Cho and Yoon *Journal of Inequalities and Applications* 2013, **2013**:83
http://www.journalofinequalitiesandapplications.com/content/2013/1/83

Journal of Inequalities and Applications
a SpringerOpen Journal

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Inclusion relationships for certain classes of analytic functions involving the Choi-Saigo-Srivastava operator

Nak Eun Cho[1] and Min Yoon[2*]

*Correspondence: myoon@pknu.ac.kr
[2]Department of Statistics, Pukyong National University, 45 Yongso-ro, Busan, Korea
Full list of author information is available at the end of the article

Abstract
The purpose of the present paper is to investigate some inclusion properties of certain classes of analytic functions associated with a family of linear operators which are defined by means of the Hadamard product (or convolution). Some invariant properties under convolution are also considered for the classes presented here. The results presented here include several previous known results as their special cases.
MSC: 30C45

Keywords: subordination; Hadamard product (or convolution); starlike function; convex function; close-to-convex function; linear operator; Choi-Saigo-Srivastava operator; Carlson-Shaffer operator; Ruscheweyh derivative operator

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w in $\mathbb{U}$ with $w(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{U}$ such that $f(z) = g(\omega(z))$. We denote by $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike, convex and close-to-convex in $\mathbb{U}$ (see, *e.g.*, Srivastava and Owa [1]).

Let $\mathcal{N}$ be a class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\operatorname{Re}\{\phi(z)\} > 0$ for $z \in \mathbb{U}$.

Making use of the principle of subordination between analytic functions, many authors investigated the subclasses $\mathcal{S}^*(\eta;\phi)$, $\mathcal{K}(\eta;\phi)$ and $\mathcal{C}(\eta,\beta;\phi,\psi)$ of the class $\mathcal{A}$ for $0 \leq \eta, \beta < 1$, $\phi, \psi \in \mathcal{N}$ (*cf.* [2] and [3]), which are defined by

$$\mathcal{S}^*(\eta;\phi) := \left\{ f \in \mathcal{A} : \frac{1}{1-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec \phi(z) \text{ in } \mathbb{U} \right\},$$
$$\mathcal{K}(\eta;\phi) := \left\{ f \in \mathcal{A} : \frac{1}{1-\eta}\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right) \prec \phi(z) \text{ in } \mathbb{U} \right\}$$

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and

$$\mathcal{C}(\eta,\beta;\phi,\psi) := \left\{ f \in \mathcal{A} : \exists g \in \mathcal{S}^*(\eta;\phi) \text{ s.t. } \frac{1}{1-\beta}\left(\frac{zf'(z)}{g(z)} - \beta\right) \prec \psi(z) \text{ in } \mathbb{U} \right\}.$$

For $\phi(z) = \psi(z) = (1+z)/(1-z)$ in the definitions above, we have the well-known classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$, respectively. Furthermore, for the function classes $\mathcal{S}^*[A,B]$ and $\mathcal{K}[A,B]$ investigated by Janowski ([4], also see [5]), it is easily seen that

$$\mathcal{S}^*\left(0;\frac{1+Az}{1+Bz}\right) = \mathcal{S}^*[A,B] \quad (-1 \le B < A \le 1)$$

and

$$\mathcal{K}\left(0;\frac{1+Az}{1+Bz}\right) = \mathcal{K}[A,B] \quad (-1 \le B < A \le 1).$$

We now define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad \bigl(z \in \mathbb{U}; a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \mathbb{Z}_0^- := \{-1,-2,\ldots\}\bigr), \tag{1.1}$$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the gamma function) by

$$(\nu)_k := \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } k = 0 \text{ and } \nu \in \mathbb{C}\backslash\{0\}, \\ \nu(\nu+1)\cdots(\nu+k-1) & \text{if } k \in \mathbb{N} := \{1,2,\ldots\} \text{ and } \nu \in \mathbb{C}. \end{cases}$$

We also denote by $L(a,c) : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$L(a,c)f(z) = \phi(a,c;z) * f(z) \quad (z \in \mathbb{U})\ (z \in \mathbb{U}; f \in \mathcal{A}), \tag{1.2}$$

where the symbol $(*)$ stands for the Hadamard product (or convolution). Then it is easily observed from definitions (1.1) and (1.2) that

$$L(2,1)f(z) = zf'(z) \quad \text{and} \quad L(n+1,1)f(z) = D^n f(z) \quad (n > -1),$$

where the symbol D^n denotes the familiar Ruscheweyh derivative [6] (also, see [7]) for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The operator $L(a,c)$, introduced and studied by Carlson-Shaffer [8], has been used widely on the space of analytic and univalent functions in $\mathbb{U}$ (see also [9]). Corresponding to the function $\phi(a,c;z)$ defined by (1.1), we also introduce a function $\phi_\lambda(a,c;z)$ given by

$$\phi(a,c;z) * \phi_\lambda(a,c;z) = \frac{z}{(1-z)^\lambda} \quad (\lambda > 0). \tag{1.3}$$

