Journal of Inequalities and Applications
a SpringerOpen Journal

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Nearly n-homomorphisms and n-derivations in fuzzy ternary Banach algebras

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Abstract
Let $n = 3k + 2$ for some $k \in \mathbb{N}$. We investigate the generalized Hyers-Ulam stability of n-homomorphisms and n-derivations on fuzzy ternary Banach algebras related to the generalized Cauchy-Jensen additive functional equation.
MSC: 39B52; 46S40; 26E50

Keywords: Hyers-Ulam-Rassias stability; fixed point theorem; fuzzy ternary Banach algebra

1 Introduction

We say a functional equation (ξ) is stable if any function g satisfying the equation (ξ) *approximately* is near to a true solution of (ξ). We say that a functional equation (ξ) is *superstable* if every approximately solution of (ξ) is an exact solution of it (see [1]).

Speaking of the stability of a functional equation, we follow the question raised in 1940 by Ulam: When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? This problem was solved in the next year for the Cauchy functional equation on Banach spaces by Hyers [2]. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$ and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$, then T is linear. It gave rise to the Hyers-Ulam type stability of functional equations. Hyers' theorem was generalized by Rassias [3] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1 (Th.M. Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality $\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which* *satisfies*

$$\left\|f(x) - L(x)\right\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$, the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

Găvruta [4] generalized the Rassias result. Beginning around the year 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [5–45]).

Some mathematicians have defined fuzzy norms on a vector space from various points of view (see [46] and [47]). Bag and Samanta [48], following Cheng and Mordeson [49], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Karmosil and Michalek type [50]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [51].

In this paper, we consider a mapping $f : X \to Y$ satisfying the following functional equation, which is introduced by Azadi Kenary [52]:

$$\sum_{\substack{1 \leq i_1 < \cdots < i_m \leq n \\ 1 \leq k_l (\neq i_j, \forall j \in \{1,\ldots,m\}) \leq n}} f\left(\frac{\sum_{j=1}^m x_{i_j}}{m} + \sum_{l=1}^{n-m} x_{k_l}\right) = \frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^n f(x_i) \tag{1.1}$$

for all $x_1, \ldots, x_n \in X$, where $m, n \in \mathbb{N}$ are fixed integers with $n \geq 2$, $1 \leq m \leq n$. Especially, we observe that in the case $m = 1$, equation (1.1) yields the Cauchy additive equation $f(\sum_{l=1}^n x_{k_l}) = \sum_{l=1}^n f(x_i)$. We observe that in the case $m = n$, equation (1.1) yields the Jensen additive equation $f(\frac{\sum_{j=1}^n x_j}{n}) = \frac{1}{n}\sum_{l=1}^n f(x_i)$. Therefore, equation (1.1) is a generalized form of the Cauchy-Jensen additive equation, and thus every solution of equation (1.1) may be analogously called general (m,n)-Cauchy-Jensen additive. For the case $m = 2$, the authors have established new theorems about the Ulam-Hyers-Rassias stability in quasi β-normed spaces [53]. Let X and Y be linear spaces. For each m with $1 \leq m \leq n$, a mapping $f : X \to Y$ satisfies equation (1.1) for all $n \geq 2$ if and only if $f(x) - f(0) = A(x)$ is Cauchy additive, where $f(0) = 0$ if $m < n$. In particular, we have $f((n-m+1)x) = (n-m+1)f(x)$ and $f(mx) = mf(x)$ for all $x \in X$.

Definition 1.1 Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, c+t) \geq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

Example 1.1 Let $(X, \|\cdot\|)$ be a normed linear space and $\alpha, \beta > 0$. Then

$$N(x,t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta\|x\|}, & t > 0, x \in X, \\ 0, & t \leq 0, x \in X, \end{cases}$$

is a fuzzy norm on X.

Definition 1.2 Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X, and we denote it by $N\text{-}\lim_{t\to\infty} x_n = x$.

Definition 1.3 Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [51]).

Definition 1.4 Let X be a ternary algebra and (X,N) be a fuzzy normed space.

(1) The fuzzy normed space (X,N) is called a fuzzy ternary normed algebra if

$$N\big([xyz], stu\big) \geq N(x,s)N(y,t)N(z,u)$$

for all $x,y,z \in X$ and all positive real numbers s, t and u.

