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A multiple Hilbert-type integral inequality with a non-homogeneous kernel

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Abstract

By using the way of weight functions and the technic of real analysis, a multiple Hilbert-type integral inequality with a non-homogeneous kernel is given. The operator expression with the norm, the reverses and some examples with the particular kernels are also considered.

MSC: 47A07; 26D15

Keywords: multiple Hilbert-type integral inequality; kernel; weight function; norm; operator

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(\geq 0) \in L^p(0,\infty)$, $g(\geq 0) \in L^q(0,\infty)$, $\|f\|_p, \|g\|_q > 0$, then we have the following equivalent inequalities (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{1}$$

$$\left\{\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx\right)^p dy\right\}^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/p)}\|f\|_p, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. (1) is the well-known Hardy-Hilbert integral inequality. Define the Hardy-Hilbert integral operator $T : L^p(0,\infty) \to L^p(0,\infty)$ as follows: for $f \in L^p(0,\infty)$, $Tf(y) := \int_0^\infty \frac{1}{x+y} f(x)\,dx$ $(y \in (0,\infty))$.

Then in view of (2), it follows $\|Tf\|_p < \frac{\pi}{\sin(\pi/p)}\|f\|_p$ and $\|T\| \leq \frac{\pi}{\sin(\pi/p)}$. Since the constant is the best possible, we find $\|T\| = \frac{\pi}{\sin(\pi/p)}$.

Inequalities (1) and (2) and the operator are important in analysis and its applications (*cf.* [2, 3]). In 2002, [4] considered the property of the Hardy-Hilbert integral operator and gave an improvement of (1) (for $p = q = 2$). In 2004, by introducing another pair of conjugate exponents (r,s) $(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$ and an independent parameter $\lambda > 0$, [5] gave the best extensions of (1) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda}\,dx\,dy < \frac{\pi}{\lambda \sin(\pi/r)}\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{3}$$

where $\phi(x) = x^{p(1-\frac{\lambda}{r})-1}$, $\psi(x) = x^{q(1-\frac{\lambda}{s})-1}$, $\|f\|_{p,\phi} = \{\int_0^\infty \phi(x)f^p(x)\,dx\}^{\frac{1}{p}} > 0$, $\|g\|_{q,\psi} > 0$. In 2007, [6] gave the following inequality with the best constant $B(\frac{\lambda}{2},\frac{\lambda}{2})$ $(\lambda > 0$; $B(u,v)$ is the beta function):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1+xy)^\lambda}\,dx\,dy < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left\{\int_0^\infty x^{1-\lambda}f^2(x)\,dx \int_0^\infty x^{1-\lambda}g^2(x)\,dx\right\}^{\frac{1}{2}}. \tag{4}$$

In 2009, [7] gave an extension of (4) in $\mathbf{R}^2$ with the kernel $\frac{1}{|1+xy|^\lambda}$ $(0<\lambda<1)$; [8] gave another extension of (4) to the general kernel $k_\lambda(1,xy)$ $(\lambda>0)$ with a pair of conjugate exponents (p,q) and obtained the following multiple Hilbert-type integral inequality. Suppose that $n \in \mathbf{N}\backslash\{1\}, p_i > 1, \sum_{i=1}^n \frac{1}{p_i} = 1, \lambda > 0, k_\lambda(x_1,\ldots,x_n) \geq 0$ is a measurable function of $-\lambda$-degree in $\mathbf{R}_+^n$, and for any $(r_1,\ldots,r_n)$ $(r_i>1)$, satisfies $\sum_{i=1}^n \frac{1}{r_i}=1$ and

$$k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\frac{\lambda}{r_j}-1}\,du_1\cdots du_{n-1} > 0.$$

If $\phi_i(x) = x^{p_i(1-\frac{\lambda}{r_i})-1}, f_i(\geq 0) \in L_{\phi_i}^{p_i}(0,\infty), \|f\|_{p_i,\phi_i} > 0$ $(i=1,\ldots,n)$, then we have the following inequality:

$$\int_{\mathbf{R}_+^n} k_\lambda(x_1,\ldots,x_n)\prod_{i=1}^n f_i(x_i)\,dx_1\cdots dx_n < k_\lambda \prod_{i=1}^n \|f_i\|_{p_i,\phi_i}, \tag{5}$$

where the constant factor k_λ is the best possible. For $n=2$, $k_\lambda(x,y) = \frac{1}{x^\lambda+y^\lambda}$ in (5), we obtain (3). Inequality (5) is an extension of the results in [9–12] and [13]. In recent years, [14] and [15] considered some Hilbert-type operators relating (1)-(3); [16] also considered a multiple Hilbert-type integral operator with the homogeneous kernel of $-n+1$-degree and the relating particular case of (5) (for $\lambda = n-1$, $\frac{1}{r_i} = \frac{1}{n-1}(1-\frac{1}{p_i})$).

In this paper, by using the way of weight functions and the technic of real analysis, a multiple Hilbert-type integral inequality with a non-homogeneous kernel is given. The operator expression with the norm, the reverses and some examples with the particular kernels are considered.

