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Strong convergence theorems for uniformly L-Lipschitzian asymptotically pseudocontractive mappings in Banach spaces

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Abstract
In this paper, we establish some strong convergence theorems of the modified Ishikawa and Mann iterations with errors of uniformly L-Lipschitzian asymptotically pseudocontractive mappings in real Banach spaces. Our results not only provide the new proof method, but also extend the known corresponding results given in (Chang in Proc. Am. Math. Soc. 129:845-853, 2001; Chang *et al.* in Appl. Math. Lett. 22:121-125, 2009; Goebel and Kirk in Proc. Am. Math. Soc. 35:171-174, 1972; Ofoedu in J. Math. Anal. Appl. 321:722-728, 2006; Schu in J. Math. Anal. Appl. 158:407-413, 1991). In order to get some applications of our results, we also provide specific examples.
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1 Introduction and preliminaries

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing, respectively. The normalized duality mapping J has the following properties:

(1) J is an odd mapping, *i.e.*, $J(-x) = -J(x)$.
(2) J is positive homogeneous, *i.e.*, for any $\lambda > 0$, $J(\lambda x) = \lambda J(x)$.
(3) J is bounded, *i.e.*, for any bounded subset A of E, $J(A)$ is a bounded subset of E^*.
(4) If E is smooth (or E^* is strictly convex), then J is single-valued.

In the sequel, we denote the single-valued normalized duality mapping by j. In a Hilbert space H, j is the identity mapping.

Let D be a nonempty closed convex subset of E. A mapping $T : D \to D$ is said to be *asymptotically nonexpansive* with a sequence $\{k_n\} \subset [1,+\infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for all

$x, y \in D$,

$$\left\| T^n x - T^n y \right\| \le k_n \|x - y\| \tag{1.1}$$

for all $n \ge 1$. The mapping T is said to be *asymptotically pseudocontractive* with a sequence $\{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for any $x, y \in D$, there exists $j(x-y) \in J(x-y)$ such that

$$\left\langle T^n x - T^n y, j(x-y) \right\rangle \le k_n \|x - y\|^2 \tag{1.2}$$

for all $n \ge 1$. Furthermore, the mapping T is said to be *uniformly L-Lipschitzian* if, for any $x, y \in D$, there exists a constant $L > 0$ such that

$$\left\| T^n x - T^n y \right\| \le L \|x - y\| \tag{1.3}$$

for all $n \ge 1$.

It is easy to see that if T is an asymptotically nonexpansive mapping, then it is both asymptotically pseudocontractive and uniformly L-Lipschitzian. The converse is not true in general. Therefore, it is interesting to study these mappings in fixed point theory and its applications. In fact, the asymptotically nonexpansive and asymptotically pseudocontractive mappings were first introduced by Goebel-Kirk [1] and Schu [2], respectively. Since then, some authors have studied several iterative sequences for asymptotically nonexpansive and asymptotically pseudocontractive mappings in Hilbert spaces and Banach spaces (see [3–11]).

In [2], Schu proved the following theorem.

Theorem 1.1 [2] *Let H be a Hilbert space, K be a nonempty bounded closed convex subset of H and $T : K \to K$ be a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ satisfying the following conditions*:

(a-1) $k_n \to 1$ *as* $n \to \infty$;

(a-2) $\sum_{n=1}^{\infty} (q_n^2 - 1) < \infty$, *where* $q_n = 2k_n - 1$.

Suppose further that $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ such that $\epsilon < \alpha_n < b$ for all $n \ge 1$, where $\epsilon > 0$ and $b \in (0, L^{-2}[(1 + L^2)^{1/2} - 1])$. For any $x_1 \in K$, let $\{x_n\}$ be an iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n$$

for all $n \ge 1$. Then $\{x_n\}$ converges strongly to a fixed point of T in K.

In [12], Chang extended above Theorem 1.1 to the setting of real uniformly smooth Banach spaces and proved the following.

