_n \in X$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying

$$N\left(\sum_{\substack{1\leq i_1<\cdots<i_m\leq n \\ 1\leq k_l\,(\neq i_j, \forall j\in\{1,\ldots,m\})\leq n}} f\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m} + \sum_{l=1}^{n-m} \mu x_{k_l}\right) - \frac{(n-m+1)\binom{n}{m}\sum_{i=1}^n \mu f(x_i)}{n}, t\right)$$
$$\geq \frac{t}{t + \varphi(x_1, \ldots, x_n)}, \tag{2}$$

$$N\big(f(x_1 \cdots x_{n-1}) - f(x_1)\cdots f(x_{n-1}), t\big) \geq \frac{t}{t + \varphi(x_1, \ldots, x_{n-1}, 0)}, \tag{3}$$

$$N\big(f\big(x_1^*\big) - f(x_1)^*, t\big) \geq \frac{t}{t + \varphi(x_1, 0, \ldots, 0)} \tag{4}$$

for all $x_1, \ldots, x_n \in X$ and all $t > 0$. Then there exists a fuzzy $*$-homomorphism $H : X \to Y$ such that

$$N\big(f(x) - H(x), t\big) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + L\varphi(x, \ldots, x)} \tag{5}$$

for all $x \in X$ and all $t > 0$.

Proof Letting $\mu = 1$ and replacing $(x_1, \ldots, x_n)$ by $(x, \ldots, x)$ in (2), we have

$$N\left(\binom{n}{m} f\big((n-m+1)x\big) - \binom{n}{m}(n-m+1)f(x), t\right) \geq \frac{t}{t + \varphi(x, \ldots, x)} \tag{6}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y; g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf\left\{\mu \in \mathbb{R}^+ : N\big(g(x) - h(x), \mu t\big) \geq \frac{t}{t + \varphi(x, \ldots, x)},\ \forall x \in X, t > 0\right\},$$

where $\inf\emptyset = +\infty$. It is easy to show that (S,d) is complete (see [30]). Now, we consider a linear mapping $J : S \to S$ such that $Jg(x) := (n-m+1)g(\frac{x}{n-m+1})$ for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \epsilon$. Then $N(g(x) - h(x), \epsilon t) \geq \frac{t}{t+\varphi(x,\ldots,x)}$ for all $x \in X$ and $t > 0$. Hence,

$$\begin{aligned}
&N\bigl(Jg(x) - Jh(x), L\epsilon t\bigr) \\
&\quad = N\left(\frac{g(\frac{x}{n-m+1})}{n-m+1} - \frac{h(\frac{x}{n-m+1})}{n-m+1}, L\epsilon t\right) \\
&\quad = N\left(g\left(\frac{x}{n-m+1}\right) - h\left(\frac{x}{n-m+1}\right), \frac{L\epsilon t}{n-m+1}\right) \\
&\quad \geq \frac{\frac{Lt}{n-m+1}}{\frac{Lt}{n-m+1} + \varphi(\frac{x}{n-m+1}, \ldots, \frac{x}{n-m+1})} \geq \frac{\frac{Lt}{n-m+1}}{\frac{Lt}{n-m+1} + \frac{L\varphi(x,\ldots,x)}{n-m+1}} = \frac{t}{t+\varphi(x,\ldots,x)}
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus, $d(g,h) = \epsilon$ implies that $d(Jg,Jh) \leq L\epsilon$. This means that $d(Jg,Jh) \leq Ld(g,h)$ for all $g,h \in S$. It follows from (6) that

$$\begin{aligned}
N\left(\frac{f(\frac{x}{n-m+1})}{(n-m+1)^{-1}} - f(x), \frac{t}{\binom{n}{m}}\right) &\geq \frac{t}{t+\varphi(\frac{x}{n-m+1}, \ldots, \frac{x}{n-m+1})} \\
&\geq \frac{t}{t+\frac{L\varphi(x,\ldots,x)}{n-m+1}}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So,

$$N\left(\frac{f(\frac{x}{n-m+1})}{(n-m+1)^{-1}} - f(x), \frac{Lt}{(n-m+1)\binom{n}{m}}\right) \geq \frac{t}{t+\varphi(x,\ldots,x)}.$$