2 Some lemmas

Lemma 1 *If* $n \in \mathbf{N}\backslash\{1\}$, $\lambda_i \in \mathbf{R}$ $(i=1,\ldots,n)$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, *then we have*

$$A := \prod_{i=1}^n \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^n x_j^{\lambda_j-1}\right]^{\frac{1}{p_i}} = 1. \tag{6}$$

Proof We find

$$A := \prod_{i=1}^n \left[x_i^{(\lambda_i-1)(1-p_i)+1-\lambda_i} \prod_{j=1}^n x_j^{\lambda_j-1}\right]^{\frac{1}{p_i}}$$

$$= \prod_{i=1}^n \left[x_i^{(1-\lambda_i)p_i} \prod_{j=1}^n x_j^{\lambda_j-1}\right]^{\frac{1}{p_i}} = \prod_{i=1}^n x_i^{1-\lambda_i}\left(\prod_{j=1}^n x_j^{\lambda_j-1}\right)^{\sum_{i=1}^n \frac{1}{p_i}},$$

and then (6) is valid. □

Definition 1 If $n \in \mathbf{N}$, $\mathbf{R}_+^n := \{(x_1, \ldots, x_n) | x_i > 0\ (i = 1, \ldots, n)\}$, $\lambda \in \mathbf{R}$, $k_\lambda(x_1, \ldots, x_n)$ is a measurable function in $\mathbf{R}_+^n$ such that for any $u > 0$ and $(x_1, \ldots, x_n) \in \mathbf{R}_+^n$, $k_\lambda(ux_1, \ldots, ux_n) = u^{-\lambda} k_\lambda(x_1, \ldots, x_n)$, then we call $k_\lambda(x_1, \ldots, x_n)$ the homogeneous function of $-\lambda$-degree in $\mathbf{R}_+^n$.

Lemma 2 *Suppose* $n \in \mathbf{N}\backslash\{1\}$, $\lambda_i \in \mathbf{R}$ $(i = 1, \ldots, n)$, $\lambda_n = \sum_{i=1}^{n-1} \lambda_i = \frac{\lambda}{2}$, $k_\lambda(x_1, \ldots, x_n) \geq 0$ *is a homogeneous function of* $-\lambda$*-degree. If*

$$H(i) := \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n) \times \prod_{j=1 (j \neq i)}^{n} u_j^{\lambda_j - 1}\, du_1 \cdots du_{i-1}\, du_{i+1} \cdots du_n \quad (i = 1, \ldots, n)$$

satisfying $k_\lambda := H(n) \in \mathbf{R}$, *then each* $H(i) = H(n) = k_\lambda$ *and for any* $i = 1, \ldots, n$,

$$\omega_i(x_i) := x_i^{\lambda_i} \int_{\mathbf{R}_+^{n-1}} k_\lambda(x_1 x_n, \ldots, x_{n-1} x_n, 1) \times \prod_{j=1 (j \neq i)}^{n} x_j^{\lambda_j - 1}\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n = k_\lambda \quad (\in \mathbf{R}). \tag{7}$$

Proof Setting $u_j = u_n v_j$ $(j \neq i, n)$ in the integral $H(i)$, we find

$$H(i) = \int_{\mathbf{R}_+^{n-1}} k_\lambda\big(v_1, \ldots, v_{i-1}, u_n^{-1}, v_{i+1}, \ldots, v_{n-1}, 1\big) \prod_{j=1 (j \neq i)}^{n-1} v_j^{\lambda_j - 1} \times u_n^{-1-\lambda_i}\, dv_1 \cdots dv_{i-1}\, dv_{i+1} \cdots dv_{n-1}\, du_n.$$

Setting $v_i = u_n^{-1}$ in the above integral, we obtain $H(i) = H(n)$. Setting $x_n' = x_n^{-1}$ in (7), since $\lambda - \lambda_n = \lambda_n$, we find

$$\begin{aligned}
\omega_i(x_i) &= x_i^{\lambda_i} \int_{\mathbf{R}_+^{n-1}} k_\lambda\big(x_1, \ldots, x_{n-1}, x_n^{-1}\big) x_n^{-\lambda_n - 1} \times \prod_{j=1 (j \neq i)}^{n-1} x_j^{\lambda_j - 1}\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n \\
&= x_i^{\lambda_i} \int_{\mathbf{R}_+^{n-1}} k_\lambda\big(x_1, \ldots, x_{n-1}, x_n'\big) \big(x_n'\big)^{\lambda_n + 1} \times \prod_{j=1 (j \neq i)}^{n-1} x_j^{\lambda_j - 1}\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_{n-1} \big(x_n'\big)^{-2}\, dx_n' \\
&= x_i^{\lambda_i} \int_{\mathbf{R}_+^{n-1}} k_\lambda\big(x_1, \ldots, x_{n-1}, x_n'\big) \big(x_n'\big)^{\lambda_n - 1} \times \prod_{j=1 (j \neq i)}^{n-1} x_j^{\lambda_j - 1}\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_{n-1}\, dx_n'.
\end{aligned}$$

Setting $u_j = x_j / x_i$ $(j \neq i, n)$ and $u_n = x_n' / x_i$ in the above integral, we find $\omega_i(x_i) = H(i) = H(n) = k_\lambda$. □

Lemma 3 *With the assumptions given in Lemma 2, then*

$$k(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1}) := \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\widetilde{\lambda}_j-1}\,du_1\cdots du_{n-1}$$

is finite in a neighborhood of $(\lambda_1,\ldots,\lambda_{n-1})$ *if any only if* $k(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1})$ *is continuous at* $(\lambda_1,\ldots,\lambda_{n-1})$.

Proof The sufficiency property is obvious. We prove the necessary property of the condition by mathematical induction in the following. For $n = 2$, there exists $I := \{\widetilde{\lambda}_1 | \widetilde{\lambda}_1 = \lambda_1 + \delta_1, |\delta_1| \le \delta_0, \delta_0 > 0\}$ such that for any $\widetilde{\lambda}_1 \in I$, $k(\widetilde{\lambda}_1) \in \mathbf{R}$. Since for $\widetilde{\lambda}_1 = \lambda_1 + \delta_1 \in I$ ($\delta_1 \neq 0$),