Theorem 1.2 [12] *Let E be a uniformly smooth Banach space, K be a nonempty bounded closed convex subset of E and $T : K \to K$ be an asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $F(T) \neq \emptyset$, where $F(T)$ is the set of fixed points of T in K. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions*:

(a-1) $\alpha_n \to 0$ *as* $n \to \infty$;

(a-2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For any $x_0 \in K$, *let* $\{x_n\}$ *be an iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n$$

for all $n \geq 0$. *If there exists a strictly increasing function* $\Phi : [0,+\infty) \to [0,+\infty)$ *with* $\Phi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x-x^*)\rangle \leq k_n \|x - x^*\|^2 - \Phi(\|x - x^*\|)$$

for all $x \in K$ *and* $n \geq 0$, *where* $x^* \in F(T)$, *then* $x_n \to x^*$ *as* $n \to \infty$.

In [13], Ofoedu extended Theorem 1.2 in a uniformly smooth Banach space to the setting of arbitrary real Banach spaces and dropped the boundedness assumption in Theorem 1.2.

Theorem 1.3 [13] *Let* E *be a real Banach space*, K *be a nonempty closed convex subset of* E *and* $T : K \to K$ *be a uniformly* L-*Lipschitzian asymptotically pseudocontractive mapping with a sequence* $\{k_n\} \subset [1,+\infty)$, $\lim_{n\to\infty} k_n = 1$ *and* $x^* \in F(T)$. *Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *satisfying the following conditions*:

(a-1) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(a-2) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;
(a-3) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.

For any $x_0 \in K$, *let* $\{x_n\}$ *be an iterative sequence defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n$$

for all $n \geq 0$. *If there exists a strictly increasing function* $\Phi : [0,+\infty) \to [0,+\infty)$ *with* $\Phi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x-x^*)\rangle \leq k_n \|x - x^*\|^2 - \Phi(\|x - x^*\|)$$

for all $x \in K$ *and* $n \geq 0$. *Then*

(1) $\{x_n\}$ *is bounded*;
(2) $\{x_n\}$ *converges strongly to* $x^* \in F(T)$.

Theorem 1.4 [13] *Let* E *be a real Banach space. Let* K *be a nonempty closed and convex subset of* E, $T : K \to K$ *be a uniformly* L-*Lipschitzian asymptotically pseudocontractive mapping with a sequence* $\{k_n\} \subset [1,+\infty)$ *with* $\lim_{n\to\infty} k_n = 1$ *and* $x^* \in F(T)$. *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be real sequences in* $[0,1]$ *satisfying the following conditions*:

(a-1) $a_n + b_n + c_n = 1$;
(a-2) $\sum_{n\geq 0}(b_n + c_n) = \infty$;
(a-3) $\sum_{n\geq 0}(b_n + c_n)^2 < \infty$;
(a-4) $\sum_{n\geq 0}(b_n + c_n)(k_n - 1) < \infty$;
(a-5) $\sum_{n\geq 0} c_n < \infty$.

For arbitrary $x_0 \in K$, *let* $\{x_n\}$ *be a sequence in* K *iteratively defined by*

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n$$

for all $n \geq 0$, *where* $\{u_n\}$ *is a bounded sequence in* K. *Suppose that there exists a strictly increasing continuous function* $\Phi : [0, +\infty) \to [0, +\infty)$ *with* $\Phi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*) \rangle \leq k_n \|x - x^*\|^2 - \Phi(\|x - x^*\|)$$

for all $x \in K$. *Then* $\{x_n\}_{n\geq 0}$ *converges strongly to* $x^* \in F(T)$.

Very recently, in [14], Chang *et al.* proved the following theorem.