This implies that $d(f,Jf) \leq \frac{L}{(n-m+1)\binom{n}{m}}$. By Theorem 2.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$H\left(\frac{x}{n-m+1}\right) = \frac{H(x)}{n-m+1} \tag{7}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that H is a unique mapping satisfying (7) such that there exists $\mu \in (0,\infty)$ satisfying $N(f(x) - H(x), \mu t) \geq \frac{t}{t+\varphi(x,\ldots,x)}$ for all $x \in X$ and $t > 0$.

(2) $d(J^p f, H) \to 0$ as $p \to \infty$. This implies the equality

$$N\text{-}\lim_{p\to\infty} \frac{f(\frac{x}{(n-m+1)^p})}{(n-m+1)^{-p}} = H(x) \tag{8}$$

for all $x \in X$.

(3) $d(f,H) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f,H) \leq \frac{L}{(n-m+1)\binom{n}{m} - (n-m+1)\binom{n}{m}L}.$$

This implies that the inequality (5) holds. Furthermore, it follows from (2) and (8) that

$$\begin{aligned} &N\left(\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l\,(\neq i_j,\forall j\in\{1,\ldots,m\})\le n}} H\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m}+\sum_{l=1}^{n-m}\mu x_{k_l}\right)-\frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^n H(\mu x_i),t\right)\\ &\quad= N\text{-}\lim_{p\to\infty}\left((n-m+1)^p\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l(\neq i_j,\forall j\in\{1,\ldots,m\})\le n}} f\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m(n-m+1)^p}+\sum_{l=1}^{n-m}\frac{\mu x_{k_l}}{(n-m+1)^p}\right)\right.\\ &\qquad\left.-\frac{(n-m+1)^{p+1}}{n}\binom{n}{m}\sum_{i=1}^n f\left(\frac{\mu x_i}{(n-m+1)^p}\right),t\right)\\ &\quad\ge \lim_{p\to\infty}\frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p}+\varphi(\frac{x_1}{(n-m+1)^p},\ldots,\frac{x_n}{(n-m+1)^p})}\\ &\quad\ge \lim_{p\to\infty}\frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p}+\frac{L^n\varphi(x_1,\ldots,x_n)}{(n-m+1)^p}}\to 1 \end{aligned}$$

for all $x_1,\ldots,x_n\in X$, all $t>0$ and all $\mu\in\mathbb{C}$. Hence,

$$\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l\,(\neq i_j,\forall j\in\{1,\ldots,m\})\le n}} H\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m}+\sum_{l=1}^{n-m}\mu x_{k_l}\right)=\frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^n H(\mu x_i)$$

for all $x_1,\ldots,x_n\in X$. So, the mapping $H:X\to Y$ is additive and $\mathbb{C}$-linear. By (3)

$$\begin{aligned} &N\left(\frac{f(\frac{x_1\cdots x_{n-1}}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}}-\frac{f(\frac{x_1}{(n-m+1)^p})\cdots f(\frac{x_{n-1}}{(n-m+1)^p})}{(n-m+1)^{-(n-1)p}},\frac{t}{(n-m+1)^{-(n-1)p}}\right)\\ &\quad\ge\frac{t}{t+\varphi(\frac{x_1}{(n-m+1)^p},\ldots,\frac{x_{n-1}}{(n-m+1)^p})} \end{aligned} \tag{9}$$

for all $x_1,\ldots,x_{n-1}\in X$ and all $t>0$. Then

$$\begin{aligned} &N\left(\frac{f(\frac{x_1\cdots x_{n-1}}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}}-\frac{f(\frac{x_1}{(n-m+1)^p})\cdots f(\frac{x_{n-1}}{(n-m+1)^p})}{(n-m+1)^{-(n-1)p}},t\right)\\ &\quad\ge\frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}}+\varphi(\frac{x_1}{(n-m+1)^p},\ldots,\frac{x_{n-1}}{(n-m+1)^p},0)}\\ &\quad\ge\frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}}+\frac{L^p\varphi(x_1,x_2,\ldots,x_{n-1},0)}{(n-m+1)^p}}\to 1\quad\text{when } p\to+\infty \end{aligned}$$