$$\begin{aligned}
&k(\lambda_1+\delta_1) = \int_0^1 k_\lambda(u_1,1)u_1^{\lambda_1+\delta_1-1}\,du_1 + \int_1^\infty k_\lambda(u_1,1)u_1^{\lambda_1+\delta_1-1}\,du_1,\\
&k_\lambda(u_1,1)u_1^{\lambda_1+\delta_1-1} \le k_\lambda(u_1,1)u_1^{\lambda_1-\delta_0-1}\,du_1, \quad u_1 \in (0,1];\\
&k_\lambda(u_1,1)u_1^{\lambda_1+\delta_1-1} \le k_\lambda(u_1,1)u_1^{\lambda_1+\delta_0-1}\,du_1, \quad u_1 \in (1,\infty),
\end{aligned}$$

and $k(\lambda_1 - \delta_0) + k(\lambda_1 + \delta_0) < \infty$, then by the Lebesgue control convergence theorem (*cf.* [17]), it follows $k(\lambda_1 + \delta_1) = k(\lambda_1) + o(1)$ ($\delta_1 \to 0$). Assuming that for n (≥ 2), $k(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1})$ is continuous at $(\lambda_1,\ldots,\lambda_{n-1})$, then for $n+1$, by the result of $n = 2$, since $k(\lambda_1+\delta_1,\ldots,\lambda_n+\delta_n)$ is finite in a neighborhood of $(\lambda_1,\ldots,\lambda_n)$, we find

$$\begin{aligned}
&\lim_{\delta_n\to 0} k(\lambda_1+\delta_1,\ldots,\lambda_n+\delta_n)\\
&\quad= \lim_{\delta_n\to 0}\int_0^\infty\left(\int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_n,1)\prod_{j=1}^{n-1} u_j^{\lambda_j+\delta_j-1}\,du_1\cdots du_{n-1}\right)u_n^{\lambda_n+\delta_n-1}\,du_n\\
&\quad= \int_0^\infty\left(\int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_n,1)\prod_{j=1}^{n-1} u_j^{\lambda_j+\delta_j-1}\,du_1\cdots du_{n-1}\right)u_n^{\lambda_n-1}\,du_n\\
&\quad= \int_{\mathbf{R}_+^{n-1}}\left(\int_0^\infty k_\lambda(u_1,\ldots,u_n,1)u_n^{\lambda_n-1}\,du_n\right)\prod_{j=1}^{n-1} u_j^{\lambda_j+\delta_j-1}\,du_1\cdots du_{n-1},
\end{aligned}$$

then by the assumption for n, it follows

$$\lim_{\delta_n\to 0} k(\lambda_1+\delta_1,\ldots,\lambda_n+\delta_n) = k(\lambda_1,\ldots,\lambda_n) + o(1) \quad (\delta_i \to 0, i = 1,\ldots,n-1).$$

By mathematical induction, we prove that for $n \in \mathbf{N}\setminus\{1\}$, $k(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{n-1})$ is continuous at $(\lambda_1,\ldots,\lambda_{n-1})$. □

Lemma 4 *With the assumptions given in Lemma 2, if there exists* $\delta > 0$ *such that for* $\max_{1\le i\le n-1}\{|\delta_i|\} < \delta$, $k(\lambda_1 + \delta_1,\ldots,\lambda_{n-1} + \delta_{n-1}) \in \mathbf{R}$, $p_i \in \mathbf{R}\setminus\{0,1\}$ $(i = 1,\ldots,n)$, $0 < \varepsilon < \min_{1\le i\le n}\{|p_i|\}\delta$, *then we have*

$$\begin{aligned}
I_\varepsilon &:= \varepsilon\int_1^\infty\cdots\int_1^\infty\left[\int_0^1 x_n^{\lambda_n+\frac{\varepsilon}{p_n}-1}k_\lambda(x_1x_n,\ldots,x_{n-1}x_n,1)\,dx_n\right]\\
&\quad\times\prod_{j=1}^{n-1} x_j^{\lambda_j-\frac{\varepsilon}{p_j}-1}\,dx_1\cdots dx_{n-1} = k_\lambda + o(1) \quad \left(\varepsilon \to 0^+\right).
\end{aligned} \tag{8}$$

Proof Setting $x_n' = x_n^{-1}$ in (8), we find

$$I_\varepsilon := \varepsilon \int_1^\infty \cdots \int_1^\infty \left[\int_1^\infty (x_n')^{-\lambda_n - \frac{\varepsilon}{p_n} - 1} k_\lambda \left(\frac{x_1}{x_n'}, \ldots, \frac{x_1}{x_n'}, 1 \right) dx_n' \right] \times \prod_{j=1}^{n-1} x_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} dx_1 \cdots dx_{n-1} = k_\lambda + o(1).$$

Setting $u_j = x_j / x_n'$ ($j = 1, \ldots, n-1$) in the above integral, since $\lambda - \lambda_n = \lambda_n$, we find (replacing x_n' by x_n)

$$I_\varepsilon = \varepsilon \int_1^\infty x_n^{-1-\varepsilon} \left[\int_{x_n^{-1}}^\infty \cdots \int_{x_n^{-1}}^\infty k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} du_1 \cdots du_{n-1} \right] dx_n. \tag{9}$$

Setting $D_j := \{(u_1, \ldots, u_{n-1}) | u_j \in (0, x_n^{-1}), u_k \in (0, \infty)\ (k \neq j)\}$ and

$$A_j(x_n) := \int \cdots \int_{D_j} k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} du_1 \cdots du_{n-1},$$

by (9), it follows

$$I_\varepsilon \geq \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} du_1 \cdots du_{n-1} - \varepsilon \sum_{j=1}^{n-1} \int_1^\infty x_n^{-1} A_j(x_n)\, dx_n. \tag{10}$$

Without loss of generality, we estimate the case of $j = n$, e.t.