Theorem 1.5 [14] *Let* E *be a real Banach space. Let* K *be a nonempty closed convex subset of* E, $T_i : K \to K$ *be two uniformly* L_i*-Lipschitzian mappings with* $F(T_1) \cap F(T_2) \neq \emptyset$ *and* $x^* \in F(T_1) \cap F(T_2)$. *Let* $\{k_n\} \subset [1, +\infty)$ *be a sequence with* $\lim_{n\to\infty} k_n = 1$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $[0, 1]$ *satisfying the following conditions*:

(a-1) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(a-2) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;

(a-3) $\sum_{n=0}^{\infty} \beta_n < \infty$;

(a-4) $\sum_{n=0}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_0 \in K$, *let* $\{x_n\}$ *be an iterative sequence in* K *defined by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T_2^n x_n \end{cases}$$

for all $n \geq 0$. *If there exists a strictly increasing function* $\Phi : [0, +\infty) \to [0, +\infty)$ *with* $\Phi(0) = 0$ *such that*

$$\langle T_i^n x - x^*, j(x - x^*) \rangle \leq k_n \|x - x^*\|^2 - \Phi(\|x - x^*\|)$$

for all $j(x - x^*) \in J(x - x^*)$ *and* $x \in K$, $i = 1, 2$, *then* $\{x_n\}$ *converges strongly to* x^*.

Also, some authors have studied the modified Halpern, Mann and Ishikawa iterative sequences for nonlinear mappings in Hilbert spaces and Banach spaces (see [15, 16]).

The aim of this paper is to give some strong convergence theorems for uniformly L-Lipschitzian and asymptotically pseudo contractive mappings in Banach spaces. Our results not only include the past ones known in [3–11], but also provide quite a different proof method.

For our main purpose, we recall some concepts and lemmas.

Definition 1.6 [17] For arbitrary $x_1 \in D$, the sequence $\{x_n\}$ in D defined by

$$\begin{cases} y_n = (1 - b_n - d_n)x_n + b_n T^n x_n + d_n v_n, \\ x_{n+1} = (1 - a_n - c_n)x_n + a_n T^n y_n + c_n u_n \end{cases} \tag{1.4}$$

for all $n \geq 1$ is called the *modified Ishikawa iteration* with errors, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0, 1]$ satisfying $a_n + c_n \leq 1$, $b_n + d_n \leq 1$ and $\{u_n\}$, $\{v_n\}$ are any bounded sequences in D.

In particular, if $b_n = d_n = 0$ in (1.4), then the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - a_n - c_n)x_n + a_n T^n x_n + c_n u_n \tag{1.5}$$

for all $n \geq 1$ is called the *modified Mann iteration* with errors.

If $c_n = d_n = 0$ in (1.4) and (1.5), then the sequence $\{x_n\}$ defined by

$$\begin{cases} y_n = (1 - b_n)x_n + b_n T^n x_n, \\ x_{n+1} = (1 - a_n)x_n + a_n T^n y_n \end{cases} \tag{1.6}$$

and

$$x_{n+1} = (1 - a_n)x_n + a_n T^n x_n \tag{1.7}$$

for all $n \geq 1$ is called the *modified Ishikawa iteration* and the *modified Mann iteration*, respectively.

Lemma 1.7 [18] *Let E be a real Banach space and $J : E \to 2^{E^*}$ be a normalized duality mapping. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\big\langle y, j(x + y)\big\rangle$$

for all $x, y \in E$ and $j(x + y) \in J(x + y)$.

Lemma 1.8 [19] *Let $\{\delta_n\}$, $\{\lambda_n\}$ and $\{\gamma_n\}$ be three nonnegative real sequences and $\Phi : [0, +\infty) \to [0, +\infty)$ be a strictly increasing continuous function with $\Phi(0) = 0$ satisfying the following inequality*:

$$\delta_{n+1}^2 \leq \delta_n^2 - \lambda_n \Phi(\delta_{n+1}) + \gamma_n$$

for all $n \geq 0$, where $\lambda_n \in [0, 1]$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\gamma_n = o(\lambda_n)$. Then $\delta_n \to 0$ as $n \to \infty$.

