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Hyperholomorphic functions and hyper-conjugate harmonic functions of octonion variables

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Abstract

We represent hyperholomorphic functions on octonionic function theory and octonionic differential operators. We research the properties of hyperholomorphic functions, hyper-conjugate harmonic functions and the integral calculus of hyperholomorphic functions by octonion forms.

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1 Introduction

The octonions in Clifford algebra are a normed division algebra with eight dimensions over the real numbers larger than the quaternions. The octonions are non-commutative and non-associative but satisfy a weaker form of associativity. The octonions were discovered in 1843 by John T. Graves and constructed in 1845 by A. Cayley. They are referred to as Cayley numbers or the Cayley algebra. The octonions have been applied in fields such as string theory, special theory of relativity and quantum theory. Dentoni and Sce [1] gave a definition of octonionic regular functions and several properties of octonionic regular functions in 1973.

In 2004 and 2006, Kajiwara, Li and Shon [2, 3] obtained some results for the regeneration in complex, quaternion and Clifford analysis, and for the inhomogeneous Cauchy-Riemann system of quaternion and Clifford analysis in ellipsoid.

In 2011, Koriyama and Nôno [4] gave three regularities (HK-holomorphy, HF-holomorphy, H_2-holomorphy) of octonionic functions based on holomorphic mappings in a domain in $\mathbf{C}^4$. Naser [5] and Nôno [6, 7] gave some properties of quaternionic hyperholomorphic functions. For any complex harmonic function f_1 in a domain of holomorphy D in $\mathbf{C}^2$, we can find a function f_2 such that $f_1 + f_2 j$ will be a function hyperholomorphic in D and the Cauchy theorem of hyperholomorphic functions in quaternion analysis. The aim of this paper is to define hyperholomorphic functions with octonion variables in $\mathbf{C}^4$ and investigate the properties of the hyperholomorphic functions of octonion variables. We give the condition of harmonicity in $\mathbf{C}^4$. Then for any complex-valued functions $g_1(z)$ and $g_2(z)$ satisfying the condition of harmonicity in a pseudoconvex domain Ω in $\mathbf{C}^4$, we can find hyper-conjugate harmonic functions $g_3(z)$ and $g_4(z)$, respectively, on Ω such that

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$g(z) = g_1(z) + g_2(z)e_2 + g_3(z)e_4 + g_4(z)e_6$ is a hyperholomorphic function on Ω. Also, we investigate the Cauchy theorem of hyperholomorphic functions in octonion analysis.

2 Preliminaries

The field $\mathcal{O} \cong \mathbf{C}^4$ of octonions

$$z = x_0 + \sum_{i=0}^{7} e_i x_i, \quad x_i (i = 0, \dots, 7) \in \mathbf{R} \tag{1}$$

is an eight-dimensional non-commutative and non-associative $\mathbf{R}$-field generated by eight base elements e_0, e_1, e_2, e_3, e_4, e_5, e_6 and e_7 with the following non-commutative multiplication rules:

$$\begin{aligned}
&e_i^2 = -1, \qquad e_i e_j = -e_j e_i, \qquad e_i e_j e_k = e_i(e_j e_k) \quad (i \neq j \neq k, i \neq 0, j \neq 0, k \neq 0),\\
&e_1 e_2 = e_3, \qquad e_3 e_5 = e_6, \qquad e_6 e_7 = e_1, \qquad e_1 e_4 = e_5,\\
&e_5 e_7 = e_2, \qquad e_2 e_6 = e_4, \qquad e_4 e_7 = e_3.
\end{aligned}$$

The element e_0 is the identities of $\mathcal{O}$ and e_1 identifies the imaginary unit $\sqrt{-1}$ in the $\mathbf{C}$-field of complex numbers. An octonion z given by (1) is regarded as $z = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6 \in \mathcal{O}$, where $z_1 := x_0 + e_1 x_1$, $z_2 := x_2 + e_1 x_3$, $z_3 := x_4 + e_1 x_5$ and $z_4 := x_6 + e_1 x_7$ are complex numbers in $\mathbf{C}$. Thus, we identify $\mathcal{O}$ with $\mathbf{C}^4$.

For the equation $z^3 + 8 = 0$ in the complex plane $\mathbf{C}$, the three solutions are -2, $1 + \sqrt{3}i$, $1 - \sqrt{3}i$ in $\mathbf{C}$.

