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Some new inequalities for the Hadamard product of M-matrices

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Abstract
If A and B are $n \times n$ nonsingular M-matrices, a new lower bound for the minimum eigenvalue $\tau(B \circ A^{-1})$ for the Hadamard product of B and A^{-1} is derived. As a consequence, a new lower bound for the minimum eigenvalue $\tau(A \circ A^{-1})$ for the Hadamard product of A and its inverse A^{-1} is given. Theoretical results and an example demonstrate that the new bounds are better than some existing ones.
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Keywords: M-matrix; lower bounds; Hadamard product; minimum eigenvalue

1 Introduction

For convenience, for any positive integer n, let $N = \{1, 2, \dots, n\}$ throughout. The set of all $n \times n$ real matrices is denoted by $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ complex matrices.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a nonnegative matrix if $a_{ij} \geq 0$. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A)$ is an eigenvalue of A.

Z_n denotes the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. An $n \times n$ matrix A is called an M-matrix if there exists an $n \times n$ nonnegative matrix B and a nonnegative real number λ such that $A = \lambda I - B$ and $\lambda \geq \rho(B)$, I is the identity matrix; if $\lambda > \rho(B)$, we call A a nonsingular M-matrix; if $\lambda = \rho(B)$, we call A a singular M-matrix. Denote by M_n the set of nonsingular M-matrices.

Let $A \in Z_n$, and let $\tau(A) = \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A)\}$. Basic for our purpose are the following simple facts (see Problems 16, 19 and 28 in Section 2.5 of [1]):

(1) $\tau(A) \in \sigma(A)$; $\tau(A)$ is called the minimum eigenvalue of A.

(2) If $A, B \in M_n$, and $A \geq B$, then $\tau(A) \geq \tau(B)$.

(3) If $A \in M_n$, then $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}, and $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of A.

For two matrices $A = (a_{ij})$ and $B = (b_{ij})$, the Hadamard product of A and B is the matrix $A \circ B = (a_{ij}b_{ij})$. If A and B are two nonsingular M-matrices, then $B \circ A^{-1}$ is also a nonsingular M-matrix [2].

Let $A, B \in M_n$ and $A^{-1} = (\beta_{ij})$, in [1, Theorem 5.7.31] the following classical result is given:

$$\tau\left(B \circ A^{-1}\right) \geq \tau(B) \min_{1\leq i\leq n} \beta_{ii}. \tag{1.1}$$

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Huang [3, Theorem 9] improved this result and obtained the following result:

$$\tau\left(B \circ A^{-1}\right) \geq \frac{1-\rho(J_A)\rho(J_B)}{1+\rho^2(J_A)} \min_{1\leq i\leq n} \frac{b_{ii}}{a_{ii}}, \tag{1.2}$$

where $\rho(J_A)$, $\rho(J_B)$ are the spectral radii of J_A and J_B.

The lower bound (1.1) is simple, but not accurate enough. The lower bound (1.2) is difficult to evaluate.

Recently, Li [4, Theorem 2.1] improved these two results and gave a new lower bound for $\tau(B \circ A^{-1})$, that is,

$$\tau\left(B \circ A^{-1}\right) \geq \min_i \left\{ \frac{b_{ii} - s_i \sum_{k\neq i} |b_{ki}|}{a_{ii}} \right\}, \tag{1.3}$$

where $r_{li} = \frac{|a_{li}|}{|a_{ll}|-\sum_{k\neq l,i}|a_{lk}|}$, $l \neq i$; $r_i = \max_{l\neq i}\{r_{li}\}$, $i \in N$; $s_{ji} = \frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|r_k}{a_{jj}}$, $j \neq i$, $j \in N$; $s_i = \max_{j\neq i}\{s_{ij}\}$, $i \in N$.

