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Viscosity iteration algorithm for a ϱ-strictly pseudononspreading mapping in a Hilbert space

Bin-Chao Deng, Tong Chen and Zhi-Fang Li*

*Correspondence:
lzf399@yahoo.com.cn
School of Management, Tianjin University, Tianjin, 300072, China

Abstract

In this paper, we discuss the strong convergence of the viscosity approximation method in Hilbert spaces relatively to the computation of fixed points of an operator in ϱ-strictly pseudononspreading. Under suitable conditions, some strong convergence theorems are proved. Our work improves previous results for nonspreading mappings.

Keywords: nonspreading mapping; ϱ-strictly pseudononspreading; demicontractive; fixed point; quasi-nonexpansive

1 Introduction

Throughout this paper, we always assume that H is a real Hilbert space endowed with an inner product and its induced norm denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. Let C be a nonempty, closed and convex subset of H and let $A : C \to H$ be a nonlinear mapping.

Definition 1.1 A is said to be

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, A is said to be α-strongly-monotone;

(iii) inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is said to be α-inverse-strongly-monotone;

(iv) k-Lipschitz continuous if there exists a constant $k \geq 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

Remark 1.2 Let $F = \mu B - \gamma f$, where B is a θ-Lipschitz and η-strongly monotone operator on H with $\theta > 0$ and f is a Lipschitz mapping on H with coefficient $L > 0$, $0 < \gamma \leq \frac{\mu\eta}{L}$. It is a simple matter to see that the operator F is $(\mu\eta - \gamma L)$-strongly monotone over H, *i.e.*,

$$\langle Fx - Fy, x - y\rangle \geq (\mu\eta - \gamma L)\|x - y\|^2, \quad \forall (x, y) \in H \times H.$$

The classical variational inequality, which is denoted by $\mathrm{VI}(A, C)$, is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The variational inequality has been extensively studied in literature (see [1–7] and the references therein).

A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A mapping T is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C;$$

see, for instance, [8–11]. It is known that a mapping $T : C \to C$ is firmly nonexpansive if and only if

$$\|Tx - Ty\|^2 + \big\|(I - T)x - (I - T)y\big\|^2 \leq \|x - y\|^2, \quad \forall x, y \in C.$$

T is said to be nonspreading in [12] if

$$\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C. \tag{1.2}$$

It is shown in [13] that (1.2) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

These mappings are generalization of a firmly nonexpansive mapping in a Hilbert space. $T : C \to C$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in C.$$

See [14–17] for more information on firmly nonexpansive mappings.

Definition 1.3 $T : H \to H$ is called demicontractive on H if there exists a constant $\alpha < 1$ such that

$$\|Tx - q\|^2 \leq \|x - q\|^2 + \alpha\|x - Tx\|^2, \quad \forall (x, q) \in H \times F_{ix}(T). \tag{1.3}$$

Definition 1.4 [18] $T : D(T) \subseteq H \to H$ is ϱ-strictly pseudononspreading if there exists $\varrho \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \varrho \big\|x - Tx - (y - Ty)\big\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{1.4}$$

for all $x, y \in D(T)$.

Remark 1.5 It is easy to claim that firmly nonexpansive mapping $\Rightarrow$ nonspreading mapping $\Rightarrow$ ϱ-strictly pseudononspreading mapping.

Indeed, from the definition of those mappings, $\forall x, y \in C$, we obtain

$$\begin{aligned}
&\|Tx - Ty\|^2 \leq \langle x - Tx, y - Ty\rangle, \quad T \text{ is firmly nonexpansive mapping.} \\
&\qquad\quad \Downarrow \\
&\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad T \text{ is nonspreading mapping.} \\
&\qquad\quad \Downarrow \\
&\|Tx - Ty\|^2 \leq \|x - y\|^2 + \varrho \big\|x - Tx - (y - Ty)\big\|^2 + 2\langle x - Tx, y - Ty\rangle, \\
&\quad T \text{ is } \varrho\text{-strictly pseudononspreading mapping.}
\end{aligned}$$

Clearly, every nonspreading mapping is ϱ-strictly pseudononspreading. The following example shows that the class of ϱ-strictly pseudononspreading mappings is more general than the class of nonspreading mappings. Let us give an example of a ϱ-strictly pseudononspreading mapping satisfying the condition of Definition 1.4.