In the octonion $\mathcal{O}$, the equation has solutions whose forms are as follows:

$$z = a + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7 \quad (a, b, c, d, e, f, g, h \in \mathbf{R}).$$

Then the equation satisfies $z^3 = (a^3 - 3ab^2 - 3ac^2 - 3ad^2 - 3ae^2 - 3af^2 - 3ag^2 - 3ah^2) + (3a^2b - b^3 - bc^2 - bd^2 - be^2 - bf^2 - bg^2 - bh^2)e_1 + (3a^2c - b^2c - c^3 - cd^2 - ce^2 - cf^2 - cg^2 - ch^2)e_2 + (3a^2d - b^2d - c^2d - d^3 - de^2 - df^2 - dg^2 - dh^2)e_3 + (3a^2e - b^2e - c^2e - d^2e - e^3 - ef^2 - eg^2 - eh^2)e_4 + (3a^2f - b^2f - c^2f - d^2f - e^2f - f^3 - fg^2 - fh^2)e_5 + (3a^2g - b^2g - c^2g - d^2g - e^2g - f^2g - g^3 - gh^2)e_6 + (3a^2h - b^2h - c^2h - d^2h - e^2h - f^2h - g^2h - h^3)e_7$. That is, $a^3 - 3ab^2 - 3ac^2 - 3ad^2 - 3ae^2 - 3af^2 - 3ag^2 - 3ah^2 = -8$, $3a^2b - b^3 - bc^2 - bd^2 - be^2 - bf^2 - bg^2 - bh^2 = 0$, $3a^2c - b^2c - c^3 - cd^2 - ce^2 - cf^2 - cg^2 - ch^2 = 0$, $3a^2d - b^2d - c^2d - d^3 - de^2 - df^2 - dg^2 - dh^2 = 0$, $3a^2e - b^2e - c^2e - d^2e - e^3 - ef^2 - eg^2 - eh^2 = 0$, $3a^2f - b^2f - c^2f - d^2f - e^2f - f^3 - fg^2 - fh^2 = 0$, $3a^2g - b^2g - c^2g - d^2g - e^2g - f^2g - g^3 - gh^2 = 0$, $3a^2h - b^2h - c^2h - d^2h - e^2h - f^2h - g^2h - h^3 = 0$. This means that the equation has infinitely many solutions

$$z = 1 + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7$$

with $b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 = 3$ in $\mathcal{O}$.

For two octonions $z = \sum_{i=0}^{7} e_i x_i$ and $w = \sum_{i=0}^{7} e_i y_i$, the inner product (z, w) is defined as follows:

$$(z, w) := \sum_{i=0}^{7} x_i y_i.$$

Also, the octonionic conjugation z^*, the absolute value $|z|$ of z and an inverse z^{-1} of z in $\mathcal{O}$ are defined, respectively, by

$$z^* = x_0 - \sum_{i=1}^{7} e_i x_i, \qquad |z| = \sqrt{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2}, \qquad z^{-1} = \frac{z^*}{|z|^2} \quad (z \neq 0).$$

Thus, the octonion $z \in \mathcal{O}$ and the octonion conjugation $z^* \in \mathcal{O}$ have the following forms:

$$z = x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7 = z_1 + z_2e_2 + z_3e_4 + z_4e_6$$

and

$$z^* = x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7 = \overline{z_1} - z_2e_2 - z_3e_4 - z_4e_6,$$

where $z_1 = x_0 + e_1x_1$, $z_2 = x_2 + e_1x_3$, $z_3 = x_4 + e_1x_5$ and $z_4 = x_6 + e_1x_7$.

We use the following differential operators:

$$D_\alpha = \frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial \overline{z_2}}, \qquad D_\alpha^* = \frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial \overline{z_2}},$$
$$D_\beta = \frac{\partial}{\partial z_3} - e_2\frac{\partial}{\partial \overline{z_4}}, \qquad D_\beta^* = \frac{\partial}{\partial \overline{z_3}} + e_2\frac{\partial}{\partial \overline{z_4}},$$

where $\partial/\partial z_j$, $\partial/\partial\overline{z_j}$ $(j = 1,2,3,4)$ are usual differential operators used in complex analysis. And we use the following octonionic differential operators:

$$D := D_\alpha - e_4D_\beta^*, \qquad D^\star := D_\alpha^* + e_4D_\beta^*.$$

The operator

$$\begin{aligned} DD^\star &= \frac{\partial^2}{\partial z_1\partial\overline{z_1}} + \frac{\partial^2}{\partial z_2\partial\overline{z_2}} + \frac{\partial^2}{\partial z_3\partial\overline{z_3}} + \frac{\partial^2}{\partial z_4\partial\overline{z_4}} \\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} + \frac{\partial^2}{\partial x_5^2} + \frac{\partial^2}{\partial x_6^2} + \frac{\partial^2}{\partial x_7^2}\right) \end{aligned}$$

is the usual complex Laplacian Δ.

