Di et al. *Journal of Inequalities and Applications* 2013, **2013**:84
http://www.journalofinequalitiesandapplications.com/content/2013/1/84

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# A note on a class of Hardy-Rellich type inequalities

Yanmei Di, Liya Jiang, Shoufeng Shen and Yongyang Jin*

*Correspondence:
yongyang@zjut.edu.cn
Department of Mathematics,
Zhejiang University of Technology,
Hangzhou, P.R. China

**Abstract**
In this note we provide simple and short proofs for a class of Hardy-Rellich type inequalities with the best constant, which extends some recent results.
**MSC:** 26D15; 35A23

**Keywords:** Hardy inequality; Hardy-Rellich inequality; Caffarelli-Kohn-Nirenberg inequality

## 1 Introduction

It is well known that Hardy's inequality and its generalizations play important roles in many areas of mathematics. The classical Hardy inequality is given by, for $N \geq 3$,

$$\int_{R^N} \left|\nabla u(x)\right|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{R^N} \frac{|u(x)|^2}{|x|^2}\,dx, \tag{1.1}$$

where $u \in C_0^\infty(R^N)$, the constant $(\frac{N-2}{2})^2$ is optimal and not attained.

Recently there has been a considerable interest in studying the Hardy-type and Rellich-type inequalities. See, for example, [1–7]. In [8] Caffarelli, Kohn and Nirenberg proved a rather general interpolation inequality with weights. That is the following so-called Caffarelli-Kohn-Nirenberg inequality. For any $u \in C_0^\infty(R^N)$, there exists $C > 0$ such that

$$\left\||x|^\gamma u\right\|_{L^r} \leq C\left\||x|^\alpha|\nabla u|\right\|_{L^p}^a \cdot \left\||x|^\beta u\right\|_{L^q}^{1-a}, \tag{1.2}$$

where

$$\frac{1}{r} + \frac{\gamma}{N} = a\left(\frac{1}{p} + \frac{\alpha - 1}{N}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{N}\right)$$

and

$$p, q \geq 1, \qquad r \geq 0, \qquad 0 \leq a \leq 1, \qquad \frac{1}{p} + \frac{\alpha}{N} > 0, \qquad \frac{1}{q} + \frac{\beta}{N} > 0,$$
$$\frac{1}{r} + \frac{\gamma}{N} > 0, \qquad \gamma = a\sigma + (1-a)\beta.$$

In [9] Costa proved the following $L^2$-case version for a class of Caffarelli-Kohn-Nirenberg inequalities with a sharp constant by an elementary method. For all $a, b \in R$

and $u \in C_0^\infty(R^N \backslash \{0\})$,

$$\hat{C} \int_{R^N} \frac{|u|^2}{|x|^{a+b+1}}\,dx \le \left(\int_{R^N} \frac{|u|^2}{|x|^{2a}}\,dx\right)^{\frac{1}{2}} \left(\int_{R^N} \frac{|\nabla u|^2}{|x|^{2b}}\,dx\right)^{\frac{1}{2}}, \tag{1.3}$$

where the constant $\hat{C} = \hat{C}(a,b) := \frac{|N-(a+b+1)|}{2}$ is sharp.

On the other hand, the Rellich inequality is a generalization of the Hardy inequality to second-order derivatives, and the classical Rellich inequality in $R^N$ states that for $N \ge 5$ and $u \in C_0^\infty(R^N \setminus \{0\})$,

$$\int_{R^N} \left|\Delta u(x)\right|^2 dx \ge \left(\frac{N(N-4)}{4}\right)^2 \int_{R^N} \frac{|u(x)|^2}{|x|^4}\,dx. \tag{1.4}$$

The constant $\frac{N^2(N-4)^2}{16}$ is sharp and never achieved. In [10] Tetikas and Zographopoulos obtained a corresponding stronger versions of the Rellich inequality which reads

$$\left(\frac{N}{2}\right)^2 \int_{R^N} \frac{|\nabla u|^2}{|x|^2}\,dx \le \int_{R^N} |\triangle u|^2\,dx \tag{1.5}$$

for all $u \in C_0^\infty$ and $N \ge 3$. In [11] Costa obtained a new class of Hardy-Rellich type inequalities which contain (1.5) as a special case. If $a + b + 3 \le N$, then

$$\hat{C} \int_{R^N} \frac{|\nabla u|^2}{|x|^{a+b+1}}\,dx \le \left(\int_{R^N} \frac{|\triangle u|^2}{|x|^{2b}}\,dx\right)^{\frac{1}{2}} \left(\int_{R^N} \frac{|\nabla u|^2}{|x|^{2a}}\,dx\right)^{\frac{1}{2}}, \tag{1.6}$$

where the constant $\hat{C} = \hat{C}(a,b) := |\frac{N+a+b-1}{2}|$ is sharp.