Analogous to $L(a,c)$, we now define the linear operator $\mathcal{I}_\lambda(a,c)$ on $\mathcal{A}$ as follows:

$$\mathcal{I}_\lambda(a,c)f(z) = \phi_\lambda(a,c;z) * f(z) \quad \bigl(a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \lambda > 0; z \in \mathbb{U}; f \in \mathcal{A}\bigr). \tag{1.4}$$

It is easily verified from definition (1.4) that

$$\mathcal{I}_2(2,1)f(z) = f(z) \quad \text{and} \quad \mathcal{I}_2(1,1)f(z) = zf'(z).$$

In particular, the operator $\mathcal{I}_\lambda(\mu+1,1)$ ($\lambda > 0$; $\mu > -1$) was introduced by Choi *et al.* [2] who investigated (among other things) several inclusion properties involving various subclasses of analytic and univalent functions. For $\mu = n$ ($n \in \mathbb{N}_0$) and $\lambda = 2$, we also note that the Choi-Saigo-Srivastava operator $\mathcal{I}_\lambda(\mu+1,1)f$ is the Noor integral operator of nth order of f studied by Liu [10] and Noor *et al.* [11–15].

By using the operator $\mathcal{I}_\lambda(a,c)$, we introduce the following classes of analytic functions for $\phi, \psi \in \mathcal{N}$, $a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $\lambda > 0$ and $0 \le \eta, \beta < 1$:

$$\mathcal{S}_{a,c}^\lambda(\eta;\phi) := \left\{f \in \mathcal{A} : \mathcal{I}_\lambda(a,c)f(z) \in \mathcal{S}^*(\eta;\phi)\right\},$$
$$\mathcal{K}_{a,c}^\lambda(\eta;\phi) := \left\{f \in \mathcal{A} : \mathcal{I}_\lambda(a,c)f(z) \in \mathcal{K}^*(\eta;\phi)\right\},$$

and

$$\mathcal{C}_{a,c}^\lambda(\eta,\beta;\phi,\psi) := \left\{f \in \mathcal{A} : \mathcal{I}_\lambda(a,c)f(z) \in \mathcal{C}^*(\eta,\beta;\phi,\psi)\right\}.$$

We also note that

$$f(z) \in \mathcal{K}_{a,c}^\lambda(\eta;\phi) \quad \Longleftrightarrow \quad zf'(z) \in \mathcal{S}_{a,c}^\lambda(\eta;\phi). \tag{1.5}$$

In particular, we set

$$\mathcal{S}_{a,c}^\lambda\left(\eta;\frac{1+Az}{1+Bz}\right) = \mathcal{S}_{a,c}^\lambda[\eta;A,B] \quad (0 \le \eta < 1; -1 \le B < A \le 1)$$

and

$$\mathcal{K}_{a,c}^\lambda\left(\eta;\frac{1+Az}{1+Bz}\right) = \mathcal{K}_{a,c}^\lambda[\eta;A,B] \quad (0 \le \eta < 1, -1 \le B < A \le 1).$$

Recently, Sokoł [16] extended the results given by Choi *et al.* [2] making use of some interesting proof techniques due to Ruscheweyh [17] and Ruscheweyh and Sheil-Small [18]. In this paper, we investigate several inclusion properties of the classes $\mathcal{S}_{a,c}^\lambda(\eta;\phi)$, $\mathcal{K}_{a,c}^\lambda(\eta;\phi)$ and $\mathcal{C}_{a,c}^\lambda(\eta,\beta;\phi,\psi)$. The integral-preserving properties in connection with the operator $\mathcal{I}_\lambda(a,c)$ are also considered. Furthermore, relevant connections of the results presented here with those obtained in earlier works are pointed out.

2 Inclusion properties involving the operator $\mathcal{I}_\lambda(a,c)$

The following lemmas will be required in our investigation.