For an M-matrix A, Fiedler *et al.* showed in [5] that $\tau(A \circ A^{-1}) \leq 1$. Subsequently, Fiedler and Markham [2, Theorem 3] gave a lower bound on $\tau(A \circ A^{-1})$,

$$\tau\left(A \circ A^{-1}\right) \geq \frac{1}{n}, \tag{1.4}$$

and proposed the following conjecture:

$$\tau\left(A \circ A^{-1}\right) \geq \frac{2}{n}. \tag{1.5}$$

Yong [6] and Song [7] have independently proved this conjecture.

Li [8, Theorem 3.1] obtained the following result:

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - t_i R_i}{1 + \sum_{j\neq i} t_{ji}} \right\}, \tag{1.6}$$

which only depends on the entries of $A = (a_{ij})$, where $R_i = \sum_{k\neq i}|a_{ik}|$; $d_i = \frac{R_i}{|a_{ii}|}$, $i \in N$; $t_{ji} = \frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|d_k}{|a_{jj}|}$, $j \neq i$, $j \in N$; $t_i = \max_{j\neq i}\{t_{ij}\}$, $i \in N$.

Li [9, Theorem 3.2] improved the bound (1.6) and obtained the following result:

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j\neq i} m_{ji}} \right\}, \tag{1.7}$$

where $r_{li} = \frac{|a_{li}|}{|a_{ll}|-\sum_{k\neq l,i}|a_{lk}|}$, $l \neq i$; $r_i = \max_{l\neq i}\{r_{li}\}$, $i \in N$; $m_{ji} = \frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|r_i}{|a_{jj}|}$, $j \neq i$, $j \in N$; $m_i = \max_{j\neq i}\{m_{ij}\}$, $i \in N$.

Recently, Li [10, Theorem 3.2] improved the bound (1.7) and gave a new lower bound for $\tau(A \circ A^{-1})$, that is,

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum_{j\neq i} T_{ji}} \right\}, \tag{1.8}$$

where $T_{ji} = \min\{m_{ji}, s_{ji}\}$, $j \neq i$; $T_i = \max_{j\neq i}\{T_{ij}\}$, $i \in N$.

In the present paper, we present a new lower bound on $\tau(B \circ A^{-1})$. As a consequence, we present a new lower bound on $\tau(A \circ A^{-1})$. These bounds improve several existing results.

The following is the list of notations that we use throughout: For $i, j, k, l \in N$,

$$R_i = \sum_{k \neq i} |a_{ik}|, \qquad C_i = \sum_{k \neq i} |a_{ki}|, \qquad d_i = \frac{R_i}{|a_{ii}|}, \qquad \hat{c}_i = \frac{C_i}{|a_{ii}|};$$

$$r_{li} = \frac{|a_{li}|}{|a_{ll}| - \sum_{k \neq l,i} |a_{lk}|}, \quad l \neq i; \qquad r_i = \max_{l \neq i}\{r_{li}\}, \quad i \in N;$$

$$c_{il} = \frac{|a_{il}|}{|a_{ll}| - \sum_{k \neq l,i} |a_{kl}|}, \quad l \neq i; \qquad c_i = \max_{l \neq i}\{c_{il}\}, \quad i \in N;$$

$$m_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_i}{|a_{jj}|}, \quad j \neq i; \qquad m_i = \max_{j \neq i}\{m_{ij}\}, \quad i \in N;$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| r_k}{|a_{jj}|}, \quad j \neq i; \qquad s_i = \max_{j \neq i}\{s_{ij}\}, \quad i \in N;$$

$$T_{ji} = \min\{m_{ji}, s_{ji}\}, \quad j \neq i; \qquad T_i = \max_{j \neq i}\{T_{ij}\}, \quad i \in N.$$

2 Some lemmas and the main results

In order to prove our results, we first give some lemmas.