The goal of this paper is to extend the above (1.3) and (1.6) to the general $L^p$ case for $1 < p < \infty$ by a different and direct approach.

## 2 Main results

In this section, we will give the proof of the main theorems.

**Theorem 1** *For all $a, b \in R$ and $u \in C_0^\infty(R^N \setminus \{0\})$, one has*

$$C \int_{R^N} \frac{|u|^p}{|x|^{a+b+1}}\,dx \le \left(\int_{R^N} \frac{|\nabla u|^p}{|x|^{ap}}\,dx\right)^{\frac{1}{p}} \left(\int_{R^N} \frac{|u|^p}{|x|^{b\frac{p}{p-1}}}\,dx\right)^{\frac{p-1}{p}}, \tag{2.1}$$

*where $1 < p < \infty$ and the constant $C = |\frac{N-(a+b+1)}{p}|$ is sharp.*

*Proof* Let $u \in C_0^\infty(R^N \setminus \{0\})$, $a, b \in R$ and $\lambda = a + b + 1$. By integration by parts and the Hölder inequality, one has

$$\begin{aligned}
\int_{R^N} \frac{|u|^p}{|x|^\lambda}\,dx &= \frac{1}{N-\lambda} \int_{R^N} |u|^p \operatorname{div}\left(\frac{x}{|x|^\lambda}\right) dx \\
&= -\frac{1}{N-\lambda} \int_{R^N} pu|u|^{p-2} \frac{x \cdot \nabla u}{|x|^\lambda}\,dx \\
&\le \left|\frac{-p}{N-\lambda}\right| \int_{R^N} \frac{|x \cdot \nabla u|}{|x|^\lambda} |u|^{p-1}\,dx
\end{aligned}$$

$$\leq \left|\frac{p}{N-\lambda}\right| \int_{R^N} \frac{|\nabla u||u|^{p-1}}{|x|^{a+b}}\,dx$$
$$\leq \left|\frac{p}{N-\lambda}\right| \left(\int_{R^N} \frac{|\nabla u|^p}{|x|^{ap}}\,dx\right)^{\frac{1}{p}} \left(\int_{R^N} \frac{|u|^p}{|x|^{b\frac{p}{p-1}}}\,dx\right)^{\frac{p-1}{p}}.$$

Then

$$\left|\frac{N-\lambda}{p}\right| \int_{R^N} \frac{|u|^p}{|x|^{\lambda}}\,dx \leq \left(\int_{R^N} \frac{|\nabla u|^p}{|x|^{ap}}\,dx\right)^{\frac{1}{p}} \left(\int_{R^N} \frac{|u|^p}{|x|^{b\frac{p}{p-1}}}\,dx\right)^{\frac{p-1}{p}}. \tag{2.2}$$

It remains to show the sharpness of the constant. By the condition with equality in the Hölder inequality, we consider the following family of functions:

$$u_\varepsilon(x) = e^{-\frac{C_\varepsilon}{\beta}|x|^\beta}, \quad \text{when } \beta = a - \frac{b}{p-1} + 1 \neq 0$$

and

$$u_\varepsilon(x) = \frac{1}{|x|^{C_\varepsilon}}, \quad \text{when } \beta = a - \frac{b}{p-1} + 1 = 0,$$

where $C_\varepsilon$ is a positive number sequence converging to $|\frac{N-(a+b+1)}{p}|$ as $\varepsilon \to 0$. By direct computation and the limit process, we know the constant $\frac{|N-(a+b+1)|}{p}$ is sharp. □

**Remark 1** When $p = 2$, the inequality (2.1) covers the inequality (2.4) in [9].

**Remark 2** When $a = 0$, $b = p - 1$, the inequality (2.1) is the classical $L^p$ Hardy inequality:

$$\left(\frac{N-p}{p}\right)^p \int_{R^N} \frac{|u|^p}{|x|^p}\,dx \leq \int_{R^N} |\nabla u|^p\,dx. \tag{2.3}$$

When we take special values for $a$, $b$, the following corollary holds.

**Corollary 1** (i) *When* $b = (a + 1)(p - 1)$, *the inequality* (2.1) *is just the weighted Hardy inequality*:

$$\left|\frac{N-p(a+1)}{p}\right|^p \int_{R^N} \frac{|u|^p}{|x|^{(a+1)p}}\,dx \leq \int_{R^N} \frac{|\nabla u|^p}{|x|^{ap}}\,dx. \tag{2.4}$$

(ii) *When* $a + b + 1 = ap$, *according to the inequality* (2.1), *we have*

$$\left|\frac{N-ap}{p}\right| \int_{R^N} \frac{|u|^p}{|x|^{ap}}\,dx \leq \left(\int_{R^N} \frac{|\nabla u|^p}{|x|^{ap}}\,dx\right)^{\frac{1}{p}} \left(\int_{R^N} \frac{|u|^p}{|x|^{ap-\frac{p}{p-1}}}\,dx\right)^{\frac{p-1}{p}}. \tag{2.5}$$

(iii) *When* $a = -p$ *and* $a + b + 1 = 0$, *we obtain the inequality*

$$\frac{N}{p} \int_{R^N} |u|^p\,dx \leq \left(\int_{R^N} |\nabla u|^p |x|^{p^2}\,dx\right)^{\frac{1}{p}} \left(\int_{R^N} \frac{|u|^p}{|x|^p}\,dx\right)^{\frac{p-1}{p}}. \tag{2.6}$$

By a similar method, we can prove the following $L^p$ case Hardy-Rellich type inequality.