Lemma 2.1 *Let* $\phi_{\lambda_i}(a,c;z)$, $\phi_\lambda(a_i,c;z)$ *and* $\phi_\lambda(a,c_i;z)$ ($i = 1,2$) *be defined by* (1.3). *Then, for* $\lambda_i > 0$, $a_i, c_i \in \mathbb{R}\backslash\mathbb{Z}_0^-$ ($i = 1,2$),

$$\phi_{\lambda_1}(a,c;z) = \phi_{\lambda_2}(a,c;z) * f_{\lambda_1,\lambda_2}(z), \tag{2.1}$$

$$\phi_\lambda(a_2,c;z) = \phi_\lambda(a_1,c;z) * f_{a_1,a_2}(z), \tag{2.2}$$

and

$$\phi_\lambda(a, c_1; z) = \phi_\lambda(a, c_2; z) * f_{c_1,c_2}(z), \tag{2.3}$$

where

$$f_{s,t}(z) = \sum_{k=0}^{\infty} \frac{(s)_k}{(t)_k} z^{k+1} \quad (z \in \mathbb{U}). \tag{2.4}$$

Proof From definition (1.3), we know that

$$\phi_\lambda(a, c; z) = \sum_{k=0}^{\infty} \frac{(c)_k(\lambda)_k}{(a)_k(1)_k} z^{k+1} \quad (z \in \mathbb{U}). \tag{2.5}$$

Therefore, equations (2.1), (2.2) and (2.3) follow from (2.5) immediately. □

Lemma 2.2 [17, pp.60-61] *Let $t \geq s > 0$. If $t \geq 2$ or $s + t \geq 3$, then the function defined by* (2.4) *belongs to the class $\mathcal{K}$.*

Lemma 2.3 [18] *Let $f \in \mathcal{K}$ and $g \in \mathcal{S}^*$. Then, for every analytic function h in $\mathbb{U}$,*

$$\frac{(f * hg)(\mathbb{U})}{(f * g)(\mathbb{U})} \subset \overline{\mathrm{co}}h(\mathbb{U}),$$

where $\overline{\mathrm{co}}h(\mathbb{U})$ denotes the closed convex hull of $h(\mathbb{U})$.

At first, the inclusion relationship involving the class $\mathcal{S}^\lambda_{a,c}(\eta; \phi)$ is contained in Theorem 2.1 below.

Theorem 2.1 *Let $\lambda_2 \geq \lambda_1 > 0$, $a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $0 \leq \eta < 1$ and $\phi \in \mathcal{N}$. If $\lambda_2 \geq 2$ or $\lambda_1 + \lambda_2 \geq 3$, then*

$$\mathcal{S}^{\lambda_2}_{a,c}(\eta; \phi) \subset \mathcal{S}^{\lambda_1}_{a,c}(\eta; \phi).$$

Proof Let $f \in \mathcal{S}^{\lambda_2}_{a,c}(\eta; \phi)$. From the definition of $\mathcal{S}^{\lambda_2}_{a,c}(\eta; \phi)$, we have

$$\frac{1}{1-\eta}\left(\frac{z(\mathcal{I}_{\lambda_2}(a,c)f(z))'}{\mathcal{I}_{\lambda_2}(a,c)f(z)} - \eta\right) = \phi\big(\omega(z)\big) \quad (z \in \mathbb{U}), \tag{2.6}$$

where w is analytic in $\mathbb{U}$ with $|\omega(z)| < 1$ ($z \in \mathbb{U}$) and $w(0) = 0$. By using (1.4), (2.1) and (2.6), we get

$$\begin{aligned}
\frac{z(\mathcal{I}_{\lambda_1}(a,c)f(z))'}{\mathcal{I}_{\lambda_1}(a,c)f(z)} &= \frac{z(\phi_{\lambda_1}(a,c;z) * f(z))'}{\phi_{\lambda_1}(a,c;z) * f(z)} = \frac{z(\phi_{\lambda_2}(a,c;z) * f_{\lambda_1,\lambda_2}(z) * f(z))'}{\phi_{\lambda_2}(a,c;z) * f_{\lambda_1,\lambda_2}(z) * f(z)} \\
&= \frac{f_{\lambda_1,\lambda_2}(z) * z(\mathcal{I}_{\lambda_2}(a,c)f(z))'}{f_{\lambda_1,\lambda_2}(z) * \mathcal{I}_{\lambda_2}(a,c)f(z)} \\
&= \frac{f_{\lambda_1,\lambda_2}(z) * [(1-\eta)\phi(\omega(z)) + \eta]\mathcal{I}_{\lambda_2}(a,c)f(z)}{f_{\lambda_1,\lambda_2}(z) * \mathcal{I}_{\lambda_2}(a,c)f(z)}.
\end{aligned} \tag{2.7}$$