Example 1.6 Let $X = l^2$ with the norm $\|\cdot\|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}, \quad \forall x = (x_1, x_2, \ldots, x_n, \ldots) \in X,$$

$C = \{x = (x_1, x_2, \ldots, x_n, \ldots) | x_i \in R^1, i = 1, 2, \ldots\}$, and let C be an orthogonal subspace of X (*i.e.*, $\forall x, y \in C$, we have $\langle x, y\rangle = 0$). Then it is obvious that C is a nonempty closed convex subset of X. Now, for any $x = (x_1, x_2, \ldots, x_n, \ldots) \in C$, define a mapping $T : C \to C$ as follows:

$$Tx = \begin{cases} (x_1, x_2, \ldots, x_n, \ldots), & \prod_{i=1}^{\infty} x_i < 0, \\ (-2x_1, -2x_2, \ldots, -2x_n, \ldots), & \prod_{i=1}^{\infty} x_i \geq 0. \end{cases} \tag{1.5}$$

To see that T is $\frac{1}{3}$-strictly pseudononspreading, we break the process of proof into three cases. $\forall x, y \in C$,

Case 1: $\prod_{i=1}^{\infty} x_i < 0$ and $\prod_{i=1}^{\infty} y_i < 0$, observe that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \frac{1}{3}\big\|x - Tx - (y - Ty)\big\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \frac{1}{3} \in [0,1),$$

since $\|Tx - Ty\|^2 = \|x - y\|^2$ and $\frac{1}{3}\|x - Tx - (y - Ty)\|^2 = 2\langle x - Tx, y - Ty\rangle = 0$.

Case 2: $\prod_{i=1}^{\infty} x_i \leq 0$ and $\prod_{i=1}^{\infty} y_i \geq 0$, we obtain $\|Tx - Ty\|^2 = \|x + 2y\|^2 = \|x\|^2 + 4\langle x, y\rangle + 4\|y\|^2$, $2\langle x - Tx, y - Ty\rangle = 0$ and $\frac{1}{3}\|x - Tx - (y - Ty)\|^2 = 3\|y\|^2$.

Hence,

$$\begin{aligned}
&\|x-y\|^2 + \frac{1}{3}\big\|x - Tx - (y - Ty)\big\|^2 + 2\langle x - Tx, y - Ty\rangle \\
&\quad = \|x\|^2 - 2\langle x, y\rangle + 4\|y\|^2 \\
&\quad = \|x\|^2 + 4\langle x, y\rangle + 4\|y\|^2 - 6\langle x, y\rangle \\
&\quad = \|x\|^2 + 4\langle x, y\rangle + 4\|y\|^2 \quad \big(\langle x, y\rangle = 0\big) \\
&\quad = \|x + 2y\|^2 = \|Tx - Ty\|^2.
\end{aligned}$$

Case 3: $\prod_{i=1}^{\infty} x_i \geq 0$ and $\prod_{i=1}^{\infty} y_i \geq 0$, we have $\|Tx - Ty\|^2 = 4\|x - y\|^2$, $\|x - Tx - (y - Ty)\|^2 = 9\|x - y\|^2$ and $2\langle x - Tx, y - Ty\rangle = 18\langle x, y\rangle = 0$. Thus

$$\begin{aligned}
\|Tx - Ty\|^2 &= 4\|x - y\|^2 \\
&= \|x - y\|^2 + \frac{1}{3}\big\|x - Tx - (y - Ty)\big\|^2 \\
&\leq \|x - y\|^2 + \frac{1}{3}\big\|x - Tx - (y - Ty)\big\|^2 + 2\langle x - Tx, y - Ty\rangle.
\end{aligned}$$

From (1), (2) and (3), we obtain that T is $\frac{1}{3}$-strictly pseudononspreading, *i.e.*,

$$\|Tx - Ty\|^2 = \|x - y\|^2 + \frac{1}{3}\big\|x - Tx - (y - Ty)\big\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in R.$$

We can easily know that $F_{ix}(T) = \{(x_1, x_2, \ldots, x_n, \ldots), \prod_{i=1}^{\infty} x_i < 0\} \cup \{0\}$, where $F_{ix}(T)$ is defined by the set of fixed points of T.