(2) A complete ternary fuzzy normed algebra is called a ternary fuzzy Banach algebra.

Example 1.2 Let $(X, \|\cdot\|)$ be a ternary normed (Banach) algebra. Let

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|}, & t > 0, x \in X, \\ 0, & t \leq 0, x \in X. \end{cases}$$

Then $N(x,t)$ is a fuzzy norm on X and (X,N) is a ternary fuzzy normed (Banach) algebra.

From now on, we suppose that $k \in \mathbb{N}$ is a fixed positive integer and $m' = 3k+2$. Also, we assume that $n \geq 3$ is a fixed positive integer.

Definition 1.5 Let (X,N) and (Y,N') be two ternary fuzzy normed algebras. Then

(1) a $\mathbb{C}$-linear mapping $H : (X,N) \to (Y,N')$ is called an m'-homomorphism if

$$\begin{aligned} &H\big(\big[\big[\cdots\big[[x_1x_2x_3]x_4x_5\big]\cdots\big]x_{m'-1}x_{m'}\big]\big) \\ &\quad = \big[\big[\cdots\big[\big[H(x_1)H(x_2)H(x_3)\big]H(x_4)H(x_5)\big]\cdots\big](x_{m'-1})H(x_{m'})\big] \end{aligned}$$

for all $x_1, x_2, \ldots, x_{m'} \in X$;

(2) a $\mathbb{C}$-linear mapping $D : (X,N) \to (X,N)$ is called an m'-derivation if

$$\begin{aligned} &D\big(\big[\big[\cdots\big[[x_1x_2x_3]x_4x_5\big]\cdots\big]x_{m'-1}x_{m'}\big]\big) \\ &\quad = \big[\big[\cdots\big[\big[D(x_1)x_2x_3\big]x_4x_5\big]\cdots\big]x_{m'-1}x_{m'}\big] \end{aligned}$$

$$\begin{aligned}
&+\bigl[\bigl[\cdots\bigl[\bigl[x_1 D(x_2) x_3\bigr] x_4 x_5\bigr]\cdots\bigr] x_{m'-1} x_{m'}\bigr] \\
&+\bigl[\bigl[\cdots\bigl[\bigl[x_1 x_2 D(x_3)\bigr] x_4 x_5\bigr]\cdots\bigr] x_{m'-1} x_{m'}\bigr] \\
&+\bigl[\bigl[\cdots\bigl[[x_1 x_2 x_3] D(x_4) x_5\bigr]\cdots\bigr] x_{m'-1} x_{m'}\bigr] \\
&+\bigl[\bigl[\cdots\bigl[[x_1 x_2 x_3] x_4 D(x_5)\bigr] x_6 \cdots\bigr] x_{m'-1} x_{m'}\bigr] \\
&+\cdots+\bigl[\bigl[\cdots\bigl[[x_1 x_2 x_3] x_4 x_5\bigr]\cdots\bigr] D(x_{m'-1}) x_{m'}\bigr] \\
&+\bigl[\bigl[\cdots\bigl[[x_1 x_2 x_3] x_4 x_5\bigr]\cdots\bigr] x_{m'-1} D(x_{m'})\bigr]
\end{aligned}$$

for all $x_1, x_2, \ldots, x_{m'} \in X$.

We apply the following theorem on weighted spaces.

Theorem 1.2 *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, we suppose that X is a ternary fuzzy normed algebra and Y is a ternary fuzzy Banach algebra. For convenience, we use the following abbreviations for a given mapping $f : X \to Y$:

$$\begin{aligned}
&\Delta f(x_1, \ldots, x_n, y_1, y_2, \ldots, y_{m'}) \\
&\quad = \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq n \\ 1 \leq k_l (\neq i_j, \forall j \in \{1,\ldots,m\}) \leq n}} f\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m} + \sum_{l=1}^{n-m} \mu x_{k_l}\right) - \frac{(n-m+1)\binom{n}{m} \sum_{i=1}^n \mu f(x_i)}{n} \\
&\qquad + f\bigl(\bigl[\bigl[\cdots\bigl[[y_1 y_2 y_3] y_4 y_5\bigr]\cdots\bigr] y_{m'-1} y_{m'}\bigr]\bigr) \\
&\qquad - \bigl[\bigl[\cdots\bigl[\bigl[f(y_1) f(y_2) f(y_3)\bigr] f(y_4) f(y_5)\bigr]\cdots\bigr] f(y_{m'-1}) f(y_{m'})\bigr],
\end{aligned}$$