$\int_1^\infty x_n^{-1} A_{n-1}(x_n)\, dx_n = O(1)$. In fact, setting $\alpha > 0$, such that $|\frac{\varepsilon}{p_{n-1}} + \alpha| < \delta$, since $-u_{n-1}^\alpha \times \ln u_{n-1} \to 0$ ($u_{n-1} \to 0^+$), there exists $M > 0$ such that $-u_{n-1}^\alpha \ln u_{n-1} \leq M$ ($u_{n-1} \in (0, 1]$), and then by the Fubini theorem, it follows

$$\begin{aligned}
0 &\leq \int_1^\infty x_n^{-1} A_{n-1}(x_n)\, dx_n \\
&= \int_1^\infty x_n^{-1} \left[\int_{\mathbf{R}_+^{n-2}} \int_0^{x_n^{-1}} k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} du_{n-1}\, du_1 \cdots du_{n-2} \right] dx_n \\
&= \int_0^1 \int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} \left(\int_1^{u_{n-1}^{-1}} x_n^{-1}\, dx_n \right) du_1 \cdots du_{n-1} \\
&= \int_0^1 \int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-1} u_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} (-\ln u_{n-1})\, du_1 \cdots du_{n-1} \\
&\leq M \int_0^1 \int_{\mathbf{R}_+^{n-2}} k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-2} u_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} u_{n-1}^{\lambda_{n-1} - (\frac{\varepsilon}{p_{n-1}} + \alpha) - 1} du_1 \cdots du_{n-1} \\
&\leq M \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1, \ldots, u_{n-1}, 1) \prod_{j=1}^{n-2} u_j^{\lambda_j - \frac{\varepsilon}{p_j} - 1} u_{n-1}^{\lambda_{n-1} - (\frac{\varepsilon}{p_{n-1}} + \alpha) - 1} du_1 \cdots du_{n-1} \\
&= M \cdot k\left(\lambda_1 - \frac{\varepsilon}{p_1}, \ldots, \lambda_{n-2} - \frac{\varepsilon}{p_{n-2}}, \lambda_{n-1} - \left(\frac{\varepsilon}{p_{n-1}} + \alpha \right) \right) < \infty.
\end{aligned}$$

Hence by (10), we have

$$I_\varepsilon \geq \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-\frac{\varepsilon}{p_j}-1}\,du_1\cdots du_{n-1} - o_1(1). \tag{11}$$

Since by Lemma 3 we find

$$\begin{aligned}
I_\varepsilon &\leq \varepsilon\int_1^\infty x_n^{-1-\varepsilon}\left[\int_0^\infty\cdots\int_0^\infty k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-\frac{\varepsilon}{p_j}-1}\,du_1\cdots du_{n-1}\right]dx_n \\
&= \int_0^\infty\cdots\int_0^\infty k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-\frac{\varepsilon}{p_j}-1}\,du_1\cdots du_{n-1} \\
&= k\left(\lambda_1-\frac{\varepsilon}{p_1},\ldots,\lambda_{n-1}-\frac{\varepsilon}{p_{n-1}}\right) = k_\lambda + o_2(1),
\end{aligned}$$

then combining with (11), we have (8). □

Lemma 5 *Suppose that* $n \in \mathbf{N}\backslash\{1\}$, $p_i \in \mathbf{R}\backslash\{0,1\}$ $(i = 1,\ldots,n)$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\frac{1}{q_n} = 1 - \frac{1}{p_n}$, $(\lambda_1,\ldots,\lambda_n) \in \mathbf{R}^n$, $\lambda_n = \sum_{i=1}^{n-1}\lambda_i = \frac{\lambda}{2}$, $k_\lambda(x_1,\ldots,x_n)$ (≥ 0) *is a measurable function of* $-\lambda$-*degree in* $\mathbf{R}_+^n$ *such that*

$$k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-1}\,du_1\cdots du_{n-1} \in \mathbf{R}.$$

If $f_i \geq 0$ *are measurable functions in* $\mathbf{R}_+$ $(i = 1,\ldots,n-1)$, $\widetilde{k}_\lambda(x_1,\ldots,x_n) := k_\lambda(x_1x_n,\ldots,x_{n-1}x_n, 1)$, *then* (1) *for* $p_i > 1$ $(i = 1,\ldots,n)$, *we have*

$$\begin{aligned}
J &:= \left\{\int_0^\infty x_n^{\frac{\lambda q_n}{2}-1}\left[\int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n-1} f_i(x_i)\,dx_1\cdots dx_{n-1}\right]^{q_n} dx_n\right\}^{\frac{1}{q_n}} \\
&\leq k_\lambda \prod_{i=1}^{n-1}\left\{\int_0^\infty x^{p_i(1-\lambda_i)-1} f^{p_i}(x)\,dx\right\}^{\frac{1}{p_i}};
\end{aligned} \tag{12}$$

(2) *for* $0 < p_1 < 1$, $p_i < 0$ $(i = 2,\ldots,n)$, *we have the reverse of* (12).

Proof (1) For $p_i > 1$ $(i = 1,\ldots,n)$, by the Hölder inequality (*cf.* [18]) and (7), it follows

$$\begin{aligned}
&\left[\int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n-1} f_i(x_i)\,dx_1\cdots dx_{n-1}\right]^{q_n} \\
&\quad = \left\{\int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n-1}\left[x_i^{(\lambda_i-1)(1-p_i)}\prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1}\right]^{\frac{1}{p_i}} f_i(x_i)\right. \\
&\qquad \left.\times\left[x_n^{(\lambda_n-1)(1-p_n)}\prod_{j=1}^{n-1} x_j^{\lambda_j-1}\right]^{\frac{1}{p_n}} dx_1\cdots dx_{n-1}\right\}^{q_n}
\end{aligned}$$