3 Some properties of hyperholomorphic functions on $\mathcal{O}$

Let Ω be an open set in $\mathbf{C}^4$. The function $g(z)$ is defined by the following form in Ω with value in $\mathcal{O}$:

$$g(z) = g_1(z) + g_2(z)e_2 + g_3(z)e_4 + g_4(z)e_6 = \{g_1(z) + g_3(z)e_4\} + \{g_2(z) + g_4(z)e_4\}e_2,$$

where $z = (z_1, z_2, z_3, z_4)$ and $g_1(z)$, $g_2(z)$, $g_3(z)$ and $g_4(z)$ are complex-valued functions.

Definition 3.1 Let Ω be an open set in $\mathbf{C}^4$. A function $g(z)$ is said to be L(R)-hyperholomorphic in Ω if the following two conditions are satisfied:

(a) $g_k(z)$ $(k = 1,2,3,4)$ are continuously differential functions in Ω, and

(b)

$$D^{\star}g = 0 \qquad \bigl(gD^{\star} = 0\bigr) \quad \text{in } \Omega. \tag{2}$$

When we deal with an L-hyperholomorphic function $g(z)$ in $\Omega \subset \mathbf{C}^4$, for simplicity, we often say that $g(z)$ is a hyperholomorphic function in $\Omega \subset \mathbf{C}^4$.

Equation (2) is applied to $g(z)$ as follows:

$$\begin{aligned} D^{\star}g &= \bigl(D_{\alpha}^{*} + e_4 D_{\beta}^{*}\bigr)\bigl(g_1(z) + g_2(z)e_2 + g_3(z)e_4 + g_4(z)e_6\bigr) \\ &= \bigl(D_{\alpha}^{*}g_1 - D_{\beta}\overline{g_3}\bigr) + \bigl(D_{\alpha}^{*}g_2 - D_{\beta}\overline{g_4}\bigr)e_2 \\ &\quad + \bigl(D_{\alpha}^{*}g_3 + D_{\beta}\overline{g_1}\bigr)e_4 + \bigl(D_{\alpha}^{*}g_4 + D_{\beta}\overline{g_2}\bigr)e_6. \end{aligned}$$

If the following equations:

$$D_{\alpha}^{*}g_1 = D_{\beta}\overline{g_3}, \qquad D_{\alpha}^{*}g_2 = D_{\beta}\overline{g_4}, \qquad D_{\alpha}^{*}g_3 = -D_{\beta}\overline{g_1}, \qquad D_{\alpha}^{*}g_4 = -D_{\beta}\overline{g_2} \tag{3}$$

are satisfied, the function $g(z)$ is a hyperholomorphic function in Ω. The equations in (3) are the corresponding o-Cauchy-Riemann equations in $\mathbf{C}^4$.

Remark 3.2 We redefine equations (3) as follows:

$$\begin{aligned} &\frac{\partial g_1}{\partial \overline{z_1}} = \frac{\partial \overline{g_3}}{\partial z_3}, \qquad \frac{\partial g_1}{\partial \overline{z_2}} = -\frac{\partial \overline{g_3}}{\partial \overline{z_4}}, \qquad \frac{\partial g_3}{\partial \overline{z_1}} = -\frac{\partial \overline{g_1}}{\partial z_3}, \qquad \frac{\partial g_3}{\partial \overline{z_2}} = \frac{\partial \overline{g_1}}{\partial \overline{z_4}}, \\ &\frac{\partial g_2}{\partial \overline{z_1}} = \frac{\partial \overline{g_4}}{\partial z_3}, \qquad \frac{\partial g_2}{\partial \overline{z_2}} = -\frac{\partial \overline{g_4}}{\partial \overline{z_4}}, \qquad \frac{\partial g_4}{\partial \overline{z_1}} = -\frac{\partial \overline{g_2}}{\partial z_3}, \qquad \frac{\partial g_4}{\partial \overline{z_2}} = \frac{\partial \overline{g_2}}{\partial \overline{z_4}}. \end{aligned} \tag{4}$$

We call that equations (4) are the condition of harmonicity.