T is not nonspreading, since for $x = \{0, 0, \ldots, 0, \ldots\}$, $y = \{1, 0, \ldots, 0, \ldots\}$, we have $\|Tx - Ty\|^2 = 4$, $\|x - y\|^2 = 1$ and $2\langle x - Tx, y - Ty\rangle = 0$, we obtain

$$\|Tx - Ty\|^2 = 4 > 1 = \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle.$$

Since our class of maps contains the class of nonspreading mappings, it also contains the class of firmly nonexpansive mappings.

Remark 1.7 [19] Let T be an α-demicontractive mapping on H with $F_{ix}(T) \neq \emptyset$ and $T_\omega = (1 - \omega)I + \omega T$ for $\omega \in (0, \infty)$:

(A1) T α-demicontractive is equivalent to

$$\langle x - T_\omega x, x - q\rangle \geq \frac{\omega}{2}\|x - Tx\|^2, \quad \forall (x, q) \in H \times F_{ix}(T).$$

(A2) $F_{ix}(T) = F_{ix}(T_\omega)$ if $\omega \neq 0$.

Remark 1.8 Observe that if T is ϱ-strictly pseudononspreading and $F_{ix}(T) \neq \emptyset$, then $\forall x \in D(T)$ and $\forall p \in F_{ix}(T)$, we obtain

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \varrho\|x - Tx\|^2.$$

Thus, every ϱ-strictly pseudononspreading mapping with a nonempty fixed point set $F_{ix}(T)$ is demicontractive (see [20, 21]).

Remark 1.9 According to Remark 1.7(A1) and the fact that the ϱ-strictly pseudononspreading mapping of T is demicontractive, let $I - T_\omega = \omega(I - T)$. Then we obtain

$$\langle x - T_\omega x, x - q\rangle \geq \frac{\omega(1-\varrho)}{2}\|x - Tx\|^2, \quad \forall (x,q) \in H \times F_{ix}(T). \tag{1.6}$$

In 2011, Osilike and Isiogugu [12] introduced the following propositions and proved a strong convergence theorem somewhat related to a Halpern-type iteration algorithm for a ϱ-strictly pseudononspreading mapping in Hilbert spaces.

Proposition 1.10 [12] *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a ϱ-strictly pseudononspreading mapping. If $F_{ix}(T) \neq \emptyset$, then it is closed and convex.*

Proposition 1.11 [12] *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a ϱ-strictly pseudononspreading mapping. Then $(I - T)$ is demiclosed at 0.*

Theorem 1.12 [12] *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a ϱ-strictly pseudononspreading mapping with a nonempty fixed point set $F_{ix}(T)$. Let $\alpha \in [\varrho, 1)$ and let $\{\alpha_n\}_{n=1}^\infty$ be a real sequence in $[0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$. Let $u \in C$, $\{x_n\}$ and $\{z_n\}$ be sequences in C generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = \alpha_n u + (I - \alpha_n)z_n, & n > 0, \\ z_n = \frac{1}{n}\sum_{k=1}^{n-1} T_\alpha^k x_n, & n \geq 1. \end{cases} \tag{1.7}$$

Then $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{F_{ix}(T)}u$, where $P_{F_{ix}(T)} : H \to F_{ix}(T)$ is a metric projection of H onto $F_{ix}(T)$.

In 2010, Tian [22] introduced the following theorem for finding an element of a set of solutions to the fixed point of a nonexpansive mapping in a Hilbert space.

Theorem 1.13 [22] *Let f be a contraction on a real Hilbert space H and T be a nonexpansive mapping on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ generated by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)Tx_n, \quad n \geq 0, \tag{1.8}$$

where B is a θ-Lipschitz and η-strongly monotone operator on H with $\theta > 0$, $\eta > 0$ and $0 < \mu < 2\eta/\theta^2$. Assume also that a sequence $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(c1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$,
(c2) $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}\alpha_{n+1}/\alpha_n = 1$.

Then the sequence $\{x_n\}$ generated by (1.8) *converges strongly to the unique solution $x^* \in F_{ix}(T)$ of the variational inequality*

$$\langle (\gamma f - \mu B)x^*, x - x^*\rangle \leq 0, \quad \forall x \in F_{ix}(T). \tag{1.9}$$

In this paper, we combine Theorem 1.12 and Theorem 1.13 and introduce the following general iterative algorithm for a ϱ-strictly pseudononspreading mapping T.