**Theorem 2** *Let* $1 < p < N$, $\frac{p-N}{p-1} \le a+b+1 \le 0$. *Then, for any* $u \in C_0^\infty(R^N \setminus \{0\})$, *the following holds*:

$$\hat{C} \int_{R^N} \frac{|\nabla u|^p}{|x|^{a+b+1}}\,dx \le \left(\int_{R^N} \frac{|\triangle_p u|^p}{|x|^{ap}}\,dx\right)^{\frac{1}{p}} \left(\int_{R^N} \frac{|\nabla u|^q}{|x|^{bq}}\,dx\right)^{\frac{1}{q}}, \tag{2.7}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$, $\hat{C} = (\frac{N-p+(p-1)(a+b+1)}{p})$ *and* $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ *is the p-Laplacian operator.*

*Proof* Set $\lambda = a + b + 1$, it is easy to see

$$\begin{aligned} \int_{R^N} \frac{|\nabla u|^p}{|x|^\lambda}\,dx &= \frac{1}{N-\lambda} \int_{R^N} |\nabla u|^p \operatorname{div}\left(\frac{x}{|x|^\lambda}\right) dx \\ &= -\frac{1}{N-\lambda} \int_{R^N} \frac{p}{2} |\nabla u|^{p-2} \frac{x}{|x|^\lambda} \cdot \nabla\left(|\nabla u|^2\right) dx \\ &= \frac{p}{2(\lambda - N)} \int_{R^N} |\nabla u|^{p-2} \frac{x \cdot \nabla(|\nabla u|^2)}{|x|^\lambda}\,dx. \end{aligned} \tag{2.8}$$

On the other hand,

$$\begin{aligned} \int_{R^N} \triangle_p u \frac{x \cdot \nabla u}{|x|^\lambda}\,dx &= \int_{R^N} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \frac{x \cdot \nabla u}{|x|^\lambda}\,dx \\ &= -\int_{R^N} |\nabla u|^{p-2} \nabla u \cdot \nabla\left(\frac{x \cdot \nabla u}{|x|^\lambda}\right) dx \\ &= -\int_{R^N} |\nabla u|^{p-2} \left(\frac{|\nabla u|^2}{|x|^\lambda} + \frac{\frac{1}{2} x \cdot \nabla(|\nabla u|^2)}{|x|^\lambda} - \lambda \frac{(x \cdot \nabla u)^2}{|x|^{\lambda+2}}\right) dx, \end{aligned}$$

which means

$$\begin{aligned} &\int_{R^N} |\nabla u|^{p-2} \cdot \frac{x \cdot \nabla(|\nabla u|^2)}{|x|^\lambda}\,dx \\ &\quad = 2\left(\lambda \int_{R^N} |\nabla u|^{p-2} \frac{(x \cdot \nabla u)^2}{|x|^\lambda}\,dx - \int_{R^N} \frac{|\nabla u|^p}{|x|^\lambda}\,dx - \int_{R^N} \triangle_p u \frac{x \cdot \nabla u}{|x|^\lambda}\right). \end{aligned} \tag{2.9}$$

Then, we can deduce from (2.8) and (2.9)

$$\begin{aligned} &\int_{R^N} \frac{|\nabla u|^p}{|x|^\lambda}\,dx \\ &\quad = \frac{p}{\lambda - N}\left(\lambda \int_{R^N} |\nabla u|^{p-2} \frac{(x \cdot \nabla u)^2}{|x|^\lambda}\,dx - \int_{R^N} \frac{|\nabla u|^p}{|x|^\lambda}\,dx - \int_{R^N} \triangle_p u \frac{x \cdot \nabla u}{|x|^\lambda}\right). \end{aligned} \tag{2.10}$$

That is,

$$\frac{N-p-\lambda}{p} \int_{R^N} \frac{|\nabla u|^p}{|x|^\lambda}\,dx + \lambda \int_{R^N} |\nabla u|^{p-2} \frac{(x \cdot \nabla u)^2}{|x|^{\lambda+2}}\,dx = \int_{R^N} \triangle_p u \frac{x \cdot \nabla u}{|x|^\lambda}\,dx. \tag{2.11}$$