It follows from (2.6) and Lemma 2.2 that $\mathcal{I}_{\lambda_2}(a,c)f(z) \in \mathcal{S}^*$ and $f_{\lambda_1,\lambda_2} \in \mathcal{K}$, respectively. Let us put $s(\omega(z)) := (1-\eta)\phi(\omega(z)) + \eta$. Then, by applying Lemma 2.3 to (2.7), we obtain

$$\frac{\{f_{\lambda_1,\lambda_2} * [(1-\eta)\phi(w)+\eta]\mathcal{I}_{\lambda_2}(a,c)f\}}{\{f_{\lambda_1,\lambda_2} * \mathcal{I}_{\lambda_2}(a,c)f\}}(\mathbb{U}) \subset \overline{\cos}\big(w(\mathbb{U})\big), \tag{2.8}$$

since s is convex univalent. Therefore, from the definition of subordination and (2.8), we have

$$\frac{1}{1-\eta}\left(\frac{z(\mathcal{I}_{\lambda_1}(a,c)f(z))'}{\mathcal{I}_{\lambda_1}(a,c)f(z)} - \eta\right) \prec \phi(z) \quad (z \in \mathbb{U}),$$

or, equivalently, $f \in \mathcal{S}_{a,c}^{\lambda_1}(\eta;\phi)$, which completes the proof of Theorem 2.1. □

By using equations (1.4), (2.2) and (2.3), we have the following theorems.

Theorem 2.2 *Let* $\lambda > 0$, $a_2 \geq a_1 > 0$, $c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $0 \leq \eta < 1$ *and* $\phi \in \mathcal{N}$. *If* $a_2 \geq 2$ *or* $a_1 + a_2 \geq 3$, *then*

$$\mathcal{S}_{a_1,c}^{\lambda}(\eta;\phi) \subset \mathcal{S}_{a_2,c}^{\lambda}(\eta;\phi).$$

Theorem 2.3 *Let* $\lambda > 0$, $a \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $c_2 \geq c_1 > 0$, $0 \leq \eta < 1$ *and* $\phi \in \mathcal{N}$. *If* $c_2 \geq 2$ *or* $c_1 + c_2 \geq 3$, *then*

$$\mathcal{S}_{a,c_2}^{\lambda}(\eta;\phi) \subset \mathcal{S}_{a,c_1}^{\lambda}(\eta;\phi).$$

Next, we prove the inclusion theorem involving the class $\mathcal{K}_{a,c}^{\lambda}(\eta;\phi)$.

Theorem 2.4 *Let* $\lambda_2 \geq \lambda_1 > 0$, $a,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $0 \leq \eta < 1$ *and* $\phi \in \mathcal{N}$. *If* $\lambda_2 \geq 2$ *or* $\lambda_1 + \lambda_2 \geq 3$, *then*

$$\mathcal{K}_{a,c}^{\lambda_2}(\eta;\phi) \subset \mathcal{K}_{a,c}^{\lambda_1}(\eta;\phi).$$

Proof Applying (1.5) and Theorem 2.1, we observe that

$$\begin{aligned}
f(z) \in \mathcal{K}_{a,c}^{\lambda_2}(\eta;\phi) \quad &\Longleftrightarrow \quad \mathcal{I}_{\lambda_2}(a,c)f(z) \in \mathcal{K}(\eta;\phi) \\
&\Longleftrightarrow \quad z\big(\mathcal{I}_{\lambda_2}(a,c)f(z)\big)' \in \mathcal{S}^*(\eta;\phi) \\
&\Longleftrightarrow \quad \mathcal{I}_{\lambda_2}(a,c)\big(zf'(z)\big) \in \mathcal{S}^*(\eta;\phi) \\
&\Longleftrightarrow \quad zf'(z) \in \mathcal{S}_{a,c}^{\lambda_2}(\eta;\phi) \\
&\Longleftrightarrow \quad zf'(z) \in \mathcal{S}_{a,c}^{\lambda_1}(\eta;\phi) \\
&\Longleftrightarrow \quad \mathcal{I}_{\lambda_1}(a,c)\big(zf'(z)\big) \in \mathcal{S}^*(\eta;\phi) \\
&\Longleftrightarrow \quad z\big(\mathcal{I}_{\lambda_1}(a,c)f(z)\big)' \in \mathcal{S}^*(\eta;\phi) \\
&\Longleftrightarrow \quad \mathcal{I}_{\lambda_1}(a,c)f(z) \in \mathcal{K}(\eta;\phi) \\
&\Longleftrightarrow \quad f(z) \in \mathcal{K}_{a,c}^{\lambda_1}(\eta;\phi),
\end{aligned}$$

which evidently proves Theorem 2.4. □

By using a similar method as in the proof of Theorem 2.4, we obtain the following two theorems.