Algorithm 1.14 Let $x_0 \in H$ be arbitrary

$$\begin{cases} x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)z_n, & n > 0, \\ z_n = \frac{1}{n}\sum_{k=1}^{n} T_\alpha^k x_n, & n \geq 1, \end{cases}$$

where $B : H \to H$ is η-strongly monotone and boundedly Lipschitzian, f is an L-Lipschitz mapping on H with coefficient $L > 0$ and $T_\alpha^k = (1-\alpha)I + \alpha T^k$, $\alpha \in (\varrho_k, \frac{1}{2})$.

Under suitable conditions, some strong convergence theorems are proved in the following chapter.

2 Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. The following lemmas are useful for main results.

Definition 2.1 A mapping T is said to be demiclosed if for any sequence $\{x_n\}$ which weakly converges to y, and if the sequence $\{Tx_n\}$ strongly converges to z, then $T(y) = z$.

Lemma 2.2 [3] *Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.3 [1] *Let $\{\mathcal{T}_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\mathcal{T}_{n_j}\}_{j\geq 0}$ of $\{\mathcal{T}_n\}$ which satisfies $\mathcal{T}_{n_j} < \mathcal{T}_{n_j+1}$ for all $j \geq 0$. Also, consider the sequence of integers $\{\delta(n)\}_{n\geq n_0}$ defined by*

$$\delta(n) = \max\{k \leq n | \mathcal{T}_k < \mathcal{T}_{k+1}\}. \tag{2.1}$$

Then $\{\delta(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty} \delta(n) = \infty$, $\forall n \geq n_0$. *It holds that $T_{\delta(n)} < \mathcal{T}_{\delta(n)+1}$, and we have*

$$\mathcal{T}_n < \mathcal{T}_{\delta(n)+1}.$$

Lemma 2.4 *Let K be a closed convex subset of a real Hilbert space H given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if the following inequality holds*:

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in K.$$

3 Main results

Let C be a nonempty closed convex subset of a real Hilbert space H and let $T^k : C \to C$ be a ϱ_k-strictly pseudononspreading mapping with a common nonempty fixed point set $\bigcap_k^n F_{ix}(T^k)$. Let f be an L-Lipschitz mapping on H with coefficient $L > 0$. Assume the set $\bigcap_k^n F_{ix}(T^k)$ is nonempty. Since $\bigcap_k^n F_{ix}(T^k)$ is closed and convex, the nearest point projection from C onto $\bigcap_k^n F_{ix}(T^k)$ is well defined. Recall $B : H \to H$ is η-strongly monotone and θ-Lipschitzian on H with $\theta > 0$, $\eta > 0$. Let $0 < \mu < 2\eta/\theta^2$, $0 < \gamma < \mu(\eta - \frac{\mu\theta^2}{2})/L = \tau/L$, consider the following sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)z_n, \quad n > 0, \\ z_n = \frac{1}{n}\sum_{k=1}^n T_\alpha^k x_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $T_\alpha^k = (1-\alpha)I + \alpha T^k$, $\alpha \in (\varrho_k, \frac{1}{2})$, $k = \{1, 2, \dots, n\}$, and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(c1) $\lim_{n\to\infty} \alpha_n = 0$,
(c2) $\sum_{n=0}^\infty \alpha_n = \infty$ or $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$.

Remark 3.1 [23] Let H be a real Hilbert space. Let B be a θ-Lipschitzian and η-strongly monotone operator on H with $\theta > 0$, $\eta > 0$. Let $0 < \mu < 2\eta/\theta^2$, let $S = (I - t\mu B)$ and $\mu(\eta - \frac{\mu\theta^2}{2}) = \tau$, then for $t \in (0, \min\{1, \frac{1}{\tau}\})$, S is a contraction with a constant $1 - t\tau$.

Before stating our main result, we introduce some lemmas for algorithm (3.1) as follows.