and

$$\begin{aligned}
&D_f(x_1, \ldots, x_n, y_1, y_2, \ldots, y_{m'}) \\
&\quad = \sum_{\substack{1 \leq i_1 < \cdots < i_m \leq n \\ 1 \leq k_l (\neq i_j, \forall j \in \{1,\ldots,m\}) \leq n}} f\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m} + \sum_{l=1}^{n-m} \mu x_{k_l}\right) - \frac{(n-m+1)\binom{n}{m} \sum_{i=1}^n \mu f(x_i)}{n} \\
&\qquad + f\bigl(\bigl[\bigl[\cdots\bigl[[y_1 y_2 y_3] y_4 y_5\bigr]\cdots\bigr] y_{m'-1} y_{m'}\bigr]\bigr) - \bigl[\bigl[\cdots\bigl[\bigl[f(y_1) y_2 y_3\bigr] y_4 y_5\bigr]\cdots\bigr] y_{m'-1} y_{m'}\bigr] \\
&\qquad - \bigl[\bigl[\cdots\bigl[\bigl[y_1 f(y_2) y_3\bigr] y_4 y_5\bigr]\cdots\bigr] y_{m'-1} y_{m'}\bigr] - \bigl[\bigl[\cdots\bigl[\bigl[y_1 y_2 f(y_3)\bigr] y_4 y_5\bigr]\cdots\bigr] y_{m'-1} y_{m'}\bigr] \\
&\qquad - \bigl[\bigl[\cdots\bigl[[y_1 y_2 y_3] f(y_4) y_5\bigr]\cdots\bigr] y_{m'-1} y_{m'}\bigr] - \bigl[\bigl[\cdots\bigl[[y_1 y_2 y_3] y_4 f(y_5)\bigr] y_6 \cdots\bigr] y_{m'-1} y_{m'}\bigr] \\
&\qquad - \cdots - \bigl[\bigl[\cdots\bigl[[y_1 y_2 y_3] y_4 y_5\bigr]\cdots\bigr] f(y_{m'-1}) y_{m'}\bigr] - \bigl[\bigl[\cdots\bigl[[y_1 y_2 y_3] y_4 y_5\bigr]\cdots\bigr] y_{m'-1} f(y_{m'})\bigr].
\end{aligned}$$

There are several recent works on stability of functional equations on Banach algebras (see [10–28]). We investigate the stability of n-homomorphisms and n-derivations on fuzzy ternary Banach algebras.

2 Main results

In this section, by using the idea of Park *et al.* [39], we prove the generalized Hyers-Ulam-Rassias stability of 5-homomorphisms and 5-derivations related to the functional equation (1.1) on fuzzy ternary Banach algebras (see also [54]). We start our main results by the stability of 5-homomorphisms.

Theorem 2.1 *Let $\varphi : X^{n+m'} \to [0,\infty)$ be a mapping such that there exists an $L < \frac{1}{(n-m+1)^{n-2}}$ with*

$$\varphi\left(\frac{x_1}{n-m+1},\dots,\frac{x_{n+m'}}{n-m+1}\right) \le \frac{L\varphi(x_1,x_2,\dots,x_{n+m'})}{n-m+1} \tag{2.1}$$

for all $x_1,\dots,x_{n+m'} \in X$. Let $f : X \to Y$ with $f(0) = 0$ be a mapping satisfying

$$N\big(\Delta f(x_1,\dots,x_{n+m'}),t\big) \ge \frac{t}{t+\varphi(x_1,\dots,x_n,0,0,\dots,0)} \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$, $x_1,\dots,x_{n+m'} \in X$ and all $t > 0$. Then there exists a unique m'-homomorphism $H : X \to Y$ such that

$$N\big(f(x)-H(x),t\big) \ge \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + L\varphi(x,\dots,x,0,0,\dots,0)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof Letting $\mu = 1$ and putting $x_{n+1} = x_{n+2} = \cdots = x_{n+m'} = 0$, $x_1 = x_2 = \cdots = x_n = x$ in (2.2), we obtain