Remark 3.3 We redefine equations (4) in $\mathbf{R}^8$ as follows:

$$\begin{aligned} &\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_4}{\partial x_4} + \frac{\partial u_5}{\partial x_5} = 0, \qquad \frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_5}{\partial x_4} + \frac{\partial u_4}{\partial x_5} = 0, \\ &\frac{\partial u_0}{\partial x_2} - \frac{\partial u_1}{\partial x_3} + \frac{\partial u_4}{\partial x_6} + \frac{\partial u_5}{\partial x_7} = 0, \qquad \frac{\partial u_1}{\partial x_2} + \frac{\partial u_0}{\partial x_3} - \frac{\partial u_5}{\partial x_6} + \frac{\partial u_4}{\partial x_7} = 0, \\ &\frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} - \frac{\partial u_6}{\partial x_4} + \frac{\partial u_7}{\partial x_5} = 0, \qquad \frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_7}{\partial x_4} + \frac{\partial u_6}{\partial x_5} = 0, \\ &\frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} + \frac{\partial u_6}{\partial x_6} + \frac{\partial u_7}{\partial x_7} = 0, \qquad \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} - \frac{\partial u_7}{\partial x_6} + \frac{\partial u_6}{\partial x_7} = 0, \\ &\frac{\partial u_4}{\partial x_0} - \frac{\partial u_5}{\partial x_1} + \frac{\partial u_0}{\partial x_4} - \frac{\partial u_1}{\partial x_5} = 0, \qquad \frac{\partial u_5}{\partial x_0} + \frac{\partial u_4}{\partial x_1} - \frac{\partial u_1}{\partial x_4} - \frac{\partial u_0}{\partial x_5} = 0, \\ &\frac{\partial u_4}{\partial x_2} - \frac{\partial u_5}{\partial x_3} - \frac{\partial u_0}{\partial x_6} - \frac{\partial u_1}{\partial x_7} = 0, \qquad \frac{\partial u_5}{\partial x_2} + \frac{\partial u_4}{\partial x_3} + \frac{\partial u_1}{\partial x_6} - \frac{\partial u_0}{\partial x_7} = 0, \\ &\frac{\partial u_6}{\partial x_0} - \frac{\partial u_7}{\partial x_1} + \frac{\partial u_2}{\partial x_4} - \frac{\partial u_3}{\partial x_5} = 0, \qquad \frac{\partial u_7}{\partial x_0} + \frac{\partial u_6}{\partial x_1} - \frac{\partial u_3}{\partial x_4} - \frac{\partial u_2}{\partial x_5} = 0, \\ &\frac{\partial u_6}{\partial x_2} - \frac{\partial u_7}{\partial x_3} - \frac{\partial u_2}{\partial x_6} - \frac{\partial u_3}{\partial x_7} = 0, \qquad \frac{\partial u_7}{\partial x_2} + \frac{\partial u_6}{\partial x_3} + \frac{\partial u_3}{\partial x_6} - \frac{\partial u_2}{\partial x_7} = 0, \end{aligned}$$

where $g_1 = u_0 + e_1u_1$, $g_2 = u_2 + e_1u_3$, $g_3 = u_4 + e_1u_5$ and $g_4 = u_6 + e_1u_7$ for real-valued functions u_i $(i = 0, \dots, 7)$.

Lemma 3.4

(i) *If the function $g(z)$ is hyperholomorphic in an open set Ω in $\mathbf{C}^4$, then the functions $g_1(z)$, $g_2(z)$, $g_3(z)$ and $g_4(z)$ are of class $\mathcal{C}^\infty$ in Ω.*

(ii) *If the function $g(z)$ satisfies the condition of harmonicity in an open set Ω in $\mathbf{C}^4$, then the functions $g_1(z)$, $g_2(z)$, $g_3(z)$ and $g_4(z)$ are harmonic in Ω.*