Lemma 3.2 *The sequence $\{x_n\}$ is generated by* (3.1) *with T^k being a ϱ-strictly pseudononspreading mapping on H and $\{\alpha_n\} \subset (0,1)$. Then $\{x_n\}$ is bounded.*

Proof Let $T_\alpha^k x = (1-\alpha)x + \alpha T^k x$ and $0 < \varrho_k < \alpha < \frac{1}{2}$. Then $\forall x, y \in C$, we have

$$\begin{aligned} \|T_\alpha^k x - T_\alpha^k y\|^2 &= \alpha\|x-y\|^2 + (1-\alpha)\|T^k x - T^k y\|^2 - \alpha(1-\alpha)\|x - T^k x - (y - T^k y)\|^2 \\ &\leq \alpha\|x-y\|^2 + (1-\alpha)\big[\|x-y\|^2 + \varrho_k\|x - T^k x - (y - T^k y)\|^2 \\ &\quad + 2\langle x - T^k x, y - T^k y\rangle\big] - \alpha(1-\alpha)\|x - T^k x - (y - T^k y)\|^2 \\ &= \|x-y\|^2 + 2(1-\alpha)\langle x - T^k x, y - T^k y\rangle \\ &\quad - (1-\alpha)(\alpha - \varrho_k)\|x - T^k x - (y - T^k y)\|^2 \\ &\leq \|x-y\|^2 + 2(1-\alpha)\langle x - T^k x, y - T^k y\rangle \\ &= \|x-y\|^2 + \frac{2(1-\alpha)}{\alpha^2}\langle x - T_\alpha^k x, y - T_\alpha^k y\rangle. \end{aligned} \tag{3.2}$$

From $p \in \bigcap_k^n F_{ix}(T^k)$ and (3.2), we also have

$$\|T_\alpha^k x_n - p\| \leq \|x_n - p\|. \tag{3.3}$$

According to (3.3), (3.1) and Remark 3.1, we obtain

$$\|z_n - p\| = \left\|\frac{1}{n}\sum_{k=1}^n T_\alpha^k x_n - p\right\| \leq \frac{1}{n}\sum_{k=1}^n \|T_\alpha^k x_n - p\| \leq \frac{1}{n}\sum_{k=1}^n \|x_n - p\| = \|x_n - p\|. \tag{3.4}$$

Thus,

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|\alpha_n\gamma\big(f(x_n)-f(p)\big)+\alpha_n\big(\gamma f(p)-\mu Bp\big)+(I-\mu\alpha_n B)(z_n-p)\big\| \\ &\le \alpha_n\gamma\big\|f(x_n)-f(p)\big\|+\alpha_n\big\|\gamma f(p)-\mu Bp\big\|+(1-\alpha_n\tau)\|z_n-p\| \\ &\le \alpha_n\gamma\big\|f(x_n)-f(p)\big\|+\alpha_n\big\|\gamma f(p)-\mu Bp\big\|+(1-\alpha_n\tau)\|x_n-p\|, \end{aligned} \tag{3.5}$$

which combined with $\|f(x_n)-f(p)\|\le L\|x_n-p\|$ amounts to

$$\|x_{n+1}-p\|\le\big(1-\alpha_n(\tau-\gamma L)\big)\|x_n-p\|+\alpha_n\big\|\gamma f(p)-\mu Bp\big\|. \tag{3.6}$$

Putting $M_1=\max\{\|x_0-p\|,\|\gamma f(p)-\mu Bp\|\}$, we clearly obtain $\|x_n-p\|\le M_1$. Hence $\{x_n\}_{n=0}^\infty$ and $\{z_n\}_{n=1}^\infty$ are bounded. From (3.3), we have that $\{T_\alpha^k x_n\}_{n=1}^\infty$ is also bounded. □

Now we are in a position to claim the main result.

Theorem 3.3 *Assume C is a nonempty closed convex subset of a real Hilbert space H and let $T^k: C\to C$ be a ϱ_k-strictly pseudononspreading mapping with a common nonempty fixed point set $\bigcap_k^n F_{ix}(T^k)$. Let f be an L-Lipschitz mapping on H with coefficient $L>0$ and $B: H\to H$ be η-strongly monotone and θ-Lipschitzian on H with $\theta>0$, $\eta>0$. Let $0<\mu<2\eta/\theta^2$, $0<\gamma<\mu(\eta-\frac{\mu\theta^2}{2})/L=\tau/L$. Consider the sequences $\{x_n\}_{n=0}^\infty$ and $\{z_n\}_{n=1}^\infty$ to be sequences in C generated from an arbitrary $x_1\in C$ by (3.1), where $T_\alpha^k=(1-\alpha)I+\alpha T^k$, $\alpha\in(\varrho_k,\frac{1}{2})$, $k=\{1,2,\ldots,n\}$, $\{\alpha_n\}_{n=1}^\infty\in[0,1)$ and $\lim_{n\to\infty}\alpha_n=0$. Then $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ converge strongly to the unique element x^* in $\bigcap_k^n F_{ix}(T^k)$ verifying*