for all $x_1,\ldots,x_{n-1}\in X$ and all $t>0$. So,

$$N\big(H(x_1\cdots x_{n-1})-H(x_1)\cdots H(x_{n-1}),t\big)=1$$

for all $x_1,\ldots,x_{n-1}\in X$ and all $t>0$. Thus, $H(x_1\cdots x_{n-1})=H(x_1)\cdots H(x_{n-1})$.

On the other hand, by (4)

$$N\left(\frac{f(\frac{x_1^*}{(n-m+1)^p})}{(n-m+1)^{-p}} - \frac{f(\frac{x_1}{(n-m+1)^p})}{(n-m+1)^{-p}}^*, \frac{t}{(n-m+1)^{-p}}\right) \geq \frac{t}{t+\varphi(\frac{x_1}{(n-m+1)^p},0,\ldots,0)}$$

for all $x_1 \in X$ and all $t > 0$. So,

$$\begin{aligned} N\left(\frac{f(\frac{x_1^*}{(n-m+1)^p})}{(n-m+1)^{-p}} - \frac{f(\frac{x_1}{(n-m+1)^p})}{(n-m+1)^{-p}}^*, t\right) &\geq \frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p} + \varphi(\frac{x_1}{(n-m+1)^p},0,\ldots,0)} \\ &\geq \frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p} + \frac{L^p}{(n-m+1)^p}\varphi(x_1,0,\ldots,0)} \end{aligned}$$

for all $x_1 \in X$ and all $t > 0$. Since $\lim_{p\to+\infty} \frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p} + \frac{L^p}{(n-m+1)^p}\varphi(x_1,0,\ldots,0)} = 1$ for all $x_1 \in X$ and $t > 0$, we get

$$N\big(H(x_1^*) - H(x_1)^*, t\big) = 1$$

for all $x_1 \in X$ and all $t > 0$. Thus, $H(x_1^*) = H(x_1)^*$ for all $x_1 \in X$. This completes the proof. □

Theorem 3.2 *Let $\varphi : X^n \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1,\ldots,x_n) \leq (n-m+1)L\varphi\left(\frac{x_1}{n-m+1},\ldots,\frac{x_n}{n-m+1}\right)$$

for all $x_1, x_2, \ldots, x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2)-(4). Then the limit $A(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$ exists for each $x \in X$ and defines a fuzzy $*$-homomorphism $H : X \to Y$ such that

$$N\big(f(x) - H(x), t\big) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + \varphi(x,\ldots,x)} \tag{10}$$

for all $x \in X$ and all $t > 0$.

Proof Let (S,d) be a generalized metric space defined as in the proof of Theorem 3.1. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{g((n-m+1)x)}{n-m+1}$ for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \epsilon$. Then $N(g(x) - h(x), \epsilon t) \geq \frac{t}{t+\varphi(x,\ldots,x)}$ for all $x \in X$ and $t > 0$. Hence,

$$\begin{aligned} N\big(Jg(x) - Jh(x), L\epsilon t\big) &= N\left(\frac{g((n-m+1)x)}{n-m+1} - \frac{h((n-m+1)x)}{n-m+1}, L\epsilon t\right) \\ &= N\big(g\big((n-m+1)x\big) - h\big((n-m+1)x\big), (n-m+1)L\epsilon t\big) \\ &\geq \frac{(n-m+1)Lt}{(n-m+1)Lt + \varphi((n-m+1)x,\ldots,(n-m+1)x)} \\ &\geq \frac{(n-m+1)Lt}{(n-m+1)Lt + (n-m+1)L\varphi(x,\ldots,x)} \\ &= \frac{t}{t+\varphi(x,\ldots,x)} \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus, $d(g,h) = \epsilon$ implies that $d(Jg,Jh) \leq L\epsilon$. This means that $d(Jg,Jh) \leq Ld(g,h)$ for all $g,h \in S$. It follows from (6) that