$$\leq \int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n) \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1} \right]^{\frac{q_n}{p_i}} f_i^{q_n}(x_i)\, dx_1 \cdots dx_{n-1}$$
$$\times \left\{ \int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n) x_n^{(\lambda_n-1)(1-p_n)} \prod_{j=1}^{n-1} x_j^{\lambda_j-1}\, dx_1 \cdots dx_{n-1} \right\}^{q_n-1}$$
$$= (k_\lambda)^{q_n-1} x_n^{1-q_n\lambda_n} \int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n)$$
$$\times \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1} \right]^{\frac{q_n}{p_i}} f_i^{q_n}(x_i)\, dx_1 \cdots dx_{n-1}, \tag{13}$$

$$J \leq (k_\lambda)^{\frac{1}{p_n}} \left\{ \int_0^\infty \int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n) \right.$$
$$\left. \times \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1} \right]^{\frac{q_n}{p_i}} f_i^{q_n}(x_i)\, dx_1 \cdots dx_{n-1}\, dx_n \right\}^{\frac{1}{q_n}}$$
$$= (k_\lambda)^{\frac{1}{p_n}} \left\{ \int_{\mathbf{R}_+^{n-1}} \left(\int_0^\infty \widetilde{k}_\lambda(x_1,\ldots,x_n) x_n^{\lambda_n-1}\, dx_n \right) \right.$$
$$\left. \times \prod_{i=1}^{n-1} \left[x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^{n-1} x_j^{\lambda_j-1} \right]^{\frac{q_n}{p_i}} f_i^{q_n}(x_i)\, dx_1 \cdots dx_{n-1} \right\}^{\frac{1}{q_n}}.$$

For $n \geq 3$, by the Hölder inequality again, it follows

$$J \leq (k_\lambda)^{\frac{1}{p_n}} \left\{ \prod_{i=1}^{n-1} \left[\int_{\mathbf{R}_+^{n-1}} \left(\int_0^\infty \widetilde{k}_\lambda(x_1,\ldots,x_n) x_n^{\lambda_n-1}\, dx_n \right) \right. \right.$$
$$\left. \left. \times\ x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^{n-1} x_j^{\lambda_j-1} f_i^{p_i}(x_i)\, dx_1 \cdots dx_{n-1} \right]^{\frac{q_n}{p_i}} \right\}^{\frac{1}{q_n}}$$
$$= (k_\lambda)^{\frac{1}{p_n}} \prod_{i=1}^{n-1} \left\{ \int_0^\infty \left[\int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n) \right. \right.$$
$$\left. \left. \times\ x_i^{\lambda_i} \prod_{j=1(j\neq i)}^{n} x_j^{\lambda_j-1}\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n \right] x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i)\, dx_i \right\}^{\frac{1}{p_i}}$$
$$= (k_\lambda)^{\frac{1}{p_n}} \prod_{i=1}^{n-1} \left\{ \int_0^\infty \omega_i(x_i) x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i)\, dx_i \right\}^{\frac{1}{p_i}}. \tag{14}$$

Then by (7), we have (12). (Note: for $n = 2$, we do not use the Hölder inequality again in the above.) (2) For $0 < p_1 < 1$, $p_i < 0$ $(i = 2,\ldots,n)$, by the reverse Hölder inequality and in the same way, we obtain the reverses of (12). □

3 Main results and applications

With the assumptions given in Lemma 5, setting $\phi_i(x) := x^{p_i(1-\lambda_i)-1}$ $(x \in (0,\infty); i = 1,\ldots,n)$, then we find $\phi_n^{1/(1-p_n)}(x) = x^{q_n\lambda_n-1}$. If $p_i > 1$ $(i = 1,\ldots,n)$, define the following real function spaces:

$$L_{\phi_i}^{p_i}(0,\infty) := \left\{f; \|f\|_{p_i,\phi_i} = \left\{\int_0^\infty \phi_i(x)\big|f(x)\big|^{p_i}\,dx\right\}^{\frac{1}{p_i}} < \infty\right\} \quad (i = 1,\ldots,n),$$
$$\prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0,\infty) := \left\{(f_1,\ldots,f_{n-1}); f_i \in L_{\phi_i}^{p_i}(0,\infty), i = 1,\ldots,n-1\right\},$$

and a multiple Hilbert-type integral operator $T : \prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0,\infty) \to L^{q_n}_{\phi_n^{1/(1-p_n)}}$ as follows: for $f = (f_1,\ldots,f_{n-1}) \in \prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0,\infty)$,

$$(Tf)(x_n) := \int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n-1} f_i(x_i)\,dx_1\cdots dx_{n-1}, \quad x_n \in (0,\infty). \tag{15}$$

Then by (12), it follows $Tf \in L^{q_n}_{\phi_n^{1/(1-p_n)}}$, T is bounded, $\|Tf\|_{q_n,\phi_n^{1/(1-p_n)}} \le k_\lambda \prod_{i=1}^{n-1}\|f_i\|_{p_i,\phi_i}$ and $\|T\| \le k_\lambda$, where

$$\|T\| := \sup_{f\in\prod_{i=1}^{n-1} L_{\phi_i}^{p_i}(0,\infty)(f_i\neq\theta, i=1,\ldots,n-1)} \frac{\|Tf\|_{q_n,\phi_n^{1/(1-p_n)}}}{\prod_{i=1}^{n-1}\|f_i\|_{p_i,\phi_i}}. \tag{16}$$

Define the formal inner product of $T(f_1,\ldots,f_{n-1})$ and f_n as

$$\big(T(f_1,\ldots,f_{n-1}),f_n\big) := \int_{\mathbf{R}_+^n} \widetilde{k}_\lambda(x_1,\ldots,x_n)\prod_{i=1}^{n} f_i(x_i)\,dx_1\cdots dx_n. \tag{17}$$