$$P_{\bigcap_k^n F_{ix}(T^k)}(I-\mu B+\gamma f)x^*=x^*, \tag{3.7}$$

which equivalently solves the following variational inequality problem:

$$x^*\in\bigcap_k^n F_{ix}\big(T^k\big),\quad \big\langle(\gamma f-\mu B)x^*,v-x^*\big\rangle\le 0,\quad \forall v\in\bigcap_k^n F_{ix}\big(T^k\big). \tag{3.8}$$

Proof According to Lemma 3.2, it is simple to know that $\{x_n\}_{n=0}^\infty$, $\{z_n\}_{n=1}^\infty$ and $\{T_\alpha^k x_n\}_{n=1}^\infty$ are bounded. Thus, for $\forall y\in C$ and $\forall k=0,1,2,\ldots,n-1$ and according to (3.2) and (3.1), we have

$$\begin{aligned}\big\|T_\alpha^{k+1}x_n-T_\alpha y\big\|^2 &= \big\|T_\alpha\big(T_\alpha^k x_n\big)-T_\alpha y\big\|^2 \\ &\le \big\|T_\alpha^k x_n-y\big\|^2+\frac{2}{(1-\alpha)}\big\langle T_\alpha^k x_n-T_\alpha^{k+1}x_n,y-T_\alpha y\big\rangle \\ &= \big\|T_\alpha^k x_n-T_\alpha y\big\|^2+\|T_\alpha y-y\|^2+2\big\langle T_\alpha^k x_n-T_\alpha y,T_\alpha y-y\big\rangle \\ &\quad+\frac{2}{(1-\alpha)}\big\langle T_\alpha^k x_n-T_\alpha^{k+1}x_n,y-T_\alpha y\big\rangle. \end{aligned} \tag{3.9}$$

Summing (3.9) from $k=0$ to n and dividing by n, we obtain

$$\begin{aligned}\frac{1}{n}\big\|T_\alpha^{k+1}x_n-T_\alpha y\big\| &\le \frac{1}{n}\|x_n-T_\alpha y\|^2+\|T_\alpha y-y\|^2+2\langle z_n-T_\alpha y,T_\alpha y-y\rangle \\ &\quad+\frac{2}{n(1-\alpha)}\big\langle x_n-T_\alpha^n x_n,T_\alpha y-y\big\rangle. \end{aligned} \tag{3.10}$$

Since $\{z_n\}_{n=1}^{\infty}$ is bounded, then there exists a subsequence $\{z_{n_j}\}_{j=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ which converges weakly to $\omega \in C$. Replacing n by n_j in (3.10), we obtain

$$\begin{aligned}
&\frac{1}{n_j}\left\|T_\alpha^k x_{n_j} - T_\alpha y\right\|^2 \\
&\quad \le \frac{1}{n_j}\|x_{n_j} - T_\alpha y\|^2 + \|T_\alpha y - y\|^2 + 2\langle z_{n_j} - T_\alpha y, T_\alpha y - y\rangle \\
&\qquad + \frac{2}{n_j(1-\alpha)}\langle x_{n_j} - T_\alpha^{n_j} x_{n_j}, T_\alpha y - y\rangle. \qquad (3.11)
\end{aligned}$$

Since $\{x_n\}_{n=1}^{\infty}$ and $\{T_\alpha^n x_n\}_{n=1}^{\infty}$ are bounded, letting $j \to \infty$ in (3.11) yields

$$0 \le \|T_\alpha y - y\|^2 + 2\langle \omega - T_\alpha y, T_\alpha y - y\rangle. \qquad (3.12)$$

Let $y = \omega$ in (3.12). We obtain that $\omega \in F_{ix}(T_\alpha) = F_{ix}(T)$.