2 Main results

Now, we give our main results in this paper.

Theorem 2.1 *Let E be a real Banach space, D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $q \in F(T) = \{x \in D : Tx = x\} \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ be four real sequences in $[0, 1]$ satisfying the following conditions*:

(A-1) $a_n, b_n, d_n \to 0$ *as* $n \to \infty$;

(A-2) $\sum_{n=1}^{\infty} a_n = \infty$;

(A-3) $c_n = o(a_n)$.

For some $x_1 \in D$, let $\{x_n\}$ be a modified Ishikawa iterative sequence with errors defined by (1.4). *Suppose that there exists a strictly increasing continuous function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that*

$$\big\langle T^n x - q, j(x - q)\big\rangle \leq k_n \|x - q\|^2 - \Phi\big(\|x - q\|\big) \tag{2.1}$$

for all $n \geq 1$, where $j(x - q) \in J(x - q)$. Then $\{x_n\}$ converges strongly to the fixed point q of T.

Proof Step 1. For any $n \geq 1$, $\{x_n\}$ is a bounded sequence.

Set $\sup\{k_n : n \geq 1\} = k$. Then there exists $x_1 \in D$ with $x_1 \neq Tx_1$ such that $r_0 = (k+L)\|x_1 - q\|^2 \in R(\Phi)$. Indeed, for any taking $x_1 \in D$ and $x_1 \neq Tx_1$, we denote $r_0 = (k+L)\|x_1 - q\|^2$. If $\Phi(r) \to +\infty$ as $r \to +\infty$, then $r_0 \in R(\Phi)$. If $\sup\{\Phi(r) : r \in [0, +\infty)\} = r_1 < +\infty$ with $r_1 < r_0$, then there exists a sequence $\{\xi_n\} \subset D$ such that $\xi_n \to q$ as $n \to \infty$ with $\xi_n \neq q$, thus there exists a positive integer n_0 such that $(k+L)\|\xi_n - q\|^2 < \frac{r_1}{2}$ for all $n \geq n_0$. We redefine $x_1 = \xi_{n_0}$ and $(k+L)\|x_1 - q\|^2 \in R(\Phi)$.

Set $R = \Phi^{-1}(r_0)$. Then we obtain $\|x_1 - q\| \leq R$. Denote

$$\begin{aligned} &B_1 = \bigl\{x \in D : \|x - q\| \leq R\bigr\}, \qquad B_2 = \bigl\{x \in D : \|x - q\| \leq 2R\bigr\}, \\ &M = \sup_n \bigl\{\|u_n - q\| + \|v_n - q\|\bigr\}. \end{aligned}$$

Next, we prove that $x_n \in B_1$ for any $n \geq 1$. If $n = 1$, then $x_1 \in B_1$. Now, we assume that it holds for some n, *i.e.*, $x_n \in B_1$. We prove that $x_{n+1} \in B_1$. Suppose that this does not hold. Then $\|x_{n+1} - q\| > R$. Now, we denote

$$\tau_0 = \min\left\{1, \frac{R}{2LR + M}, \frac{\Phi(R)}{10R^2}, \frac{\Phi(R)}{10MR}, \frac{\Phi(R)}{10R[2LR + 3L^2R + 2L(M+R)]}\right\}. \tag{2.2}$$