Theorem 1 *With the assumptions given in Lemma* 5, *suppose that for any* $(\lambda_1,\ldots,\lambda_n) \in \mathbf{R}^n$, *it satisfies* $\lambda_n = \sum_{i=1}^{n-1}\lambda_i = \frac{\lambda}{2}$, *and*

$$0 < k_\lambda = \int_{\mathbf{R}_+^{n-1}} k_\lambda(u_1,\ldots,u_{n-1},1)\prod_{j=1}^{n-1} u_j^{\lambda_j-1}\,du_1\cdots du_{n-1} < \infty. \tag{18}$$

If $f_i(\ge 0) \in L_{\phi_i}^{p_i}(0,\infty)$, $\|f\|_{p_i,\phi_i} > 0$ $(i = 1,\ldots,n)$, *then* (i) *for* $p_i > 1$ $(i = 1,\ldots,n)$, *we have* $\|T\| = k_\lambda$ *and the following equivalent inequalities*:

$$\big\|T(f_1,\ldots,f_{n-1})\big\|_{q_n,\phi_n^{1/(1-p_n)}} < k_\lambda\prod_{i=1}^{n-1}\|f_i\|_{p_i,\phi_i}, \tag{19}$$
$$\big(T(f_1,\ldots,f_{n-1}),f_n\big) < k_\lambda\prod_{i=1}^{n}\|f_i\|_{p_i,\phi_i}, \tag{20}$$

where the constant factor k_λ *is the best possible*; (ii) *for* $0 < p_1 < 1$, $p_i < 0$ $(i = 2,\ldots,n)$, *using the formal symbols in the case of* (i), *we have the equivalent reverses of* (19) *and* (20) *with the same best constant factor.*

Proof (i) For all $p_i > 1$, if (12) takes the form of equality, then for $n \geq 3$ in (14), there exist C_i and C_k ($i \neq k$) such that they are not all zero and

$$C_i x_i^{(\lambda_i-1)(1-p_i)} \prod_{j=1(j\neq i)}^{n-1} x_j^{\lambda_j-1} f_j^{p_j}(x_j) \\ = C_k x_k^{(\lambda_k-1)(1-p_k)} \prod_{j=1(j\neq k)}^{n-1} x_j^{\lambda_j-1} f_j^{p_j}(x_j) \quad \text{a.e. in } \mathbf{R}_+^n,$$

e.t. $C_i x_i^{p_i(1-\lambda_i)} f_i^{p_i}(x_i) = C_k x_k^{p_k(1-\lambda_k)} f_k^{p_k}(x_k) = C$ a.e. in $\mathbf{R}_+^n$. Assuming that $C_i > 0$, then $x_i^{p_i(1-\lambda_i)-1} f_i^{p_i}(x_i) = C/(C_i x_i)$, which contradicts $\|f\|_{p_i,\phi_i} > 0$. (Note: for $n = 2$, we consider (13) for $f_k^{p_i}(x_k) = 1$ in the above.) Hence we have (19). By the Hölder inequality, it follows

$$\begin{aligned} (Tf, f_n) &= \int_0^\infty \left(x_n^{\lambda_n - \frac{1}{q_n}} \int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i)\, dx_1 \cdots dx_{n-1} \right) \left(x_n^{\frac{1}{q_n} - \lambda_n} f_n(x_n) \right) dx_n \\ &\leq \left\| T(f_1, \ldots, f_{n-1}) \right\|_{q_n, \phi_n^{1/(1-p_n)}} \|f_n\|_{p_n,\phi_n}, \end{aligned} \tag{21}$$

and then by (19), we have (20). Assuming that (20) is valid, setting

$$f_n(x_n) := x_n^{q_n\lambda_n - 1} \left[\int_{\mathbf{R}_+^{n-1}} \widetilde{k}_\lambda(x_1, \ldots, x_n) \prod_{i=1}^{n-1} f_i(x_i)\, dx_1 \cdots dx_{n-1} \right]^{q_n - 1},$$

then $J = \{\int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n)\, dx_n\}^{\frac{1}{q_n}}$. By (12), it follows $J < \infty$. If $J = 0$, then (19) is naturally valid. Assuming that $0 < J < \infty$, by (20), it follows

$$\int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n)\, dx_n = J^{q_n} = (Tf, f_n) < k_\lambda \prod_{i=1}^{n} \|f_i\|_{p_i,\phi_i},$$

$$\left\{ \int_0^\infty x_n^{p_n(1-\lambda_n)-1} f_n^{p_n}(x_n)\, dx_n \right\}^{\frac{1}{q_n}} = J < k_\lambda \prod_{i=1}^{n-1} \|f_i\|_{p_i,\phi_i},$$

and then (19) is valid, which is equivalent to (20).

For $\varepsilon > 0$ small enough, setting $\widetilde{f}_i(x)$ as: $\widetilde{f}_i(x) = 0$, $x \in (0,1)$; $\widetilde{f}_i(x) = x^{\lambda_i - \frac{\varepsilon}{p_i} - 1}$, $x \in [1,\infty)$ ($i = 1, \ldots, n-1$), $\widetilde{f}_n(x) = x^{\lambda_n + \frac{\varepsilon}{p_n} - 1}$, $x \in (0,1)$; $\widetilde{f}_n(x) = 0$, $x \in [1,\infty)$, if there exists $k \leq k_\lambda$ such that (20) is still valid as we replace k_λ by k, then in particular, by Lemma 4, we have

$$k_\lambda + o(1) = I_\varepsilon = \varepsilon\left(T(\widetilde{f}_1, \ldots, \widetilde{f}_{n-1}), \widetilde{f}_n \right) < \varepsilon k \prod_{i=1}^{n} \|\widetilde{f}_i\|_{p_i,\phi_i} = k$$

and $k_\lambda \leq k (\varepsilon \to 0^+)$. Hence $k = k_\lambda$ is the best value of (20). We confirm that the constant factor k_λ in (19) is the best possible, otherwise we can get a contradiction by (21) that the constant factor in (20) is not the best possible. Therefore $\|T\| = k_\lambda$.