Proof We have

$$\begin{aligned} DD^\star g_1 &= \frac{\partial^2 g_1}{\partial z_1 \partial \overline{z_1}} + \frac{\partial^2 g_1}{\partial z_2 \partial \overline{z_2}} + \frac{\partial^2 g_1}{\partial z_3 \partial \overline{z_3}} + \frac{\partial^2 g_1}{\partial z_4 \partial \overline{z_4}} \\ &= \frac{\partial}{\partial z_1}\left(\frac{\partial \overline{g_3}}{\partial z_3}\right) + \frac{\partial}{\partial z_2}\left(-\frac{\partial \overline{g_3}}{\partial \overline{z_4}}\right) + \frac{\partial}{\partial z_3}\left(-\frac{\partial \overline{g_3}}{\partial z_1}\right) + \frac{\partial}{\partial \overline{z_4}}\left(\frac{\partial \overline{g_3}}{\partial z_2}\right) \\ &= 0, \end{aligned}$$

and the functions g_2, g_3 and g_4 are proved by a similar method as in the proof of the case of g_1. And, by (i), $g_j(z)$ $(j = 1, 2, 3, 4)$ are of class $\mathcal{C}^\infty$ functions in Ω. □

Definition 3.5 Let $\Omega \subset \mathbf{C}^n$ be an open set with a $\mathcal{C}^2$ boundary. Let $\Omega = \{z; \rho(z) < 0\}$, where ρ is in $\mathcal{C}^2$ in a neighborhood of $\overline{\Omega}$ and grad $\rho \neq 0$ on $b\Omega$. Then Ω is pseudoconvex if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(z) w_j \overline{w_k} \geq 0,$$

for all $z \in b\Omega$ and $w \in \mathbf{C}^n$ satisfying $\sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z) w_j = 0$.

Consider an automorphism γ:

$$(z_1, z_2, z_3, s_4) = \gamma(z_1, z_2, z_3, z_4) := (z_1, z_2, z_3, \overline{z_4})$$

of $\mathbf{C}^4$. A domain Ω in $\mathbf{C}^4 \cong \mathcal{O}$ is said to be pseudoconvex with respect to the complex variables $z_1, z_2, z_3, \overline{z_4}$, if $\gamma(\Omega)$ is a pseudoconvex domain of the space $\mathbf{C}^4$ of four complex variables z_1, z_2, z_3, s_4 in the sense of complex analysis.

Theorem 3.6 *Let Ω be a domain in $\mathbf{C}^4 \cong \mathcal{O}$, which is a pseudoconvex domain with respect to the complex variables $z_1, z_2, z_3, \overline{z_4}$ and let $g_1(z)$ and $g_2(z)$ be complex-valued functions of class $\mathcal{C}^2$ on Ω satisfying the condition of harmonicity* (4). *Then there exist hyper-conjugate harmonic functions $g_3(z)$ and $g_4(z)$, respectively, of class $\mathcal{C}^2$ on Ω such that $g(z)$ is a hyperholomorphic function on Ω.*

Proof We consider the 1-forms and the differential operator on $\gamma(\Omega)$:

$$\begin{aligned} \psi_1 &:= -\frac{\partial \overline{g_1}}{\partial z_3} d\overline{z_1} + \frac{\partial \overline{g_1}}{\partial \overline{z_4}} d\overline{z_2} + \frac{\partial \overline{g_1}}{\partial z_1} d\overline{z_3} - \frac{\partial \overline{g_1}}{\partial z_2} dz_4, \\ \psi_2 &:= -\frac{\partial \overline{g_2}}{\partial z_3} d\overline{z_1} + \frac{\partial \overline{g_2}}{\partial \overline{z_4}} d\overline{z_2} + \frac{\partial \overline{g_2}}{\partial z_1} d\overline{z_3} - \frac{\partial \overline{g_2}}{\partial z_2} dz_4, \end{aligned}$$

and

$$\delta = \frac{\partial}{\partial \overline{z_1}} d\overline{z_1} + \frac{\partial}{\partial \overline{z_2}} d\overline{z_2} + \frac{\partial}{\partial \overline{z_3}} d\overline{z_3} + \frac{\partial}{\partial z_4} dz_4.$$