$$N\left(f(x) - \frac{f((n-m+1)x)}{n-m+1}, \frac{t}{(n-m+1)\binom{n}{m}}\right) \geq \frac{t}{t+\varphi(x,\ldots,x)} \tag{11}$$

for all $x \in X$ and $t > 0$. So, $d(f,Jf) \leq \frac{1}{(n-m+1)\binom{n}{m}}$. By Theorem 2.1, there exists a mapping $H : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$(n-m+1)H(x) = H\big((n-m+1)x\big) \tag{12}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that H is a unique mapping satisfying (12) such that there exists $\mu \in (0,\infty)$ satisfying $N(f(x) - H(x), \mu t) \geq \frac{t}{t+\varphi(x,\ldots,x)}$ for all $x \in X$ and $t > 0$.

(2) $d(J^p f, H) \to 0$ as $p \to \infty$. This implies the equality

$$H(x) = N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$$

for all $x \in X$.

(3) $d(f,H) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f,H) \leq \frac{1}{(n-m+1)\binom{n}{m} - (n-m+1)\binom{n}{m}L}.$$

This implies that the inequality (10) holds.

The rest of the proof is similar to the proof of Theorem 3.1. □

From now on, we assume that X has a unit e and a unitary group $\mathcal{U}(X) := \{u \in X : u^*u = uu^* = e\}$.

Theorem 3.3 *Let $\varphi : X^n \to [0,\infty)$ be a function such that there exists an $L < \frac{1}{(n-m+1)^{n-2}}$ with*

$$\varphi\left(\frac{x_1}{n-m+1}, \ldots, \frac{x_n}{n-m+1}\right) \leq \frac{L\varphi(x_1,x_2,\ldots,x_n)}{n-m+1}$$

for all $x_1,\ldots,x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2) *and*

$$N\big(f(u_1\cdots u_{n-1}) - f(u_1)\cdots f(u_{n-1}), t\big) \geq \frac{t}{t+\varphi(u_1,\ldots,u_{n-1},0)}, \tag{13}$$

$$N\big(f(u_1^*) - f(u_1)^*, t\big) \geq \frac{t}{t+\varphi(u_1,0,\ldots,0)} \tag{14}$$

for all $u_1,\ldots,u_n \in \mathcal{U}(X)$ and all $t > 0$. Then there exists a fuzzy $*$-homomorphism $H : X \to Y$ satisfying (5).

Proof By the same reasoning as in the proof of Theorem 3.1, there is a $\mathbb{C}$-linear mapping $H : X \to Y$ satisfying (5). The mapping $H : X \to Y$ is given by

$$N\text{-}\lim_{p\to\infty} \frac{f(\frac{x}{(n-m+1)^p})}{(n-m+1)^{-p}} = H(x)$$

for all $x \in X$. By (13)

$$N\left(\frac{f(\frac{u_1\cdots u_{n-1}}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}} - \frac{f(\frac{u_1}{(n-m+1)^p})\cdots f(\frac{u_{n-1}}{(n-m+1)^p})}{(n-m+1)^{-(n-1)p}}, \frac{t}{(n-m+1)^{-(n-1)p}}\right) \\ \geq \frac{t}{t+\varphi(\frac{u_1}{(n-m+1)^p},\ldots,\frac{u_{n-1}}{(n-m+1)^p})}$$

for all $u_1,\ldots,u_{n-1} \in \mathcal{U}(X)$ and all $t > 0$. Then

$$N\left(\frac{f(\frac{u_1\cdots u_{n-1}}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}} - \frac{f(\frac{u_1}{(n-m+1)^p})\cdots f(\frac{u_{n-1}}{(n-m+1)^p})}{(n-m+1)^{-(n-1)p}}, t\right) \\ \geq \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \varphi(\frac{u_1}{(n-m+1)^p},\ldots,\frac{u_{n-1}}{(n-m+1)^p},0)} \\ \geq \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \frac{L^p\varphi(u_1,u_2,\ldots,u_{n-1},0)}{(n-m+1)^p}} \to 1$$