Theorem 2.5 *Let* $\lambda > 0$, $a_2 \geq a_1 > 0$, $c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $0 \leq \eta < 1$ *and* $\phi \in \mathcal{N}$. *If* $a_2 \geq 2$ *or* $a_1 + a_2 \geq 3$, *then*

$$\mathcal{K}^{\lambda}_{a_1,c}(\eta;\phi) \subset \mathcal{K}^{\lambda}_{a_2,c}(\eta;\phi).$$

Theorem 2.6 *Let* $\lambda > 0$, $a \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $c_2 \geq c_1 > 0$, $0 \leq \eta < 1$ *and* $\phi \in \mathcal{N}$. *If* $c_2 \geq 2$ *or* $c_1 + c_2 \geq 3$, *then*

$$\mathcal{K}^{\lambda}_{a,c_2}(\eta;\phi) \subset \mathcal{K}^{\lambda}_{a,c_1}(\eta;\phi).$$

Taking $\phi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1; z \in \mathbb{U})$ in Theorems 2.1-2.6, we have the following corollaries below.

Corollary 2.1 *Let* $\lambda_2 \geq \lambda_1 > 0$ *and let* $\lambda_2 \geq \min\{2, 3 - \lambda_1\}$, *and* $a_2 \geq a_1 > 0$ *and* $a_2 \geq \min\{2, 3 - a_1\}$. *Then*

$$\mathcal{S}^{\lambda_2}_{a_1,c}[\eta;A,B] \subset \mathcal{S}^{\lambda_1}_{a_1,c}[\eta;A,B] \subset \mathcal{S}^{\lambda_1}_{a_2,c}[\eta;A,B] \quad \bigl(c \in \mathbb{R}\backslash\mathbb{Z}_0^-; 0 \leq \eta < 1; -1 \leq B < A \leq 1\bigr),$$

and

$$\mathcal{K}^{\lambda_2}_{a_1,c}[\eta;A,B] \subset \mathcal{K}^{\lambda_1}_{a_1,c}[\eta;A,B] \subset \mathcal{K}^{\lambda_1}_{a_2,c}[\eta;A,B] \quad \bigl(c \in \mathbb{R}\backslash\mathbb{Z}_0^-; 0 \leq \eta < 1; -1 \leq B < A \leq 1\bigr).$$

Corollary 2.2 *Let* $a_2 \geq a_1 > 0$ *and let* $a_2 \geq \min\{2, 3 - a_1\}$, *and* $c_2 \geq c_1 > 0$ *and* $c_2 \geq \min\{2, 3 - c_1\}$. *Then*

$$\mathcal{S}^{\lambda}_{a_1,c_2}[\eta;A,B] \subset \mathcal{S}^{\lambda}_{a_1,c_1}[\eta;A,B] \subset \mathcal{S}^{\lambda}_{a_2,c_1}[\eta;A,B] \quad (\lambda > 0; 0 \leq \eta < 1; -1 \leq B < A \leq 1),$$

and

$$\mathcal{K}^{\lambda}_{a_1,c_2}[\eta;A,B] \subset \mathcal{K}^{\lambda}_{a_1,c_1}[\eta;A,B] \subset \mathcal{K}^{\lambda}_{a_2,c_1}[\eta;A,B] \quad (\lambda > 0; 0 \leq \eta < 1; -1 \leq B < A \leq 1).$$

Corollary 2.3 *Let* $\lambda_2 \geq \lambda_1 > 0$ *and let* $\lambda_2 \geq \min\{2, 3 - \lambda_1\}$, *and* $c_2 \geq c_1 > 0$ *and* $c_2 \geq \min\{2, 3 - c_1\}$. *Then*

$$\mathcal{S}^{\lambda_2}_{a,c_2}[\eta;A,B] \subset \mathcal{S}^{\lambda_2}_{a,c_1}[\eta;A,B] \subset \mathcal{S}^{\lambda_1}_{a,c_1}[\eta;A,B] \quad \bigl(a \in \mathbb{R}\backslash\mathbb{Z}_0^-; 0 \leq \eta < 1; -1 \leq B < A \leq 1\bigr),$$

and

$$\mathcal{K}^{\lambda_2}_{a,c_2}[\eta;A,B] \subset \mathcal{K}^{\lambda_2}_{a,c_1}[\eta;A,B] \subset \mathcal{K}^{\lambda_1}_{a,c_1}[\eta;A,B] \quad \bigl(a \in \mathbb{R}\backslash\mathbb{Z}_0^-; 0 \leq \eta < 1; -1 \leq B < A \leq 1\bigr).$$

To prove the theorems below, we need the following lemma.

Lemma 2.4 [16] *Let* $\phi \in \mathcal{N}$. *If* $f \in \mathcal{K}$ *and* $q \in \mathcal{S}^*(\eta;\phi)$, *then* $f * q \in \mathcal{S}^*(\eta;\phi)$.