We operate the operator δ from the left-hand side of the 1-forms ψ_1 and ψ_2 on $\gamma(\Omega)$:

$$\begin{aligned} \delta\psi_1 = {} & \left(\frac{\partial^2 \overline{g_1}}{\partial \overline{z_1}\, \partial \overline{z_4}} + \frac{\partial^2 \overline{g_1}}{\partial \overline{z_2}\, \partial z_3} \right) d\overline{z_1} \wedge d\overline{z_2} + \left(\frac{\partial^2 \overline{g_1}}{\partial \overline{z_1}\, \partial z_1} + \frac{\partial^2 \overline{g_1}}{\partial \overline{z_3}\, \partial z_3} \right) d\overline{z_1} \wedge d\overline{z_3} \\ & + \left(-\frac{\partial^2 \overline{g_1}}{\partial \overline{z_1}\, \partial z_2} + \frac{\partial^2 \overline{g_1}}{\partial z_4\, \partial z_3} \right) d\overline{z_1} \wedge dz_4 + \left(\frac{\partial^2 \overline{g_1}}{\partial \overline{z_2}\, \partial z_1} - \frac{\partial^2 \overline{g_1}}{\partial \overline{z_3}\, \partial \overline{z_4}} \right) d\overline{z_2} \wedge d\overline{z_3} \\ & + \left(-\frac{\partial^2 \overline{g_1}}{\partial \overline{z_2}\, \partial z_2} - \frac{\partial^2 \overline{g_1}}{\partial z_4\, \partial \overline{z_4}} \right) d\overline{z_2} \wedge dz_4 + \left(-\frac{\partial^2 \overline{g_1}}{\partial \overline{z_3}\, \partial z_2} - \frac{\partial^2 \overline{g_1}}{\partial z_4\, \partial z_1} \right) d\overline{z_3} \wedge dz_4, \end{aligned}$$

and

$$\begin{aligned} \delta\psi_2 = {} & \left(\frac{\partial^2 \overline{g_2}}{\partial \overline{z_1}\, \partial \overline{z_4}} + \frac{\partial^2 \overline{g_2}}{\partial \overline{z_2}\, \partial z_3} \right) d\overline{z_1} \wedge d\overline{z_2} + \left(\frac{\partial^2 \overline{g_2}}{\partial \overline{z_1}\, \partial z_1} + \frac{\partial^2 \overline{g_2}}{\partial \overline{z_3}\, \partial z_3} \right) d\overline{z_1} \wedge d\overline{z_3} \\ & + \left(-\frac{\partial^2 \overline{g_2}}{\partial \overline{z_1}\, \partial z_2} + \frac{\partial^2 \overline{g_2}}{\partial z_4\, \partial z_3} \right) d\overline{z_1} \wedge dz_4 + \left(\frac{\partial^2 \overline{g_2}}{\partial \overline{z_2}\, \partial z_1} - \frac{\partial^2 \overline{g_2}}{\partial \overline{z_3}\, \partial \overline{z_4}} \right) d\overline{z_2} \wedge d\overline{z_3} \\ & + \left(-\frac{\partial^2 \overline{g_2}}{\partial \overline{z_2}\, \partial z_2} - \frac{\partial^2 \overline{g_2}}{\partial z_4\, \partial \overline{z_4}} \right) d\overline{z_2} \wedge dz_4 + \left(-\frac{\partial^2 \overline{g_2}}{\partial \overline{z_3}\, \partial z_2} - \frac{\partial^2 \overline{g_2}}{\partial z_4\, \partial z_1} \right) d\overline{z_3} \wedge dz_4. \end{aligned}$$

By the condition of harmonicity (4), all coefficients vanish. From Hörmander [8], the δ-closed forms ψ_1 and ψ_2 of z_1, z_2, z_3, z_4 are δ-exact forms on $\gamma(\Omega)$. Since Ω is a pseudoconvex domain, there exist hyper-conjugate harmonic functions $g_3(z)$ and $g_4(z)$ of class $\mathcal{C}^\infty$ on Ω with $\overline{\partial}$-closed forms $\gamma^{-1}\psi_1 = \overline{\partial} g_3(z)$ and $\gamma^{-1}\psi_2 = \overline{\partial} g_4(z)$ on Ω of z_1, z_2, z_3, s_4 are $\overline{\partial}$-exact $(0,1)$-forms on Ω such that $g(z)$ is a hyperholomorphic function on Ω (see Krantz [9]). □

Theorem 3.7 *Let Ω be a domain in $\mathbf{C}^4 \cong \mathcal{O}$, which is a pseudoconvex domain with respect to the complex variables $z_1, z_2, z_3, \overline{z_4}$ and let $J_1(z) = g_1(z) + g_3(z)e_4$ be a complex-valued function of class $\mathcal{C}^2$ on Ω satisfying the condition of harmonicity* (4). *Then there exists a hyper-conjugate harmonic function $J_2(z) = g_2(z) + g_4(z)e_4$ of class $\mathcal{C}^2$ on Ω such that $g(z) = J_1(z) + J_2(z)e_2$ is a hyperholomorphic function on Ω.*