as $p \to +\infty$ for all $u_1,\ldots,u_{n-1} \in \mathcal{U}(X)$ and all $t > 0$. So,

$$N\bigl(H(u_1\cdots u_{n-1}) - H(u_1)\cdots H(u_{n-1}), t\bigr) = 1$$

for all $u_1,\ldots,u_{n-1} \in \mathcal{U}(X)$ and all $t > 0$. Thus,

$$H(u_1\cdots u_{n-1}) = H(u_1)\cdots H(u_{n-1}). \tag{15}$$

Since H is $\mathbb{C}$-linear and each $x \in X$ is a finite linear combination of unitary elements, *i.e.*,

$$x = \sum_{j=1}^{m} \lambda_j u_j \quad \bigl(\lambda_j \in \mathbb{C}, u_j \in U(X)\bigr),$$

it follows from (15) that

$$H(xv) = H\left(\sum_{j=1}^{m} \lambda_j u_j v\right) = \sum_{j=1}^{n} \lambda_j H(u_j v) = \sum_{j=1}^{n} \lambda_j H(u_j)H(v) = H\left(\sum_{j=1}^{m} \lambda_j u_j\right)H(v)$$

for all $v \in \mathcal{U}(X)$. So, $H(xv) = H(x)H(v)$. Similarly, one can obtain that $H(xy) = H(x)H(y)$ for all $x, y \in X$. Thus by induction, one can easily show that $H(x_1\cdots x_{n-1}) = H(x_1)\cdots H(x_{n-1})$. By (4)

$$N\left(\frac{f(\frac{u_1^*}{(n-m+1)^p})}{(n-m+1)^{-p}} - \frac{f(\frac{u_1}{(n-m+1)^p})}{(n-m+1)^{-p}}^*, \frac{t}{(n-m+1)^{-p}}\right) \geq \frac{t}{t+\varphi(\frac{u_1}{(n-m+1)^p},0,\ldots,0)}$$

for all $u_1 \in \mathcal{U}(X)$ and all $t > 0$. So,

$$N\left(\frac{f(\frac{u_1^*}{(n-m+1)^p})}{(n-m+1)^{-p}} - \frac{f(\frac{u_1}{(n-m+1)^p})}{(n-m+1)^{-p}}^*, t\right) \geq \frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p} + \varphi(\frac{u_1}{(n-m+1)^p}, 0, \dots, 0)} \geq \frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p} + \frac{L^p\varphi(u_1,0,\dots,0)}{(n-m+1)^p}}$$

for all $u_1 \in \mathcal{U}(X)$ and all $t > 0$. Since $\lim_{p\to+\infty} \frac{\frac{t}{(n-m+1)^p}}{\frac{t}{(n-m+1)^p} + \frac{L^p\varphi(u_1,0,\dots,0)}{(n-m+1)^p}} = 1$ for all $u_1 \in \mathcal{U}(X)$ and all $t > 0$, we get

$$N\bigl(H\bigl(u_1^*\bigr) - H(u_1)^*, t\bigr) = 1$$

for all $u_1 \in \mathcal{U}(X)$ and all $t > 0$. Thus,

$$H\bigl(u_1^*\bigr) = H(u_1)^* \tag{16}$$

for all $u_1 \in \mathcal{U}(X)$. Since H is $\mathbb{C}$-linear and each $x \in X$ is a finite linear combination of unitary elements, *i.e.*, $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(X)$), it follows from (16) that

$$H\bigl(x^*\bigr) = H\left(\sum_{j=1}^m \overline{\lambda_j} u_j^*\right) = \sum_{j=1}^n \overline{\lambda_j} H\bigl(u_j^*\bigr) = \sum_{j=1}^n \overline{\lambda_j} H(u_j)^* = H\left(\sum_{j=1}^m \lambda_j u_j\right)^* = H(x)^*$$

for all $x \in X$. So, $H(x^*) = H(x)^*$ for all $x \in X$. Therefore, the mapping $H : X \to Y$ is a $*$-homomorphism. □

Similarly, we have the following. We will omit the proof.