By the Hölder inequality,

$$\int_{R^N} \triangle_p u \frac{x \cdot \nabla u}{|x|^{\lambda+2}}\,dx \le \left(\int_{R^N} \frac{|\triangle_p u|^q}{|x|^{aq}}\right)^{\frac{1}{q}} \left(\int_{R^N} \frac{|\nabla u|^p}{|x|^{bp}}\right)^{\frac{1}{p}}, \tag{2.12}$$

note that $\frac{p-N}{p-1} \le \lambda \le 0$. Thus

$$\frac{N-p+(p-1)\lambda}{p} \int_{R^N} \frac{|\nabla u|^p}{|x|^{\lambda}}\,dx \le \left(\int_{R^N} \frac{|\triangle_p u|^p}{|x|^{ap}}\right)^{\frac{1}{p}} \left(\int_{R^N} \frac{|\nabla u|^q}{|x|^{bq}}\right)^{\frac{1}{q}}. \tag{2.13}$$

We mention that we do not know whether the constant $(\frac{N-p+(p-1)(a+b+1)}{p})$ in (2.7) is optimal or not. □

**Corollary 2** *When $a+b+1=0$, we have the following inequalities*:

(i) *when $a=-1$, $b=0$, the inequality* (2.7) *is equivalent to the inequality*

$$\left(\frac{N-p}{p}\right)^p \int_{R^N} |\nabla u|^p\,dx \le \int_{R^N} |\triangle_p u|^p |x|^p\,dx. \tag{2.14}$$

(ii) *When $a=1$, $b=-2$, we obtain the inequality*

$$\left(\frac{N-p}{p}\right) \int_{R^N} |\nabla u|^p\,dx \le \left(\int_{R^N} \frac{|\triangle_p u|^p}{|x|^p}\,dx\right)^{\frac{1}{p}} \left(\int_{R^N} |\nabla u|^q |x|^{2q}\,dx\right)^{\frac{1}{q}}. \tag{2.15}$$

(iii) *When $a=0$, $b=-1$, we get*

$$\left(\frac{N-p}{p}\right) \int_{R^N} |\nabla u|^p\,dx \le \left(\int_{R^N} |\triangle_p u|^p\,dx\right)^{\frac{1}{p}} \left(\int_{R^N} |\nabla u|^q |x|^q\,dx\right)^{\frac{1}{q}}. \tag{2.16}$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors jointly worked on the results and they read and approved the final manuscript.

**Acknowledgements**
This work is supported by NNSF of China (11001240), ZJNSF (LQ12A01023) and the foundation of the Zhejiang University of the Technology (20100229).

Received: 31 May 2012 Accepted: 18 February 2013 Published: 4 March 2013

**References**

1. Adimurthi, AS: Role of the fundamental solution in Hardy-Sobolev type inequalities. Proc. R. Soc. Edinb., Sect. A **136**, 1111-1130 (2006)
2. Garofalo, N, Lanconelli, E: Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation. Ann. Inst. Fourier (Grenoble) **40**, 313-356 (1990)
3. Goldstein, JA, Kombe, I: Nonlinear degenerate parabolic equations on the Heisenberg group. Int. J. Evol. Equ. **1**, 1-22 (2005)
4. Goldstein, JA, Zhang, QS: On a degenerate heat equation with a singular potential. J. Funct. Anal. **186**, 342-359 (2001)
5. Jin, Y, Han, Y: Weighted Rellich inequality on *H*-type groups and nonisotropic Heisenberg groups. J. Inequal. Appl. **2010**, Article ID 158281 (2010)
6. Jin, Y, Zhang, G: Degenerate *p*-Laplacian operators and Hardy type inequalities on *h*-type groups. Can. J. Math. **62**, 1116-1130 (2010)
7. García Azorero, JP, Peral Alonso, I Hardy inequalities and some critical elliptic and parabolic problems. J. Differ. Equ. **144**, 441-476 (1998)

8. Caffarelli, L, Kohn, R, Nirenberg, L: First order interpolation inequalities with weights. Compos. Math. **53**, 259-275 (1984)
9. Costa, DG: Some new and short proofs for a class of Caffarelli-Kohn-Nirenberg type inequalities. J. Math. Anal. Appl. **337**, 311-317 (2008)
10. Tertikas, A, Zographopoulos, NB: Best constants in the Hardy-Rellich inequalities and related improvements. Adv. Math. **209**, 407-459 (2007)
11. Costa, DG: On Hardy-Rellich type inequalities in $R^N$. Appl. Math. Lett. **22**, 902-905 (2009)

doi:10.1186/1029-242X-2013-84
**Cite this article as:** Di et al.: **A note on a class of Hardy-Rellich type inequalities.** *Journal of Inequalities and Applications* 2013 **2013**:84.