Proof Let $q \in \mathcal{S}^*(\eta;\phi)$. Then

$$zq'(z) = \big[(1-\eta)\phi\big(\omega(z)\big) + \eta\big]q(z) \quad (z \in \mathbb{U}),$$

where ω is an analytic function in $\mathbb{U}$ with $|\omega(z)| < 1$ $(z \in \mathbb{U})$ and $w(0) = 0$. Thus we have

$$\frac{z(f(z) * q(z))'}{f(z) * q(z)} = \frac{f(z) * zq'(z)}{f(z) * q(z)} = \frac{f(z) * [(1-\eta)\phi(\omega(z)) + \eta]q(z)}{f(z) * q(z)} \quad (z \in \mathbb{U}). \tag{2.9}$$

By using similar arguments to those used in the proof of Theorem 2.1, we conclude that (2.9) is subordinated to ϕ in $\mathbb{U}$ and so $f * q \in \mathcal{S}^*(\eta;\phi)$. □

Finally, we give the inclusion properties involving the class $\mathcal{C}_{a,c}^{\lambda}(\eta,\beta;\phi,\psi)$.

Theorem 2.7 *Let* $\lambda_2 \geq \lambda_1 > 0$ *and* $\lambda_2 \geq \min\{2, 3-\lambda_1\}$, *and let* $a_2 \geq a_1 > 0$ *and* $a_2 \geq \min\{2, 3-a_1\}$. *Then*

$$\mathcal{C}_{a_1,c}^{\lambda_2}(\eta,\beta;\phi,\psi) \subset \mathcal{C}_{a_1,c}^{\lambda_1}(\eta,\beta;\phi,\psi) \subset \mathcal{C}_{a_2,c}^{\lambda_1}(\eta,\beta;\phi,\psi)$$
$$\big(c \in \mathbb{R}\backslash\mathbb{Z}_0^-; 0 \leq \eta,\beta < 1; \phi,\psi \in \mathcal{N}\big).$$

Proof We begin by proving that

$$\mathcal{C}_{a_1,c}^{\lambda_2}(\eta,\beta;\phi,\psi) \subset \mathcal{C}_{a_1,c}^{\lambda_1}(\eta,\beta;\phi,\psi).$$

Let $f \in \mathcal{C}_{a_1,c}^{\lambda_2}(\eta,\beta;\phi,\psi)$. Then there exists a function $q_2 \in \mathcal{S}^*(\eta;\phi)$ such that

$$\frac{1}{1-\beta}\left(\frac{z(\mathcal{I}_{\lambda_2}(a_1,c)f(z))'}{q_2(z)} - \beta\right) \prec \psi(z) \quad (z \in \mathbb{U}). \tag{2.10}$$

From (2.10), we obtain

$$z\big(\mathcal{I}_{\lambda_2}(a_1,c)f(z)\big)' = \big[(1-\beta)\psi\big(\omega(z)\big) + \beta\big]q_2(z) \quad (z \in \mathbb{U}),$$

where w is an analytic function in $\mathbb{U}$ with $|\omega(z)| < 1$ $(z \in \mathbb{U})$ and $w(0) = 0$. By virtue of (2.3), Lemma 2.2 and Lemma 2.4, we see that $f_{\lambda_1,\lambda_2}(z) * q_2(z) \equiv q_1(z)$ belongs to $\mathcal{S}^*(\eta;\phi)$. Then, making use of (2.1), we have

$$\begin{aligned}\frac{1}{1-\beta}\left(\frac{z(\mathcal{I}_{\lambda_1}(a_1,c)f(z))'}{q_1(z)} - \beta\right) &= \frac{1}{1-\beta}\left(\frac{f_{\lambda_1,\lambda_2}(z) * z(\mathcal{I}_{\lambda_2}(a_1,c)f(z))'}{f_{\lambda_1,\lambda_2}(z) * q_2(z)} - \beta\right)\\ &= \frac{1}{1-\beta}\left(\frac{f_{\lambda_1,\lambda_2}(z) * [(1-\beta)\psi(\omega(z)) + \beta]q_2(z)}{f_{\lambda_1,\lambda_2}(z) * q_2(z)} - \beta\right)\\ &\prec \psi(z) \quad (z \in \mathbb{U}).\end{aligned}$$

Therefore we prove that $f \in \mathcal{C}_{a_1,c}^{\lambda_1}(\eta,\beta;\phi,\psi)$.