(ii) For $0 < p_1 < 1$, $p_i < 0$ ($i = 2, \ldots, n$), by using the reverse Hölder inequality and in the same way, we have the equivalent reverses of (19) and (20) with the same best constant factor. □

Example 1 For $\lambda > 0$, $\lambda_i = \frac{\lambda}{r_i}$ $(i = 1, \ldots, n)$, $r_n = 2$, $\sum_{i=1}^{n} \frac{1}{r_i} = 1$, $k_\lambda(x_1, \ldots, x_n) = \frac{1}{(\sum_{i=1}^{n} x_i)^\lambda}$, by mathematical induction, we can show

$$k_\lambda = \int_{R_+^{n-1}} \frac{1}{(\sum_{i=1}^{n-1} u_i + 1)^\lambda} \prod_{j=1}^{n-1} u_j^{\frac{\lambda}{r_j}-1} \, du_1 \cdots du_{n-1} = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(\frac{\lambda}{r_i}\right). \tag{22}$$

In fact, for $n = 2$, we obtain

$$k_\lambda = \int_{R_+} \frac{1}{(u_1 + 1)^\lambda} u_1^{\frac{\lambda}{r_1}-1} \, du_1 = \frac{1}{\Gamma(\lambda)} \Gamma\left(\frac{\lambda}{r_1}\right) \Gamma\left(\frac{\lambda}{r_2}\right).$$

Assuming that for n (≥ 2) (22) is valid, then for $n + 1$, it follows

$$\begin{aligned}
k_\lambda &= \int_{R_+^{n}} \frac{1}{(\sum_{i=1}^{n} u_i + 1)^\lambda} \prod_{j=1}^{n} u_j^{\frac{\lambda}{r_j}-1} \, du_1 \cdots du_n \\
&= \int_{R_+^{n-1}} \prod_{j=2}^{n} u_j^{\frac{\lambda}{r_j}-1} \left[\int_{R_+} \frac{1}{[u_1 + (\sum_{i=2}^{n} u_i + 1)]^\lambda} u_1^{\frac{\lambda}{r_1}-1} \, du_1 \right] du_2 \cdots du_n \\
&= \int_{R_+^{n-1}} \frac{1}{(\sum_{i=2}^{n} u_i + 1)^\lambda} \prod_{j=2}^{n} u_j^{\frac{\lambda}{r_j}-1} \left[\int_{R_+} \frac{1}{(v_1 + 1)^\lambda} v_1^{\frac{\lambda}{r_1}-1} \, dv_1 \right] du_2 \cdots du_n \\
&= \frac{\Gamma(\frac{\lambda}{r_1}) \Gamma(\lambda - \frac{\lambda}{r_1})}{\Gamma(\lambda)} \int_{R_+^{n-1}} \frac{1}{(\sum_{i=2}^{n} u_i + 1)^{\lambda(1-\frac{1}{r_1})}} \prod_{j=2}^{n} u_j^{\frac{\lambda}{r_j}-1} \, du_2 \cdots du_n \\
&= \frac{\Gamma(\frac{\lambda}{r_1}) \Gamma(\lambda - \frac{\lambda}{r_1})}{\Gamma(\lambda)} \frac{1}{\Gamma(\lambda - \frac{\lambda}{r_1})} \prod_{i=2}^{n+1} \Gamma\left(\frac{\lambda}{r_i}\right) = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n+1} \Gamma\left(\frac{\lambda}{r_i}\right).
\end{aligned}$$

Then by mathematical induction, (22) is valid for $n \in \mathbf{N} \backslash \{1\}$.

Example 2 For $\lambda > 0$, $\lambda_i = \frac{\lambda}{r_i}$ $(i = 1, \ldots, n)$, $r_n = 2$, $\sum_{i=1}^{n} \frac{1}{r_i} = 1$, $k_\lambda(x_1, \ldots, x_n) = \frac{1}{(\max_{1 \leq i \leq n}\{x_i\})^\lambda}$, by mathematical induction, we can show

$$k_\lambda = \int_{R_+^{n-1}} \frac{1}{(\max_{1 \leq i \leq n-1}\{u_i\} + 1)^\lambda} \prod_{j=1}^{n-1} u_j^{\frac{\lambda}{r_j}-1} \, du_1 \cdots du_{n-1} = \frac{\prod_{i=1}^{n} r_i}{\lambda^{n-1}}. \tag{23}$$

In fact, for $n = 2$, we obtain

$$k_\lambda = \int_{R_+} \frac{u_1^{\frac{\lambda}{r_1}-1}}{(\max\{u_1, 1\})^\lambda} \, du_1 = \int_0^1 u_1^{\frac{\lambda}{r_1}-1} \, du_1 + \int_1^\infty u_1^{\frac{-\lambda}{r_2}-1} \, du_1 = \frac{1}{\lambda} r_1 r_2.$$

Assuming that for n (≥ 2), (22) is valid, then for $n + 1$, it follows

$$\begin{aligned}
k_\lambda &= \int_{R_+^{n-1}} \prod_{j=2}^{n} u_j^{\frac{\lambda}{r_j}-1} \left[\int_0^\infty \frac{1}{(\max_{1 \leq i \leq n}\{u_i, 1\})^\lambda} u_1^{\frac{\lambda}{r_1}-1} \, du_1 \right] du_2 \cdots du_n \\
&= \int_{R_+^{n-1}} \prod_{j=2}^{n} u_j^{\frac{\lambda}{r_j}-1} \left[\int_0^{\max\{u_2, \ldots, u_n, 1\}} \frac{1}{(\max_{2 \leq i \leq n}\{u_i, 1\})^\lambda} u_1^{\frac{\lambda}{r_1}-1} \, du_1 \right.
\end{aligned}$$