Since $a_n, b_n, c_n, \frac{c_n}{a_n}, d_n, k_n - 1 \to 0$ as $n \to \infty$, without loss of generality, let $0 \leq a_n, b_n, c_n, \frac{c_n}{a_n}$, $d_n, k_n - 1 \leq \tau_0$ for any $n \geq 1$. Thus, we have

$$\begin{aligned} \|y_n - q\| &\leq (1 - b_n - d_n)\|x_n - q\| + b_n\bigl\|T^n x_n - T^n q\bigr\| + d_n\|v_n - q\| \\ &\leq (1 - b_n - d_n + b_n L)\|x_n - q\| + d_n M \\ &\leq (1 + b_n L)R + d_n M \\ &\leq R + \tau_0(LR + M) \\ &< 2R, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1 - a_n - c_n)\|x_n - q\| + a_n\bigl\|T^n y_n - T^n q\bigr\| + c_n\|u_n - q\| \\ &\leq \|x_n - q\| + a_n L\|y_n - q\| + c_n\|u_n - q\| \\ &\leq R + a_n L\bigl[(1 + b_n L)R + d_n M\bigr] + c_n M \\ &\leq R + 2a_n LR + c_n M \\ &\leq R + \tau_0(2LR + M) \\ &\leq 2R \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\bigl\|T^n x_{n+1} - T^n y_n\bigr\| \\ &\quad\leq L\|x_{n+1} - y_n\| \\ &\quad\leq a_n L\bigl\|T^n y_n - x_n\bigr\| + c_n L\|u_n - x_n\| + b_n L\bigl\|x_n - T^n x_n\bigr\| + d_n L\|x_n - v_n\| \\ &\quad\leq a_n L\bigl(\|x_n - q\| + \bigl\|T^n y_n - T^n q\bigr\|\bigr) + c_n L\bigl(\|u_n - q\| + \|x_n - q\|\bigr) \\ &\qquad + b_n L\bigl(\|x_n - q\| + \bigl\|T^n x_n - T^n q\bigr\|\bigr) + d_n L\bigl(\|v_n - q\| + \|x_n - q\|\bigr) \end{aligned}$$

$$
\begin{aligned}
&\le a_nL\big(\|x_n-q\|+L\|y_n-q\|\big)+c_nL(M+R)+b_nL(1+L)\|x_n-q\|+d_nL(M+R)\\
&\le a_nL(1+2L)R+c_nL(M+R)+b_nL(1+L)R+d_nL(M+R)\\
&\le \tau_0\big[L(1+2L)R+L(M+R)+L(1+L)R+L(M+R)\big]\\
&\le \tau_0\big[2LR+3L^2R+2L(M+R)\big]\\
&\le \frac{\Phi(R)}{10R}.
\end{aligned}
\tag{2.5}
$$

Applying Lemma 1.7 and the formulas above, we obtain

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \big\|(1-a_n-c_n)(x_n-q)+a_n\big(T^ny_n-q\big)+d_n(u_n-q)\big\|^2\\
&\le (1-a_n)^2\|x_n-q\|^2+2a_n\big\langle T^ny_n-q,j(x_{n+1}-q)\big\rangle+2c_n\big\langle u_n-q,j(x_{n+1}-q)\big\rangle\\
&= (1-a_n)^2\|x_n-q\|^2+2a_n\big\langle T^nx_{n+1}-q,j(x_{n+1}-q)\big\rangle\\
&\quad+2a_n\big\langle T^ny_n-T^nx_{n+1},j(x_{n+1}-q)\big\rangle+2c_n\big\langle u_n-q,j(x_{n+1}-q)\big\rangle\\
&\le (1-a_n)^2\|x_n-q\|^2+2a_n\big[k_n\|x_{n+1}-q\|^2-\Phi\big(\|x_{n+1}-q\|\big)\big]\\
&\quad+2a_n\big\|T^nx_{n+1}-T^ny_n\big\|\cdot\|x_{n+1}-q\|+2c_n\|u_n-q\|\cdot\|x_{n+1}-q\|\\
&\le (1-a_n)^2R^2+2a_n\big[k_n\|x_{n+1}-q\|^2-\Phi(R)\big]+\frac{2a_n}{10R}\Phi(R)2R+4c_nMR\\
&\le (1-a_n)^2R^2+2a_n\big[k_n\|x_{n+1}-q\|^2-\Phi(R)\big]+\frac{4a_n}{5}\Phi(R).
\end{aligned}
\tag{2.6}
$$