For the second part, by using arguments similar to those detailed above with (2.2), we obtain

$$\mathcal{C}_{a_1,c}^{\lambda_1}(\eta,\beta;\phi,\psi) \subset \mathcal{C}_{a_2,c}^{\lambda_1}(\eta,\beta;\phi,\psi).$$

Thus the proof of Theorem 2.7 is completed. □

The following results can be obtained by using the same techniques as in the proof of Theorem 2.7, and so we omit the detailed proofs involved.

Theorem 2.8 *Let* $a_2 \geq a_1 > 0$ *and* $a_2 \geq \min\{2, 3 - a_1\}$, *and let* $c_2 \geq c_1 > 0$ *and* $c_2 \geq \min\{2, 3 - c_1\}$. *Then*

$$\mathcal{C}^{\lambda}_{a_1,c_2}(\eta,\beta;\phi,\psi) \subset \mathcal{C}^{\lambda}_{a_1,c_1}(\eta,\beta;\phi,\psi) \subset \mathcal{C}^{\lambda}_{a_2,c_1}(\eta,\beta;\phi,\psi)$$
$$(\lambda > 0; 0 \leq \eta,\beta < 1; \phi,\psi \in \mathcal{N}).$$

Theorem 2.9 *Let* $\lambda_2 \geq \lambda_1 > 0$ *and* $\lambda_2 \geq \min\{2, 3 - \lambda_1\}$, *and let* $c_2 \geq c_1 > 0$ *and* $c_2 \geq \min\{2, 3 - c_1\}$. *Then*

$$\mathcal{C}^{\lambda_2}_{a,c_2}(\eta,\beta;\phi,\psi) \subset \mathcal{C}^{\lambda_2}_{a,c_1}(\eta,\beta;\phi,\psi) \subset \mathcal{C}^{\lambda_1}_{a,c_1}(\eta,\beta;\phi,\psi)$$
$$\left(a \in \mathbb{R}\backslash\mathbb{Z}^-_0; 0 \leq \eta,\beta < 1; \phi,\psi \in \mathcal{N}\right).$$

Remark 2.1 (i) Taking $\lambda_2 = \lambda_1 + 1$ ($\lambda_1 \geq 1$), $a_2 = a_1 + 1$ ($a_1 \geq 1$), $c = 1$ and $\eta = \beta = 0$ in Theorems 2.1-2.2, Theorems 2.4-2.5 and Theorem 2.7, we have the results obtained by Choi *et al.* [2], which extend the results earlier given by Noor *et al.* [12, 14] and Liu [10].

(ii) For $a = \mu + 1$ ($\mu > -1$), $c = 1$ and $\eta = \beta = 0$, Theorems 2.1-2.2, Theorems 2.4-2.5 and Theorem 2.7 reduce the corresponding results obtained by Sokoł [16].

3 Inclusion properties involving various operators

The next theorem shows that the classes $\mathcal{S}^{\lambda}_{a,c}(\eta;\phi)$, $\mathcal{K}^{\lambda}_{a,c}(\eta;\phi)$ and $\mathcal{C}^{\lambda}_{a,c}(\eta,\beta;\phi,\psi)$ are invariant under convolution with convex functions.

Theorem 3.1 *Let* $\lambda > 0$, $a > 0$, $c \in \mathbb{R}\backslash\mathbb{Z}^-_0$, $0 \leq \eta,\beta < 1$, $\phi,\psi \in \mathcal{N}$ *and let* $g \in \mathcal{K}$. *Then*

(i) $f \in \mathcal{S}^{\lambda}_{a,c}(\eta;\phi) \Longrightarrow g * f \in \mathcal{S}^{\lambda}_{a,c}(\eta;\phi)$,

(ii) $f \in \mathcal{K}^{\lambda}_{a,c}(\eta;\phi) \Longrightarrow g * f \in \mathcal{K}^{\lambda}_{a,c}(\eta;\phi)$,

(iii) $f \in \mathcal{C}^{\lambda}_{a,c}(\eta,\beta;\phi,\psi) \Longrightarrow g * f \in \mathcal{C}^{\lambda}_{a,c}(\eta,\beta;\phi,\psi)$.

Proof (i) Let $f \in \mathcal{S}^{\lambda}_{a,c}(\eta;\phi)$. Then we have

$$\frac{1}{1-\eta}\left(\frac{z(\mathcal{I}_{\lambda}(a,c)(g*f)(z))'}{\mathcal{I}_{\lambda}(a,c)(g*f)(z)} - \eta\right) = \frac{1}{1-\eta}\left(\frac{g(z) * z(\mathcal{I}_{\lambda}(a,c)f(z))'}{g(z) * \mathcal{I}_{\lambda}(a,c)f(z)} - \eta\right).$$

By using the same techniques as in the proof of Theorem 2.1, we obtain (i).