$$+\int_{\max\{u_2,\ldots,u_n,1\}}^{\infty}\frac{1}{u_1^{\lambda}}u_1^{\frac{\lambda}{r_1}-1}\,du_1\Bigg]\,du_2\cdots du_n$$

$$=\frac{r_1^2}{\lambda(r_1-1)}\int_{R_+^{n-1}}\frac{1}{(\max_{2\le i\le n}\{u_i,1\})^{\lambda(1-\frac{1}{r_1})}}\prod_{j=2}^{n}u_j^{\frac{\lambda}{r_j}-1}\,du_2\cdots du_n$$

$$=\frac{r_1^2}{\lambda(r_1-1)}\left(\frac{r_1}{r_1-1}\right)^{n-1}\int_{R_+^{n-1}}\frac{1}{(\max_{2\le i\le n}\{v_i,1\})^{\lambda}}\prod_{j=2}^{n}v_j^{\frac{\lambda}{r_j}\frac{r_1}{r_1-1}-1}\,dv_2\cdots dv_n$$

$$=\frac{r_1^2}{\lambda(r_1-1)}\left(\frac{r_1}{r_1-1}\right)^{n-1}\frac{1}{\lambda^{n-1}}\prod_{i=2}^{n+1}\frac{r_1-1}{r_1}r_i=\frac{1}{\lambda^n}\prod_{i=1}^{n+1}r_i.$$

Then by mathematical induction, (23) is valid for $n\in\mathbf{N}\setminus\{1\}$.

Example 3 For $\lambda>0$, $\lambda_i=\frac{-\lambda}{r_i}$ $(i=1,\ldots,n)$, $r_n=2$, $\sum_{i=1}^{n}\frac{1}{r_i}=1$, $k_\lambda(x_1,\ldots,x_n)=(\min_{1\le i\le n}\{x_i\})^{\lambda}$, by mathematical induction, we can show

$$k_{-\lambda}=\int_{R_+^{n-1}}\bigl(\min\{u_1,\ldots,u_{n-1},1\}\bigr)^{\lambda}\prod_{j=1}^{n-1}u_j^{\frac{-\lambda}{r_j}-1}\,du_1\cdots du_{n-1}=\frac{\prod_{i=1}^{n}r_i}{\lambda^{n-1}}.\tag{24}$$

In fact, for $n=2$, we obtain

$$k_{-\lambda}=\int_0^1 u_1^{\frac{\lambda}{r_2}-1}\,du_1+\int_1^{\infty}u_1^{\frac{-\lambda}{r_1}-1}\,du_1=\frac{1}{\lambda}r_1r_2.$$

Assuming that for n (≥ 2), (24) is valid, then for $n+1$, it follows

$$k_{-\lambda}=\int_{R_+^{n-1}}\prod_{j=2}^{n}u_j^{\frac{-\lambda}{r_j}-1}\left[\int_0^{\infty}\bigl(\min\{u_1,\ldots,u_n,1\}\bigr)^{\lambda}u_1^{\frac{-\lambda}{r_1}-1}\,du_1\right]du_2\cdots du_n$$

$$=\int_{R_+^{n-1}}\prod_{j=2}^{n}u_j^{\frac{-\lambda}{r_j}-1}\Bigg[\int_0^{\min\{u_2,\ldots,u_n,1\}}u_1^{\lambda}u_1^{\frac{-\lambda}{r_1}-1}\,du_1$$

$$+\int_{\min\{u_2,\ldots,u_n,1\}}^{\infty}\bigl(\min\{u_2,\ldots,u_n,1\}\bigr)^{\lambda}u_1^{\frac{-\lambda}{r_1}-1}\,du_1\Bigg]\,du_2\cdots du_n$$

$$=\frac{r_1^2}{\lambda(r_1-1)}\int_{R_+^{n-1}}\bigl(\min\{u_2,\ldots,u_n,1\}\bigr)^{\lambda(1-\frac{1}{r_1})}\prod_{j=2}^{n}u_j^{\frac{-\lambda(1-\frac{1}{r_1})}{(1-\frac{1}{r_1})r_j}-1}\,du_2\cdots du_n$$

$$=\frac{r_1^2}{\lambda(r_1-1)}\frac{1}{[\lambda(1-\frac{1}{r_1})]^{n-1}}\prod_{i=2}^{n+1}\left(1-\frac{1}{r_1}\right)r_i=\frac{1}{\lambda^n}\prod_{i=1}^{n+1}r_i.$$

Then, by mathematical induction, (24) is valid for $n\in\mathbf{N}\setminus\{1\}$.

Remarks (i) In particular, for $n=2$ in (20), we have

$$\int_0^{\infty}\int_0^{\infty}k_\lambda(1,xy)f(x)g(y)\,dx\,dy$$

$$<k_\lambda\left\{\int_0^{\infty}x^{p(1-\frac{\lambda}{2})-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^{\infty}x^{q(1-\frac{\lambda}{2})-1}g^q(x)\,dx\right\}^{\frac{1}{q}},\tag{25}$$

where $k_\lambda = \int_0^\infty k_\lambda(u,1)u^{\frac{\lambda}{2}-1}\,du > 0$ ($\lambda \in \mathbf{R}$) is the best possible. Inequality (25) is an extension of (4) and (8.1.7) in [8].

(ii) In Examples 1 and 2, by Theorem 1, since for any $(\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n(\lambda_n = \sum_{i=1}^n \lambda_i = \frac{\lambda}{2})$, we obtain $0 < k_\lambda < \infty$, then we have $\|T\| = k_\lambda$ and the equivalent inequalities (19) and (20) with the particular kernels and some equivalent reverses. In Example 3, still using Theorem 1, we find $0 < \|T\| = k_{-\lambda} < \infty$ and the relating particular inequalities.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QH carried out the study, and wrote the manuscript. BY participated in the design of the study, and reformed the manuscript. All authors read and approved the final manuscript.

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