Since $a_n\to 0$ and $k_n\to 1$ as $n\to\infty$, we have $2k_na_n\to 0$ as $n\to\infty$. Thus, without loss of generality, let $1-2k_na_n>0$ for any $n\ge 1$. Then (2.6) implies that

$$
\begin{aligned}
&\|x_{n+1}-q\|^2\\
&\quad\le R^2+\frac{2a_n}{1-2k_na_n}\left[(k_n-1)+\frac{a_n}{2}\right]R^2-\frac{2a_n}{1-2k_na_n}\Phi(R)+\frac{2a_n}{1-2k_na_n}\left[\frac{2}{5}\Phi(R)\right]\\
&\quad\le R^2-\frac{2a_n}{5(1-2k_na_n)}\Phi(R)\\
&\quad\le R^2,
\end{aligned}
\tag{2.7}
$$

which is a contradiction. Hence, $x_{n+1}\in B_1$, *i.e.*, $\{x_n\}$ is a bounded sequence.

Step 2. We prove that $\|x_n-q\|\to 0$ as $n\to\infty$.

By Step 1, we obtain $\{\|x_n-q\|\}$ is a bounded sequence and so is $\{\|y_n-q\|\}$. Let

$$
M_0=\sup_n\big\{\|x_n-q\|\big\}+\sup_n\big\{\|y_n-q\|\big\}+\sup_n\big\{\|u_n-q\|\big\}+\sup_n\big\{\|v_n-q\|\big\}.
$$

Observe that

$$
\begin{aligned}
\|x_{n+1}-y_n\| &\le a_nL\big\|T^ny_n-x_n\big\|+c_nL\|u_n-x_n\|+b_nL\big\|x_n-T^nx_n\big\|+d_nL\|x_n-v_n\|\\
&\le a_nL\big(\|x_n-q\|+\big\|T^ny_n-T^nq\big\|\big)+c_nL\big(\|u_n-q\|+\|x_n-q\|\big)\\
&\quad+b_nL\big(\|x_n-q\|+\big\|T^nx_n-T^nq\big\|\big)+d_nL\big(\|v_n-q\|+\|x_n-q\|\big)\\
&\le (a_n+b_n)L(1+L)M_0+(c_n+d_n)LM_0.
\end{aligned}
\tag{2.8}
$$

Using Lemma 1.7, (2.6) and (2.8), we have

$$
\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \le (1 - a_n - c_n)^2 \|x_n - q\|^2 + 2a_n \langle T^n y_n - q, j(x_{n+1} - q) \rangle + 2c_n \langle u_n - q, j(x_{n+1} - q) \rangle \\
&\quad \le (1 - a_n)^2 \|x_n - q\|^2 + 2a_n \langle T^n x_{n+1} - q, j(x_{n+1} - q) \rangle \\
&\qquad + 2a_n \langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - q) \rangle + 2c_n \langle u_n - q, j(x_{n+1} - q) \rangle \\
&\quad \le (1 - a_n)^2 \|x_n - q\|^2 + 2a_n \bigl[k_n \|x_{n+1} - q\|^2 - \Phi\bigl(\|x_{n+1} - q\|\bigr)\bigr] \\
&\qquad + 2a_n \bigl\| T^n x_{n+1} - T^n y_n \bigr\| \cdot \|x_{n+1} - q\| + 2c_n \|u_n - q\| \cdot \|x_{n+1} - q\| \\
&\quad \le \|x_n - q\|^2 + 2a_n(k_n - 1)M_0^2 + a_n^2 M_0^2 - a_n \Phi\bigl(\|x_{n+1} - q\|\bigr) \\
&\qquad + 2a_n L \|x_{n+1} - y_n\| M_0 + 2c_n M_0^2 \\
&\quad \le \|x_n - q\|^2 + 2a_n(k_n - 1)M_0^2 + a_n^2 M_0^2 - a_n \Phi\bigl(\|x_{n+1} - q\|\bigr) \\
&\qquad + 2a_n L \bigl[(a_n + b_n)L(1 + L)M_0 + (c_n + d_n)LM_0\bigr] M_0 + 2c_n M_0^2 \\
&\quad \le \|x_n - q\|^2 - a_n \Phi\bigl(\|x_{n+1} - q\|\bigr) + A_n,
\end{aligned} \tag{2.9}
$$