(ii) Let $f \in \mathcal{K}^{\lambda}_{a,c}(\eta;\phi)$. Then, by (1.5), $zf'(z) \in \mathcal{S}^{\lambda}_{a,c}(\eta;\phi)$ and hence from (i), $g(z) * zf'(z) \in \mathcal{S}^{\lambda}_{a,c}(\eta;\phi)$. Since

$$g(z) * zf'(z) = z(g*f)'(z),$$

we have (ii) applying (1.5) once again.

(iii) Let $f \in \mathcal{C}^{\lambda}_{a,c}(\eta,\beta;\phi,\psi)$. Then there exists a function $q \in \mathcal{S}^*(\eta;\phi)$ such that

$$z\bigl(\mathcal{I}_{\lambda}(a,c)f(z)\bigr)' = \bigl[(1-\beta)\psi\bigl(\omega(z)\bigr) + \beta\bigr]q(z) \quad (z \in \mathbb{U}),$$

where w is an analytic function in $\mathbb{U}$ with $|\omega(z)| < 1$ ($z \in \mathbb{U}$) and $w(0) = 0$. From Lemma 2.4, we have that $g * q \in \mathcal{S}^*(\eta; \phi)$. Since

$$\begin{aligned}\frac{1}{1-\beta}\left(\frac{z(\mathcal{I}_\lambda(a,c)(g*f)(z))'}{(g*q)(z)} - \beta\right) &= \frac{1}{1-\beta}\left(\frac{g(z)*z(\mathcal{I}_\lambda(a,c)f(z))'}{g(z)*q(z)} - \beta\right) \\ &= \frac{1}{1-\beta}\left(\frac{g(z)*[(1-\beta)\psi(\omega(z))+\beta]q(z)}{g(z)*q(z)} - \beta\right) \\ &\prec \psi(z) \quad (z \in \mathbb{U}),\end{aligned}$$

we obtain (iii).

Now we consider the following operators defined by

$$\Psi_1(z) = \sum_{k=1}^{\infty} \frac{1+c}{k+c} z^k \quad \bigl(\mathrm{Re}\{c\} \geq 0; z \in \mathbb{U}\bigr) \tag{3.1}$$

and

$$\Psi_2(z) = \frac{1}{1-x} \log\left[\frac{1-xz}{1-z}\right] \quad \bigl(\log 1 = 0; |x| \leq 1, x \neq 1; z \in \mathbb{U}\bigr). \tag{3.2}$$

It is well known ([19], see also [6, 20]) that the operators Ψ_1 and Ψ_2 are convex univalent in $\mathbb{U}$. Therefore we have the following result, which can be obtained from Theorem 3.1 immediately. □

Corollary 3.1 *Let* $a > 0$, $\lambda > 0$, $c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $0 \leq \eta, \beta < 1$, $\phi, \psi \in \mathcal{N}$ *and let* Ψ_i ($i = 1, 2$) *be defined by* (3.1) *and* (3.2). *Then*

(i) $f \in \mathcal{S}_{a,c}^{\lambda}(\eta; \phi) \Longrightarrow \Psi_i * f \in \mathcal{S}_{a,c}^{\lambda}(\eta; \phi)$,
(ii) $f \in \mathcal{K}_{a,c}^{\lambda}(\eta; \phi) \Longrightarrow \Psi_i * f \in \mathcal{K}_{a,c}^{\lambda}(\eta; \phi)$,
(iii) $f \in \mathcal{C}_{a,c}^{\lambda}(\eta, \beta; \phi, \psi) \Longrightarrow \Psi_i * f \in \mathcal{C}_{a,c}^{\lambda}(\eta, \beta; \phi, \psi)$.

Remark 3.1 Letting $a = \mu + 1$ ($\mu > -1$), $c = 1$ and $\eta = \beta = 0$ in Theorem 3.1, we have the corresponding results given by Sokoł [16].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

Author details
[1]Department of Applied Mathematics, Pukyong National University, 45 Yongso-ro, Busan, Korea. [2]Department of Statistics, Pukyong National University, 45 Yongso-ro, Busan, Korea.

Acknowledgements
Dedicated to Professor Hari M Srivastava.
The authors would like to express their thanks to the referees for some valuable comments regarding a previous version of this paper. This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2012-0002619).

Received: 27 November 2012 Accepted: 4 February 2013 Published: 1 March 2013

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doi:10.1186/1029-242X-2013-83
Cite this article as: Cho and Yoon: **Inclusion relationships for certain classes of analytic functions involving the Choi-Saigo-Srivastava operator.** *Journal of Inequalities and Applications* 2013 **2013**:83.