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A new application of quasi power increasing sequences. I

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Abstract

In (Rocky Mt. J. Math. 38:801-807, 2008), we proved a theorem dealing with an application of quasi-σ-power increasing sequences. In the present paper, we prove that theorem under less and more weaker conditions. This theorem also includes some new and known results.

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1 Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. A positive sequence $X = (X_n)$ is said to be a quasi-σ-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^{\sigma}X_n \geq m^{\sigma}X_m$ holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi-σ-power increasing sequence for any nonnegative σ, but the converse may not be true as can be seen by taking an example, say $X_n = n^{-\sigma}$ for $\sigma > 0$ (see [2]). Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by z_n^{α} and t_n^{α} the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is,

$$z_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^{\alpha} = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} = O(n^{\alpha}), \qquad A_{-n}^{\alpha} = 0 \quad \text{for } n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $\varphi - |C,\alpha|_k$, $k \geq 1$ and $\alpha > -1$, if (see [3])

$$\sum_{n=1}^{\infty} \left|\varphi_n\left(z_n^{\alpha} - z_{n-1}^{\alpha}\right)\right|^k = \sum_{n=1}^{\infty} n^{-k} \left|\varphi_n t_n^{\alpha}\right|^k < \infty. \tag{4}$$

In the special case, if we take $\varphi_n = n^{1-\frac{1}{k}}$, then $\varphi - |C,\alpha|_k$ summability is the same as $|C,\alpha|_k$ summability (see [4]). Also, if we take $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then $\varphi - |C,\alpha|_k$ summability reduces to $|C,\alpha;\delta|_k$ summability (see [5]).

2 Known result

In [6], we have proved the following theorem.

Theorem A *Let* $(\lambda_n) \in \mathcal{BV}$ *and let* (X_n) *be a quasi-*σ*-power increasing sequence for some* σ $(0 < \sigma < 1)$. *Suppose also that there exist sequences* (β_n) *and* (λ_n) *such that*

$$|\Delta\lambda_n| \le \beta_n, \tag{5}$$

$$\beta_n \to 0 \quad \text{as } n \to \infty, \tag{6}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{7}$$

$$|\lambda_n|X_n = O(1) \quad \text{as } n \to \infty. \tag{8}$$

If there exists an $\epsilon > 0$ *such that the sequence* $(n^{\epsilon-k}|\varphi_n|^k)$ *is non-increasing and if the sequence* (w_n^α) *defined by* (*see* [7])

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1, \\ \max_{1\le v\le n} |t_v^\alpha|, & 0 < \alpha < 1, \end{cases} \tag{9}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{(|\varphi_n|w_n^\alpha)^k}{n^k} = O(X_m) \quad \text{as } m \to \infty, \tag{10}$$

then the series $\sum a_n\lambda_n$ *is summable* $\varphi - |C,\alpha|_k$, $k \ge 1$, $0 < \alpha \le 1$ *and* $k\alpha + \epsilon > 1$.

It should be remarked that we have added the condition '$(\lambda_n) \in \mathcal{BV}$' in the statement of Theorem A because it is necessary.

3 The main result

The aim of this paper is to prove Theorem A under less and weaker conditions. Now, we will prove the following theorem.

Theorem *Let* (X_n) *be a quasi-*σ*-power increasing sequence for some* σ $(0 < \sigma < 1)$. *If there exists an* $\epsilon > 0$ *such that the sequence* $(n^{\epsilon-k}|\varphi_n|^k)$ *is non-increasing and if the conditions from* (5) *to* (8) *are satisfied and if the condition*

$$\sum_{n=1}^{m} \frac{(|\varphi_n|w_n^\alpha)^k}{n^k X_n^{k-1}} = O(X_m) \quad \text{as } m \to \infty \tag{11}$$

is satisfied, then the series $\sum a_n\lambda_n$ *is summable* $\varphi - |C,\alpha|_k$, $k \ge 1$, $0 < \alpha \le 1$, *and* $k(\alpha - 1) + \epsilon > 1$.

Remark It should be noted that condition (11) is the same as condition (10) when $k = 1$. When $k > 1$, condition (11) is weaker than condition (10), but the converse is not true. As in [8] we can show that if (10) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{(|\varphi_n| w_n^{\alpha})^k}{n^k X_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^{m} \frac{(|\varphi_n| w_n^{\alpha})^k}{n^k} = O(X_m).$$

If (11) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} \frac{(|\varphi_n| w_n^{\alpha})^k}{n^k} = \sum_{n=1}^{m} X_n^{k-1} \frac{(|\varphi_n| w_n^{\alpha})^k}{n^k X_n^{k-1}} = O\left(X_m^{k-1}\right) \sum_{n=1}^{m} \frac{(|\varphi_n| w_n^{\alpha})^k}{n^k X_n^{k-1}} = O\left(X_m^k\right) \neq O(X_m).$$

Also, it should be noted that the condition '$(\lambda_n) \in \mathcal{BV}$' has been removed.