where

$$A_n = 2a_n(k_n - 1)M_0^2 + a_n^2 M_0^2 + 2a_n L\bigl[(a_n + b_n)L(1 + L)M_0 + (c_n + d_n)LM_0\bigr]M_0 + 2c_n M_0^2.$$

Let $\delta_n = \|x_n - q\|^2$, $\lambda_n = a_n$ and $\gamma_n = A_n$. Then (2.9) leads to

$$\delta_{n+1}^2 \le \delta_n^2 - \lambda_n \Phi(\delta_{n+1}) + \gamma_n.$$

Therefore, by Lemma 1.8, we obtain $\lim_{n\to\infty} \delta_n = 0$, *i.e.*, $x_n \to q$ as $n \to \infty$. This completes the proof. □

From Theorem 2.1, we have the following corollary.

Corollary 2.2 *Let E be a real Banach space. Let D be a nonempty closed convex subset of E, $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence* $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ *and* $q \in F(T)$. *Let* $\{a_n\}$ *and* $\{c_n\}$ *be two real sequences in* $[0, 1]$ *satisfying the following conditions*:

(A-1) $a_n \to 0$ *as* $n \to \infty$;
(A-2) $\sum_{n=1}^{\infty} a_n = \infty$;
(A-3) $c_n = o(a_n)$.

For some $x_1 \in D$, *let* $\{x_n\}$ *be a modified Mann iterative sequence with errors defined by* (1.5). *Suppose that there exists a strictly increasing function* $\Phi : [0, +\infty) \to [0, +\infty)$ *with* $\Phi(0) = 0$ *such that*

$$\bigl\langle T^n x - q, j(x - q) \bigr\rangle \le k_n \|x - q\|^2 - \Phi\bigl(\|x - q\|\bigr) \tag{2.10}$$

for all $n \ge 1$, *where* $j(x - q) \in J(x - q)$. *Then* $\{x_n\}$ *converges strongly to the fixed point* q *of* T.

Proof In Theorem 2.1, letting $b_n = 0$, $d_n = 0$, we can get the convergence of the modified Mann iteration (1.5). □

Theorem 2.3 *Let E be a real Banach space. Let D be a nonempty closed convex subset of E and $T_i : K \to K$ be two uniformly L_i-Lipschitzian mappings with $q \in F(T_1) \cap F(T_2)$. Let $\{k_n\} \subset [1, +\infty)$ be a sequence with $k_n \to 1$ as $n \to \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ be four real sequences in $[0, 1]$ satisfying the following conditions*:

(A-1) $a_n, b_n \to 0$ *as* $n \to \infty$;
(A-2) $\sum_{n=1}^{\infty} a_n = \infty$;
(A-3) $c_n = o(a_n)$, $d_n \to 0$ *as* $n \to \infty$.

For some $x_1 \in D$, let $\{x_n\}$ be an iterative sequence with errors defined by

$$\begin{cases} y_n = (1 - b_n - d_n)x_n + b_n T_1^n x_n + d_n v_n, \\ x_{n+1} = (1 - a_n - c_n)x_n + a_n T_2^n y_n + c_n u_n \end{cases} \tag{2.11}$$

for all $n \geq 1$. Suppose that there exists a strictly increasing function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(0) = 0$ such that

$$\langle T_i^n x - q, j(x - q)\rangle \leq k_n\|x - q\|^2 - \Phi(\|x - q\|) \tag{2.12}$$

for all $n \geq 1$ and $i = 1, 2$, where $j(x - q) \in J(x - q)$. Then $\{x_n\}$ converges strongly to the fixed point q of $T_1 \cap T_2$.