Theorem 3.4 *Let $\varphi : X^n \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1, \dots, x_n) \leq (n-m+1)L\varphi\left(\frac{x_1}{n-m+1}, \dots, \frac{x_n}{n-m+1}\right)$$

for all $x_1, x_2, \dots, x_n \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2), (13) *and* (14). *Then the limit $A(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$ exists for each $x \in X$ and defines a fuzzy $*$-homomorphism $H : X \to Y$ such that*

$$N\bigl(f(x) - H(x), t\bigr) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + \varphi(x, \dots, x)}$$

for all $x \in X$ and all $t > 0$.

4 Approximate derivations on fuzzy Banach $*$-algebras

In this section, we assume that (X, N) is a fuzzy Banach $*$-algebra. Using fixed point method, we prove the Hyers-Ulam stability of derivations on fuzzy Banach $*$-algebras related to functional equation (1).

Theorem 4.1 *Let $\varphi : X^n \to [0,\infty)$ be a function such that there exists an $L < \frac{1}{(n-m+1)^{n-2}}$ with*

$$\varphi\left(\frac{x_1}{n-m+1},\ldots,\frac{x_n}{n-m+1}\right) \le \frac{L\varphi(x_1,x_2,\ldots,x_n)}{n-m+1}$$

for all $x_1,\ldots,x_n \in X$. Let $f : X \to X$ be a mapping satisfying $f(0) = 0$,

$$N\left(\sum_{\substack{1\le i_1<\cdots<i_m\le n\\ 1\le k_l\,(\neq i_j,\forall j\in\{1,\ldots,m\})\le n}} f\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m}+\sum_{l=1}^{n-m}\mu x_{k_l}\right)-\frac{(n-m+1)\binom{n}{m}\sum_{i=1}^n \mu f(x_i)}{n},t\right)$$
$$\ge \frac{t}{t+\varphi(x_1,\ldots,x_n)}, \tag{17}$$

$$N\left(f\left(x_1^*\right)-f(x_1)^*,t\right) \ge \frac{t}{t+\varphi(x_1,0,\ldots,0)}, \tag{18}$$

$$N\left(f\left(\prod_{i=1}^{n-1}x_i\right)-\sum_{i=1}^{n-1}\sum_{j=1,j\neq i}^{n-1}x_i f(x_j),t\right) \ge \frac{t}{t+\varphi(x_1,\ldots,x_{n-1},0)} \tag{19}$$

for all $x_1,\ldots,x_{n-1} \in X$ and all $t > 0$. Then $D(x) := N\text{-}\lim_{p\to\infty} \frac{f(\frac{x}{(n-m+1)^p})}{(n-m+1)^{-p}}$ exists for all $x \in X$ and defines a fuzzy $*$-derivation $D : X \to X$ such that

$$N\left(f(x)-D(x),t\right) \ge \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t+L\varphi(x,\ldots,x)}$$

for all $x \in X$ and all $t > 0$.

Proof The proof is similar to the proof of Theorem 3.1. □

Theorem 4.2 *Let $\varphi : X^n \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1,\ldots,x_n) \le (n-m+1)L\varphi\left(\frac{x_1}{n-m+1},\ldots,\frac{x_n}{n-m+1}\right)$$

for all $x_1,x_2,\ldots,x_n \in X$. Let $f : X \to X$ be a mapping satisfying $f(0) = 0$, (17), (18) *and* (19). *Then the limit $D(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$ exists for all $x \in X$ and defines a fuzzy $*$-derivation $D : X \to X$ such that*

$$N\left(f(x)-D(x),t\right) \ge \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t+\varphi(x,\ldots,x)}$$

for all $x \in X$ and all $t > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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