Proof We consider the 1-form and the differential operator on $\gamma(\Omega)$:

$$\begin{aligned} \psi := {} & \left(\frac{\partial \overline{g_4}}{\partial z_3} - \frac{\partial \overline{g_2}}{\partial z_3} e_4 \right) d\overline{z_1} + \left(-\frac{\partial \overline{g_4}}{\partial \overline{z_4}} + \frac{\partial \overline{g_2}}{\partial \overline{z_4}} e_4 \right) d\overline{z_2} \\ & + \left(-\frac{\partial \overline{g_4}}{\partial z_1} + \frac{\partial \overline{g_2}}{\partial z_1} e_4 \right) d\overline{z_3} + \left(\frac{\partial \overline{g_4}}{\partial z_2} - \frac{\partial \overline{g_2}}{\partial z_2} e_4 \right) dz_4 \end{aligned}$$

and

$$\delta = \frac{\partial}{\partial \overline{z_1}} d\overline{z_1} + \frac{\partial}{\partial \overline{z_2}} d\overline{z_2} + \frac{\partial}{\partial \overline{z_3}} d\overline{z_3} + \frac{\partial}{\partial z_4} dz_4.$$

We operate the operator δ from the left-hand side of the 1-form ψ on $\gamma(\Omega)$:

$$\begin{aligned}
\delta\psi = {} & \left\{\left(-\frac{\partial^2\overline{g_4}}{\partial\overline{z_1}\,\partial\overline{z_4}} - \frac{\partial^2\overline{g_4}}{\partial\overline{z_2}\,\partial z_3}\right) + \left(\frac{\partial^2\overline{g_2}}{\partial\overline{z_1}\,\partial\overline{z_4}} + \frac{\partial^2\overline{g_2}}{\partial\overline{z_2}\,\partial z_3}\right)e_4\right\} d\overline{z_1}\wedge d\overline{z_2} \\
& + \left\{\left(-\frac{\partial^2\overline{g_4}}{\partial\overline{z_1}\,\partial z_1} - \frac{\partial^2\overline{g_4}}{\partial\overline{z_3}\,\partial z_3}\right) + \left(\frac{\partial^2\overline{g_2}}{\partial\overline{z_1}\,\partial z_1} + \frac{\partial^2\overline{g_2}}{\partial\overline{z_3}\,\partial z_3}\right)e_4\right\} d\overline{z_1}\wedge d\overline{z_3} \\
& + \left\{\left(\frac{\partial^2\overline{g_4}}{\partial\overline{z_1}\,\partial z_2} - \frac{\partial^2\overline{g_4}}{\partial z_4\,\partial z_3}\right) + \left(-\frac{\partial^2\overline{g_2}}{\partial\overline{z_1}\,\partial z_2} + \frac{\partial^2\overline{g_2}}{\partial z_4\,\partial z_3}\right)e_4\right\} d\overline{z_1}\wedge dz_4 \\
& + \left\{\left(-\frac{\partial^2\overline{g_4}}{\partial\overline{z_2}\,\partial z_1} + \frac{\partial^2\overline{g_4}}{\partial\overline{z_3}\,\partial\overline{z_4}}\right) + \left(\frac{\partial^2\overline{g_2}}{\partial\overline{z_2}\,\partial z_1} - \frac{\partial^2\overline{g_2}}{\partial\overline{z_3}\,\partial\overline{z_4}}\right)e_4\right\} d\overline{z_2}\wedge d\overline{z_3} \\
& + \left\{\left(\frac{\partial^2\overline{g_4}}{\partial\overline{z_2}\,\partial z_2} + \frac{\partial^2\overline{g_4}}{\partial z_4\,\partial\overline{z_4}}\right) + \left(-\frac{\partial^2\overline{g_2}}{\partial\overline{z_2}\,\partial z_2} - \frac{\partial^2\overline{g_2}}{\partial z_4\,\partial\overline{z_4}}\right)e_4\right\} d\overline{z_2}\wedge dz_4 \\
& + \left\{\left(\frac{\partial^2\overline{g_4}}{\partial\overline{z_3}\,\partial z_2} + \frac{\partial^2\overline{g_4}}{\partial z_4\,\partial z_1}\right) + \left(-\frac{\partial^2\overline{g_2}}{\partial\overline{z_3}\,\partial z_2} - \frac{\partial^2\overline{g_2}}{\partial z_4\,\partial z_1}\right)e_4\right\} d\overline{z_3}\wedge dz_4.
\end{aligned}$$