Observe that since $\{x_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are bounded, and $\lim_{n\to\infty}\alpha_n = 0$, then

$$\begin{aligned}
\|x_{n+1} - z_n\| &= \alpha_n\left\|\gamma f(x_n) - \mu B z_n\right\| \\
&\le \alpha_n\gamma\left\|f(x_n) - f(p)\right\| + \alpha_n\left\|\gamma f(p) - \mu Bp\right\| + \alpha_n\left\|\mu B(z_n - p)\right\| \\
&\le \alpha_n\gamma L\|x_n - p\| + \alpha_n\left\|\gamma f(p) - \mu Bp\right\| + \alpha_n\tau\|z_n - p\|,
\end{aligned}$$

then

$$\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0. \qquad (3.13)$$

We next show that

$$\limsup_{n\to\infty}\langle(\gamma f - \mu B)z, x_{n+1} - z\rangle \le 0. \qquad (3.14)$$

Indeed, take $\{x_{n_j+1}\}_{n=1}^{\infty}$ of $\{x_{n+1}\}_{n=1}^{\infty}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - \mu B)x^*, x_{n+1} - x^*\rangle = \lim_{j\to\infty}\langle(\gamma f - \mu B)x^*, x_{n_j+1} - x^*\rangle,$$

where x^* is obtained in (3.7). We may assume that $x_{n_j+1} \rightharpoonup z$ as $j \to \infty$. From (3.13), we have $z_{n_j} \rightharpoonup z$ as $j \to \infty$, then to arbitrary bounded linear functional g on H, we have

$$\begin{aligned}
\left\|g(z_{n_j}) - g(z)\right\| &\le \left\|g(z_{n_j}) - g(x_{n_j+1})\right\| + \left\|g(x_{n_j+1}) - g(z)\right\| \\
&\le \|g\|\|z_{n_j} - x_{n_j+1}\| + \left\|g(x_{n_j+1}) - g(z)\right\| \\
&\to 0, \quad \text{as } j \to 0.
\end{aligned}$$

Thus, we obtain $z_{n_j} \to z$ as $j \to 0$, and $z \in F_{ix}(T)$. Hence, we have

$$\lim_{j\to\infty}\langle(\gamma f - \mu B)x^*, x_{n_j+1} - x^*\rangle = \langle(\gamma f - \mu B)x^*, z - x^*\rangle \le 0. \qquad (3.15)$$

Moreover, from (3.1), (3.13) and (3.14), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle(\gamma f-\mu B)z, z_n - z\rangle \\
&\quad = \limsup_{n\to\infty}\langle(\gamma f-\mu B)z, x_{n+1} - z\rangle + \limsup_{n\to\infty}\langle(\gamma f-\mu B)z, z_n - x_{n+1}\rangle \\
&\quad \le \limsup_{n\to\infty}\langle(\gamma f-\mu B)z, x_{n+1} - z\rangle \le 0.
\end{aligned} \tag{3.16}$$

As required, finally we show that $x_n \to x^*$ and $z_n \to x^*$.

According to (3.1), (3.4) and (3.16), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(\gamma f(x_n)-\mu Bx^*) + (I-\mu\alpha_n B)z_n - (I-\mu\alpha_n B)x^*\|^2 \\
&= \alpha_n^2\|\gamma f(x_n)-\mu Bx^*\|^2 + \|(I-\mu\alpha_n B)z_n - (I-\mu\alpha_n B)x^*\|^2 \\
&\quad + 2\alpha_n\langle(I-\mu\alpha_n B)z_n - (I-\mu\alpha_n B)x^*, \gamma f(x_n)-\mu Bx^*\rangle \\
&\le \alpha_n^2\|\gamma f(x_n)-\mu Bx^*\|^2 + (1-\alpha_n\tau)^2\|z_n - x^*\|^2 \\
&\quad + 2\alpha_n\bigl[\langle z_n - x^*, \gamma f(x_n)-\mu Bx^*\rangle - \mu\alpha_n\langle Bz_n - Bx^*, \gamma f(x_n)-\mu Bx^*\rangle\bigr] \\
&\le \bigl[(1-\alpha_n\tau)^2 + 2\alpha_n\gamma L\bigr]\|x_n - x^*\|^2 + \alpha_n\bigl[2\langle z_n - x^*, \gamma f(x_n)-\mu Bx^*\rangle \\
&\quad + \alpha_n\|\gamma f(x_n)-\mu Bx^*\|^2 + 2\mu\alpha_n\|Bz_n - Bx^*\|\|\gamma f(x_n)-\mu Bx^*\|\bigr] \\
&\le \bigl[1-2\alpha_n(\tau-\gamma L)\bigr]\|x_n - x^*\|^2 + \alpha_n\bigl[2\langle x_n - x^*, \gamma f(x_n)-\mu Bx^*\rangle \\
&\quad + \alpha_n\|\gamma f(x_n)-\mu Bx^*\|^2 + 2\mu\alpha_n\|Bz_n - Bx^*\|\|\gamma f(x_n)-\mu Bx^*\| \\
&\quad + \alpha_n\tau^2\|x_n - x^*\|^2\bigr] \\
&= (1-\bar{\alpha_n})\|x_n - x^*\|^2 + \bar{\alpha_n}\bar{\beta_n},
\end{aligned}$$