We need the following lemmas for the proof of our theorem.

Lemma 1 [9] *If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$, *then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p\right| \leq \max_{1 \leq m \leq v} \left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p\right|. \tag{12}$$

Lemma 2 [2] *Under the conditions on* (X_n), (β_n) *and* (λ_n), *as expressed in the statement of the theorem, we have the following*:

$$n\beta_n X_n = O(1) \quad \text{as } n \to \infty, \tag{13}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{14}$$

4 Proof of the theorem

Let (T_n^{α}) be the nth (C, α), with $0 < \alpha \leq 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (2), we have

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{15}$$

First, applying Abel's transformation and then using Lemma 1, we get that

$$\begin{aligned}
T_n^{\alpha} &= \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \\
\left|T_n^{\alpha}\right| &\leq \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p\right| + \frac{|\lambda_n|}{A_n^{\alpha}} \left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\right| \\
&\leq \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} A_v^{\alpha} w_v^{\alpha} |\Delta\lambda_v| + |\lambda_n| w_n^{\alpha} \\
&= T_{n,1}^{\alpha} + T_{n,2}^{\alpha}.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k} \left|\varphi_n T_{n,r}^{\alpha}\right|^k < \infty \quad \text{for } r = 1, 2.$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-k} \left|\varphi_n T_{n,1}^{\alpha}\right|^k &\leq \sum_{n=2}^{m+1} n^{-k} \left(A_n^{\alpha}\right)^{-k} |\varphi_n|^k \left\{\sum_{v=1}^{n-1} A_v^{\alpha} w_v^{\alpha} |\Delta\lambda_v|\right\}^k \\
&\leq \sum_{n=2}^{m+1} n^{-k} n^{-\alpha k} |\varphi_n|^k \sum_{v=1}^{n-1} v^{\alpha k} \left(w_v^{\alpha}\right)^k |\Delta\lambda_v|^k \times \left\{\sum_{v=1}^{n-1} 1\right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} \left(w_v^{\alpha}\right)^k (\beta_v)^k \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k} |\varphi_n|^k}{n^{k(\alpha-1)+\epsilon+1}} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} \left(w_v^{\alpha}\right)^k \beta_v (\beta_v)^{k-1} v^{\epsilon-k} |\varphi_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{k(\alpha-1)+\epsilon+1}} \\
&= O(1) \sum_{v=1}^{m} v^{\alpha k} \left(w_v^{\alpha}\right)^k \beta_v (\beta_v)^{k-1} v^{\epsilon-k} |\varphi_v|^k \int_v^{\infty} \frac{dx}{x^{k(\alpha-1)+\epsilon+1}} \\
&= O(1) \sum_{v=1}^{m} \beta_v (\beta_v)^{k-1} \left(w_v^{\alpha} |\varphi_v|\right)^k \\
&= O(1) \sum_{v=1}^{m} \beta_v \left(\frac{1}{vX_v}\right)^{k-1} \left(w_v^{\alpha} |\varphi_v|\right)^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{(|\varphi_r| w_r^{\alpha})^k}{r^k X_r^{k-1}} + O(1) m\beta_m \sum_{v=1}^{m} \frac{(|\varphi_v| w_v^{\alpha})^k}{v^k X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} \left|\Delta(v\beta_v)\right| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} \left|(v+1)\Delta\beta_v - \beta_v\right| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad \text{as } m \to \infty
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2. Finally, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{-k} \left|\varphi_n T_{n,2}^{\alpha}\right|^k &= \sum_{n=1}^{m} |\lambda_n| |\lambda_n|^{k-1} n^{-k} \left(w_n^{\alpha} |\varphi_n|\right)^k \\
&= O(1) \sum_{n=1}^{m} |\lambda_n| \left(\frac{1}{X_n}\right)^{k-1} n^{-k} \left(w_n^{\alpha} |\varphi_n|\right)^k
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{n=1}^{m}\frac{(|\varphi_n|w_n^\alpha)^k}{n^kX_n^{k-1}}+O(1)|\lambda_m|\sum_{n=1}^{m}\frac{(|\varphi_n|w_n^\alpha)^k}{n^kX_n^{k-1}}\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m\\
&= O(1)\sum_{n=1}^{m-1}\beta_nX_n+O(1)|\lambda_m|X_m=O(1)\quad \text{as } m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 2. This completes the proof of the theorem. If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\epsilon = 1$, $\alpha = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$), then we get a new result dealing with $|C,\alpha|_k$ (resp. $|C,1|_k$) summability factors. Also, if we set $\epsilon = 1$ and $\varphi_n = n^{\delta+1-\frac{1}{k}}$, then we get another new result concerning the $|C,\alpha;\delta|_k$ summability factors. Finally, if we take (X_n) as an almost increasing sequence, then we get the result of Bor and Seyhan under weaker conditions (see [10]).

Competing interests

The author declares that he has no competing interests.

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