By the same method as the proof of Theorem 3.6, our result is proved. □

Theorem 3.8 *Let $g(z)$ be a hyperholomorphic function in a domain G of $\mathcal{O}$ and let*

$$\begin{aligned}
\tau = {} & dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_2}\wedge d\overline{z_3}\wedge d\overline{z_4} \\
& + dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_3}\wedge d\overline{z_4} \\
& - dz_1\wedge dz_2\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_2}\wedge d\overline{z_3}\wedge d\overline{z_4}e_4 \\
& - dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_2}\wedge d\overline{z_3}e_4.
\end{aligned}$$

Then for any domain $\Omega\subset G$ with smooth boundary $b\Omega$,

$$\int_{b\Omega}\tau g = 0, \tag{5}$$

where τg is the octonion product of the form τ on the function $g(z)$.

Proof Let

$$\begin{aligned}
\tau_{(1)} &= dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_2}\wedge d\overline{z_3}\wedge d\overline{z_4}, \\
\tau_{(2)} &= dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_3}\wedge d\overline{z_4}, \\
\tau_{(3)} &= dz_1\wedge dz_2\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_2}\wedge d\overline{z_3}\wedge d\overline{z_4}, \\
\tau_{(4)} &= dz_1\wedge dz_2\wedge dz_3\wedge dz_4\wedge d\overline{z_1}\wedge d\overline{z_2}\wedge d\overline{z_3}.
\end{aligned}$$

By the rule of octonion multiplications,

$$\begin{aligned}
\tau g &= (\tau_{(1)} + \tau_{(2)} - \tau_{(3)}e_4 - \tau_{(4)}e_4)(g_1 + g_2e_2 + g_3e_4 + g_4e_6) \\
&= g_1\tau_{(1)} + g_1\tau_{(2)} - \overline{g_1}\tau_{(3)}e_4 - \overline{g_1}\tau_{(4)}e_4 \\
&\quad + g_2\tau_{(1)}e_2 + g_2\tau_{(2)}e_2 - \overline{g_2}\tau_{(3)}e_6 - \overline{g_2}\tau_{(4)}e_6
\end{aligned}$$

$$+ g_3\tau_{(1)}e_4 + g_3\tau_{(2)}e_4 + \overline{g_3}\tau_{(3)} + \overline{g_3}\tau_{(4)}$$

$$+ g_4\tau_{(1)}e_6 + g_4\tau_{(2)}e_6 + \overline{g_4}\tau_{(3)}e_2 + \overline{g_4}\tau_{(4)}e_2.$$

Hence,

$$d(\tau g) = \left(-\frac{\partial g_1}{\partial \overline{z_1}} - \frac{\partial g_1}{\partial \overline{z_2}} + \frac{\partial \overline{g_3}}{\partial z_3} - \frac{\partial \overline{g_3}}{\partial \overline{z_4}}\right) dV + \left(-\frac{\partial g_2}{\partial \overline{z_1}} - \frac{\partial g_2}{\partial \overline{z_2}} + \frac{\partial \overline{g_4}}{\partial z_3} - \frac{\partial \overline{g_4}}{\partial \overline{z_4}}\right) dVe_2$$
$$+ \left(-\frac{\partial \overline{g_1}}{\partial z_3} + \frac{\partial \overline{g_1}}{\partial \overline{z_4}} - \frac{\partial g_3}{\partial \overline{z_1}} - \frac{\partial g_3}{\partial \overline{z_2}}\right) dVe_4 + \left(-\frac{\partial \overline{g_2}}{\partial z_3} + \frac{\partial \overline{g_2}}{\partial \overline{z_4}} - \frac{\partial g_4}{\partial \overline{z_1}} - \frac{\partial g_4}{\partial \overline{z_2}}\right) dVe_6,$$

where $dV = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \wedge d\overline{z_1} \wedge d\overline{z_2} \wedge d\overline{z_3} \wedge d\overline{z_4}$, and by the condition of harmonicity (4), $d(\tau g) = 0$. By Stoke's theorem, we have

$$\int_{b\Omega} \tau g = \int_{\Omega} d(\tau g) = 0. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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