where $\bar{\alpha_n} = 2\alpha_n(\tau-\gamma L)$,

$$\begin{aligned}
\bar{\beta_n} = {}& \frac{1}{2(\tau-\gamma L)}\bigl[2\langle x_n - x^*, \gamma f(x_n)-\mu Bx^*\rangle + \alpha_n\|\gamma f(x_n)-\mu Bx^*\|^2 \\
&+ 2\mu\alpha_n\|Bz_n - Bx^*\|\|\gamma f(x_n)-\mu Bx^*\| + \alpha_n\tau^2\|x_n - x^*\|^2\bigr].
\end{aligned}$$

It is easily seen that $\lim_{n\to\infty}\bar{\alpha_n}$, $\sum\bar{\alpha_n} = \infty$ and $\limsup_{n\to\infty}\bar{\beta_n} \le 0$. By Lemma 2.2, we conclude that $x_n \to x^*$ as $n\to\infty$, and z_n also converges strongly to the unique element x^* in $F_{ix}(T)$. In addition, the variational inequality (3.15) can be written as

$$\langle(I-\mu B+\gamma f)x^* - x^*, z - x^*\rangle \ge 0, \quad z \in \bigcap_k^n F_{ix}(T^k).$$

So, by Lemma 2.4, it is equivalent to the fixed point equation

$$P_{\bigcap_k^n F_{ix}(T^k)}(I-\mu B+\gamma f)x^* = x^*. \qquad \square$$

Remark 3.4 For a nonspreading mapping T, we have $\varrho = 0$ in Theorem 3.3 to obtain the following corollary.

Corollary 3.5 *Assume C is a nonempty closed convex subset of a real Hilbert space H and let $T^k : C \to C$ be a nonspreading mapping with a common nonempty fixed point set $\bigcap_k^n F_{ix}(T^k)$. Let f be an L-Lipschitz mapping on H with coefficient $L > 0$ and $B : H \to H$ be η-strongly monotone and θ-Lipschitzian on H with $\theta > 0$, $\eta > 0$. Let $0 < \mu < 2\eta/\theta^2$, $0 < \gamma < \mu(\eta - \frac{\mu\theta^2}{2})/L = \tau/L$, consider the sequences $\{x_n\}_{n=0}^\infty$ and $\{z_n\}_{n=1}^\infty$ to be sequences in C generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)z_n, & n > 0, \\ z_n = \frac{1}{n}\sum_{k=1}^n T_\alpha^k x_n, & n \geq 1, \end{cases}$$

where $T_\alpha^k = (1-\alpha)I + \alpha T^k$, $\alpha \in (0, \frac{1}{2})$, $\{\alpha_n\}_{n=1}^\infty \in [0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$. Then $\{x_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ converge strongly to the unique element x^* in $\bigcap_k^n F_{ix}(T^k)$ verifying

$$P_{\bigcap_k^n F_{ix}(T^k)}(I - \mu B + \gamma f)x^* = x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in \bigcap_k^n F_{ix}(T^k), \quad \langle (\gamma f - \mu B)x^*, \nu - x^* \rangle \leq 0, \quad \forall \nu \in \bigcap_k^n F_{ix}(T^k).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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