$$\begin{aligned} &N\left(\binom{n}{m} f\big((n-m+1)x\big) - \binom{n}{m}(n-m+1)f(x),t\right) \\ &\quad \ge \frac{t}{t+\varphi(x,\dots,x,0,0,\dots,0)} \end{aligned} \tag{2.4}$$

for all $x \in X$ and $t > 0$. Set $S := \{h : X \to Y; h(0) = 0\}$ and define $d : S \times S \to [0,\infty]$ by

$$d(f,g) = \inf\left\{\mu \in \mathbb{R}^+ : N\big(g(x)-h(x),\mu t\big) \ge \frac{t}{t+\varphi(x,\dots,x,0,0,\dots,0)}, \forall x \in X, t > 0\right\},$$

where $\inf \emptyset = +\infty$. By using the same technique as in the proof of Theorem 3.2 of [54], we can show that (S,d) is a complete generalized metric space. We define $J : S \to S$ by

$$Jg(x) := (n-m+1)g\left(\frac{x}{n-m+1}\right)$$

for all $x \in X$. It is easy to see that $d(Jg,Jh) \le Ld(g,h)$ for all $g,h \in S$. This implies that

$$d(f,Jf) \le \frac{L}{(n-m+1)\binom{n}{m}}.$$

By Banach's fixed point approach, J has a unique fixed point $H : X \to Y$ in $S_0 := \{h \in S : d(h,f) < \infty\}$ satisfying

$$H\left(\frac{x}{n-m+1}\right) = \frac{H(x)}{n-m+1} \tag{2.5}$$

for all $x \in X$. This implies that H is a unique mapping such that (2.5) and that there exists $\mu \in (0,\infty)$ satisfying $N(f(x) - H(x), \mu t) \geq \frac{t}{t+\varphi(x,\ldots,x,0,0,\ldots,0)}$ for all $x \in X$ and $t > 0$. Moreover, we have $d(J^p f, H) \to 0$ as $p \to \infty$. This implies the equality

$$N\text{-}\lim_{p\to\infty} \frac{f(\frac{x}{(n-m+1)^p})}{(n-m+1)^{-p}} = H(x) \tag{2.6}$$

for all $x \in X$.

It follows from (2.2) and (2.6) that

$$\sum_{\substack{1\leq i_1<\cdots<i_m\leq n\\ 1\leq k_l(\neq i_j,\forall j\in\{1,\ldots,m\})\leq n}} H\left(\frac{\sum_{j=1}^m \mu x_{i_j}}{m} + \sum_{l=1}^{n-m} \mu x_{k_l}\right) = \frac{(n-m+1)}{n}\binom{n}{m}\sum_{i=1}^n \mu H(x_i)$$

for all $\mu \in \mathbb{T}^1$, $x_1,\ldots,x_n \in X$. This means that $H : X \to Y$ is additive. By using the same technique as in the proof of Theorem 2.1 of [55], we can show that H is $\mathbb{C}$-linear. By (2.2), we have

$$\begin{aligned}
N\Bigg(&\frac{f(\frac{[[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}} \\
&- \frac{1}{(n-m+1)^{-(n-1)p}}\left[\left[\cdots\left[\left[f\left(\frac{y_1}{(n-m+1)^p}\right)f\left(\frac{y_2}{(n-m+1)^p}\right)f\left(\frac{y_3}{(n-m+1)^p}\right)\right]\right.\right.\right. \\
&\left.\left.\left.\times f\left(\frac{y_4}{(n-m+1)^p}\right)f\left(\frac{y_5}{(n-m+1)^p}\right)\right]\cdots\right]f\left(\frac{y_{m'-1}}{(n-m+1)^p}\right)f\left(\frac{y_{m'}}{(n-m+1)^p}\right)\right], \\
&\frac{t}{(n-m+1)^{-(n-1)p}}\Bigg) \\
\geq{}& \frac{t}{t+\varphi(0,0,\ldots,0,\frac{y_1}{(n-m+1)^p},\frac{y_2}{(n-m+1)^p},\ldots,\frac{y_{m'}}{(n-m+1)^p})}
\end{aligned}$$