Proof Similarly, we can obtain the result of Theorem 2.3 by using the proof method of Theorem 2.1. □

Remark 2.4 Theorem 2.1 extends, improves and unifies Theorems 3.1, 3.2, 3.3 of [13] and Theorem 3.5 of [14] in the following sense:

(1) The modified Mann iteration and modified Ishikawa iteration are replaced by the modified Ishikawa iteration with errors introduced by Xu [17].
(2) The proof method of Theorem 2.1 is quite different from the method of [13, 14].
(3) In [13], the author did not require the function Φ to be surjective. Since x_0 is an arbitrary point chosen in D, it is possible that $\Phi^{-1}(a_0)$ is not well defined.
(4) The conditions $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$, $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$, $\sum_{n=0}^{\infty} \beta_n < \infty$ in [13, Theorem 3.1, Theorem 3.2] and [14, Theorem 2.1] are replaced by the more general conditions $\alpha_n, \beta_n \to 0$ as $n \to \infty$. Also, the conditions $\sum_{n\geq 0}(b_n + c_n)^2 < \infty$, $\sum_{n\geq 0}(b_n + c_n)(k_n - 1) < \infty$, $\sum_{n\geq 0} c_n < \infty$ in [13, Theorem 3.3] are replaced by $b_n \to 0$ as $n \to \infty$ and $c_n = o(b_n)$ of Corollary 2.2.

Remark 2.5 A mapping T is said to be *weak uniformly Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{2.13}$$

for all $n \geq 1$, $x \in D$ and $y \in F(T)$. Then, using the same methods, we can also prove that Theorem 2.1 holds for the more general class of weak uniformly Lipschitzian asymptotically pseudocontractive mappings. In practical application, it can be seen from the following example.

Example 2.6 Let $E = \Re$ be the set of real numbers with the usual norm and $D = [0, +\infty)$. Define a mapping $T : D \to D$ by

$$Tx = \frac{x^3}{1 + x^2}$$

for all $x \in D$. Then T has a fixed point $q = 0 \in D$ and T is a strictly monotone increasing mapping. Thus, $Tx \leq x$ for any $x \in D$, which implies that $T^n x \leq T^{n-1} x \leq \cdots \leq Tx$. Define a function $\Phi : [0, +\infty) \to [0, +\infty)$ by $\Phi(t) = \frac{t^2}{1+t^2}$. Then Φ is a strictly increasing continuous function with $\Phi(0) = 0$. For all $x \in D$ and $q \in F(T)$, if $k_n = 1$ and $L = 1$, then we obtain

$$\begin{aligned}
\langle T^n x - T^n q, j(x - q)\rangle &= \langle T^n x - 0, j(x - 0)\rangle \\
&= \langle T^n x, x\rangle \\
&\leq \langle Tx, x\rangle \\
&= \left\langle \frac{x^3}{1 + x^2}, x\right\rangle \\
&= \frac{x^4}{1 + x^2} \\
&= k_n|x - q|^2 - \frac{|x - q|^2}{1 + |x - q|^2} \\
&= k_n|x - q|^2 - \Phi(|x - q|)
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
|T^n x - T^n q| &= |T^n x - 0| \\
&\leq |Tx - 0| \\
&= \left|\frac{x^3}{1 + x^2} - 0\right| \\
&\leq |x - 0| \\
&\leq L|x - q|.
\end{aligned} \tag{2.15}$$

Therefore, T is weakly uniform Lipschitzian and satisfies (2.1) of Theorem 2.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this paper and read and approved the final manuscript.

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