for all $y_1, y_2, \ldots, y_{m'} \in X$ and all $t > 0$. Then

$$\begin{aligned}
&N\big(H([[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]) \\
&\quad - [[\cdots[[H(y_1)H(y_2)H(y_3)]H(y_4)H(y_5)]\cdots]H(y_{m'-1})H(y_{m'})], t\big) \\
&= \lim_{p\longrightarrow\infty} N\Bigg(\frac{f(\frac{[[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}} \\
&\quad - \frac{1}{(n-m+1)^{-(n-1)p}}\left[\left[\cdots\left[\left[f\left(\frac{y_1}{(n-m+1)^p}\right)f\left(\frac{y_2}{(n-m+1)^p}\right)f\left(\frac{y_3}{(n-m+1)^p}\right)\right]\right.\right.\right. \\
&\quad \left.\left.\times f\left(\frac{y_4}{(n-m+1)^p}\right)f\left(\frac{y_5}{(n-m+1)^p}\right)\right]\cdots\right]
\end{aligned}$$

$$\times f\left(\frac{y_{m'-1}}{(n-m+1)^p}\right)f\left(\frac{y_{m'}}{(n-m+1)^p}\right)\Big], t\Big)$$

$$\geq \lim_{p\longrightarrow\infty} \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \varphi(0,0,\ldots,0,\frac{y_1}{(n-m+1)^p},\frac{y_2}{(n-m+1)^p},\ldots,\frac{y_{m'}}{(n-m+1)^p})}$$

$$\geq \lim_{p\longrightarrow\infty} \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \frac{L^p\varphi(0,0,\ldots,0,y_1,y_2,\ldots,y_{m'})}{(n-m+1)^p}} = 1$$

for all $y_1, y_2, \ldots, y_{m'} \in X$ and all $t > 0$. Hence

$$N\big(H([[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]) - [[\cdots[[H(y_1)H(y_2)H(y_3)]H(y_4)H(y_5)]\cdots]H(y_{m'-1})H(y_{m'})], t\big) = 1$$

for all $y_1, y_2, \ldots, y_{m'} \in X$ and all $t > 0$. Hence

$$H([[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]) - [[\cdots[[H(y_1)H(y_2)H(y_3)]H(y_4)H(y_5)]\cdots]H(y_{m'-1})H(y_{m'})]$$

for all $y_1, y_2, \ldots, y_{m'} \in X$. This means that H is an m'-homomorphism. This completes the proof. □

Theorem 2.2 *Let $\varphi : X^{n+m'} \to [0,\infty)$ be a mapping such that there exists an $L < 1$ with*

$$\varphi(x_1,\ldots,x_{n+m'}) \leq (n-m+1)L\varphi\left(\frac{x_1}{n-m+1},\ldots,\frac{x_{n+m'}}{n-m+1}\right)$$

for all $x_1, x_2, \ldots, x_{n+m'} \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (2.2). *Then the limit $H(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$ exists for all $x \in X$ and defines an m'-homomorphism $H : X \to Y$ such that*

$$N\big(f(x)-H(x),t\big) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + \varphi(x,x,\ldots,x,0,0,\ldots,0)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof Let (S,d) be the metric space defined as in the proof of Theorem 2.1. Consider the mapping $T : S \to S$ by $Tg(x) := \frac{g((n-m+1)x)}{n-m+1}$ for all $x \in X$. One can show that $d(g,h) = \epsilon$ implies that $d(Tg,Th) \leq L\epsilon$ for all positive real numbers ϵ. This means that T is a contraction on (S,d). The mapping

$$H(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$$

is the unique fixed point of T in S. H has the following property:

$$(n-m+1)H(x) = H\big((n-m+1)x\big) \tag{2.8}$$

for all $x \in X$. This implies that H is a unique mapping satisfying (2.8) such that there exists $\mu \in (0,\infty)$ satisfying $N(f(x)-H(x),\mu t) \geq \frac{t}{t+\varphi(x,x,\ldots,x,0,0,\ldots,0)}$ for all $x \in X$ and $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Now, we investigate the Hyers-Ulam-Rassias stability of m'-derivations in ternary fuzzy Banach algebras.

Theorem 2.3 *Let* $\varphi : X^{n+m'} \to [0,\infty)$ *be a mapping such that there exists an* $L < \frac{1}{(n-m+1)^{n-2}}$ *with* (2.1) *Let* $f : X \to Y$ *with* $f(0) = 0$ *be a mapping satisfying*

$$N\big(Df(x_1,\dots,x_{n+m'}),t\big) \geq \frac{t}{t+\varphi(x_1,\dots,x_n,0,0,\dots,0)}, \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$, $x_1,\dots,x_{n+m'} \in X$ *and all* $t > 0$. *Then there exists a unique* m'*-derivation* $D : X \to Y$ *such that*

$$N\big(f(x)-D(x),t\big) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + L\varphi(x,\dots,x,0,0,\dots,0)}. \tag{2.10}$$

for all $x \in X$ *and all* $t > 0$.

Proof By the same reasoning as that in the proof of Theorem 2.1, the mapping $D : X \to X$ defined by

$$D(x) := N\text{-}\lim_{p\to\infty} \frac{f(\frac{x}{(n-m+1)^p})}{(n-m+1)^{-p}} \quad (x \in X),$$

is a unique $\mathbb{C}$-linear mapping which satisfies (2.10). We show that D is an m'-derivation. By (2.9),

$$\begin{aligned}
&N\Bigg(\frac{f(\frac{[[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}}\\
&\quad - \frac{1}{(n-m+1)^{-(n-1)p}}\Bigg(\Bigg[\Bigg[\cdots\Bigg[\Bigg[f\bigg(\frac{y_1}{(n-m+1)^p}\bigg)y_2y_3\Bigg]y_4y_5\Bigg]\cdots\Bigg]y_{m'-1}y_{m'}\Bigg]\\
&\quad + \Bigg[\Bigg[\cdots\Bigg[\Bigg[y_1f\bigg(\frac{y_2}{(n-m+1)^p}\bigg)y_3\Bigg]y_4y_5\Bigg]\cdots\Bigg]y_{m'-1}y_{m'}\Bigg]\\
&\quad + \Bigg[\Bigg[\cdots\Bigg[\Bigg[y_1y_2f\bigg(\frac{y_3}{(n-m+1)^p}\bigg)\Bigg]y_4y_5\Bigg]\cdots\Bigg]y_{m'-1}y_{m'}\Bigg]\Bigg)\\
&\quad - \frac{[[\cdots[[y_1y_2y_3]f(\frac{y_4}{(n-m+1)^p})y_5]\cdots]y_{m'-1}y_{m'}]}{(n-m+1)^{-(n-1)p}}\\
&\quad - \frac{[[\cdots[[y_1y_2y_3]y_4f(\frac{y_5}{(n-m+1)^p})]\cdots]y_{m'-1}y_{m'}]}{(n-m+1)^{-(n-1)p}}\\
&\quad - \cdots - \frac{[[\cdots[[y_1y_2y_3]y_4y_5]\cdots]f(\frac{y_{m'-1}}{(n-m+1)^p})y_{m'}]}{(n-m+1)^{-(n-1)p}}\\
&\quad - \frac{[[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}f(\frac{y_m}{(n-m+1)^p})]}{(n-m+1)^{-(n-1)p}},\\
&\quad \frac{t}{(n-m+1)^{-(n-1)p}}\Bigg)\\
&\quad \geq \frac{t}{t+\varphi(0,0,\dots,0,\frac{y_1}{(n-m+1)^p},\frac{y_2}{(n-m+1)^p},\dots,\frac{y_{m'}}{(n-m+1)^p})}
\end{aligned} \tag{2.11}$$

for all $a, b, c \in X$ and all $t > 0$. Then we have

$$\begin{aligned}
&N\bigl(D([[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]) - [[\cdots[[D(y_1)y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]\\
&\quad - [[\cdots[[y_1D(y_2)y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}] - [[\cdots[[y_1y_2D(y_3)]y_4y_5]\cdots]y_{m'-1}y_{m'}]\\
&\quad - [[\cdots[[y_1y_2y_3]D(y_4)y_5]\cdots]y_{m'-1}y_{m'}] - [[\cdots[[y_1y_2y_3]y_4D(y_5)]y_6\cdots]y_{m'-1}y_{m'}]\\
&\quad - \cdots - [[\cdots[[y_1y_2y_3]y_4y_5]\cdots]D(y_{m'-1})y_{m'}]\\
&\quad - [[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}D(y_{m'})], t\bigr)\\
&= \lim_{p\longrightarrow\infty} N\Biggl(\frac{f(\frac{[[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}y_{m'}]}{(n-m+1)^{(n-1)p}})}{(n-m+1)^{-(n-1)p}}\\
&\quad - \frac{1}{(n-m+1)^{-(n-1)p}}\Biggl(\Biggl[\Biggl[\cdots\Biggl[\Biggl[f\Biggl(\frac{y_1}{(n-m+1)^p}\Biggr)y_2y_3\Biggr]y_4y_5\Biggr]\cdots\Biggr]y_{m'-1}y_{m'}\Biggr]\\
&\quad + \Biggl[\Biggl[\cdots\Biggl[\Biggl[y_1f\Biggl(\frac{y_2}{(n-m+1)^p}\Biggr)y_3\Biggr]y_4y_5\Biggr]\cdots\Biggr]y_{m'-1}y_{m'}\Biggr]\\
&\quad + \Biggl[\Biggl[\cdots\Biggl[\Biggl[y_1y_2f\Biggl(\frac{y_3}{(n-m+1)^p}\Biggr)\Biggr]y_4y_5\Biggr]\cdots\Biggr]y_{m'-1}y_{m'}\Biggr]\Biggr)\\
&\quad - \frac{[[\cdots[[y_1y_2y_3]f(\frac{y_4}{(n-m+1)^p})y_5]\cdots]y_{m'-1}y_{m'}]}{(n-m+1)^{-(n-1)p}}\\
&\quad - \frac{[[\cdots[[y_1y_2y_3]y_4f(\frac{y_5}{(n-m+1)^p})]\cdots]y_{m'-1}y_{m'}]}{(n-m+1)^{-(n-1)p}}\\
&\quad - \cdots - \frac{[[\cdots[[y_1y_2y_3]y_4y_5]\cdots]f(\frac{y_{m'-1}}{(n-m+1)^p})y_{m'}]}{(n-m+1)^{-(n-1)p}}\\
&\quad - \frac{[[\cdots[[y_1y_2y_3]y_4y_5]\cdots]y_{m'-1}f(\frac{y_m}{(n-m+1)^p})]}{(n-m+1)^{-(n-1)p}}, t\Biggr)\\
&\geq \lim_{p\longrightarrow\infty} \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \varphi(0,0,\ldots,0,\frac{y_1}{(n-m+1)^p},\frac{y_2}{(n-m+1)^p},\ldots,\frac{y_{m'}}{(n-m+1)^p})}\\
&\geq \lim_{p\longrightarrow\infty} \frac{\frac{t}{(n-m+1)^{(n-1)p}}}{\frac{t}{(n-m+1)^{(n-1)p}} + \frac{L^p\varphi(0,0,\ldots,0,y_1,y_2,\ldots,y_{m'})}{(n-m+1)^p}} = 1
\end{aligned}$$

for all $y_1, y_2, \ldots, y_{m'} \in X$ and all $t > 0$. It follows that D is an m'-derivation. □

Theorem 2.4 *Let $\varphi : X^{n+m'} \to [0,\infty)$ be a mapping such that there exists an $L < 1$ with*

$$\varphi(x_1,\ldots,x_{n+m'}) \leq (n-m+1)L\varphi\left(\frac{x_1}{n-m+1},\ldots,\frac{x_{n+m'}}{n-m+1}\right)$$

for all $x_1, x_2, \ldots, x_{n+m'} \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (2.9). Then the limit $D(x) := N\text{-}\lim_{p\to\infty} \frac{f((n-m+1)^p x)}{(n-m+1)^p}$ exists for all $x \in X$ and defines an m'-derivation $D : X \to Y$ such that

$$N\bigl(f(x) - D(x), t\bigr) \geq \frac{(n-m+1)\binom{n}{m}(1-L)t}{(n-m+1)\binom{n}{m}(1-L)t + \varphi(x,x,\ldots,x,0,0,\ldots,0)} \tag{2.12}$$

for all $x \in X$ and all $t > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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Received: 2 March 2012 Accepted: 7 February 2013 Published: 27 February 2013

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doi:10.1186/1029-242X-2013-71
Cite this article as: Hassani et al.: **Nearly** ***n*****-homomorphisms and** ***n*****-derivations in fuzzy ternary Banach algebras.** *Journal of Inequalities and Applications* 2013 **2013**:71.