Lemma 2.1 [11] *If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an M-matrix, then there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix.*

Lemma 2.2 [1] *Let $A, B = (a_{ij}) \in \mathbb{C}^{n \times n}$ and suppose that $D \in \mathbb{C}^{n \times n}$ and $E \in \mathbb{C}^{n \times n}$ are diagonal matrices. Then*

$$D(A \circ B)E = (DAE) \circ B = (DA) \circ (BE) = (AE) \circ (DB) = A \circ (DBE).$$

Lemma 2.3 [10] *If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a strictly row diagonally dominant M-matrix, then $A^{-1} = (\beta_{ij})$ satisfies*

$$\beta_{ji} \leq T_{ji} \beta_{ii}, \quad i, j \in N, i \neq j.$$

Lemma 2.4 [12] *If A^{-1} is a doubly stochastic matrix, then $Ae = e$, $A^T e = e$, where $e = (1, 1, \ldots, 1)^T$.*

Lemma 2.5 [9] *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (\beta_{ij})$, we have*

$$\beta_{ii} \geq \frac{1}{a_{ii}}, \quad i \in N.$$

Lemma 2.6 [10] *If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an M-matrix and $A^{-1} = (\beta_{ij})$ is a doubly stochastic matrix, then*

$$\beta_{ii} \geq \frac{1}{1 + \sum_{j \neq i} T_{ji}}, \quad i \in N.$$

Lemma 2.7 [13] *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_{\substack{i,j=1\\ i\neq j}}^{n}\left\{z\in\mathbb{C}: |z-a_{ii}||z-a_{jj}| \le \left(x_i\sum_{k\neq i}\frac{1}{x_k}|a_{ki}|\right)\left(x_j\sum_{k\neq j}\frac{1}{x_k}|a_{kj}|\right)\right\}.$$

Theorem 2.1 *Let $A, B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be two nonsingular M-matrices, and let $A^{-1} = (\beta_{ij})$. Then*

$$\tau\left(B\circ A^{-1}\right) \ge \min_{i\neq j}\frac{1}{2}\left\{b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 + 4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}. \quad (2.1)$$

Proof It is evident that (2.1) is an equality for $n = 1$.

We next assume that $n \ge 2$.

If A is an M-matrix, then by Lemma 2.1 we know that there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix and satisfies

$$\tau\left(B\circ A^{-1}\right) = \tau\left(D^{-1}\left(B\circ A^{-1}\right)D\right) = \tau\left(B\circ\left(D^{-1}AD\right)^{-1}\right).$$

So, for convenience and without loss of generality, we assume that A is a strictly row diagonally dominant M-matrix. Therefore, $0 < T_i < 1$, $i \in N$.

If $B\circ A^{-1}$ is irreducible, then B and A are irreducible. Let $\tau(B\circ A^{-1}) = \lambda$, so that $0 < \lambda < b_{ii}\beta_{ii}$, $\forall i \in N$. Thus, by Lemma 2.7, there is a pair (i,j) of positive integers with $i \neq j$ such that

$$|\lambda - b_{ii}\beta_{ii}||\lambda - b_{jj}\beta_{jj}| \le \left(T_i\sum_{k\neq i}\frac{1}{T_k}|b_{ki}\beta_{ki}|\right)\left(T_j\sum_{k\neq j}\frac{1}{T_k}|b_{kj}\beta_{kj}|\right).$$

Observe that

$$\begin{aligned}
&\left(T_i\sum_{k\neq i}\frac{1}{T_k}|b_{ki}\beta_{ki}|\right)\left(T_j\sum_{k\neq j}\frac{1}{T_k}|b_{kj}\beta_{kj}|\right)\\
&\quad\le \left(T_i\sum_{k\neq i}\frac{1}{T_k}|b_{ki}|T_{ki}\beta_{ii}\right)\left(T_j\sum_{k\neq j}\frac{1}{T_k}|b_{kj}|T_{kj}\beta_{jj}\right)\\
&\quad\le \left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right).
\end{aligned}$$

Thus, we have

$$|\lambda - b_{ii}\beta_{ii}||\lambda - b_{jj}\beta_{jj}| \le \left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right).$$

Then we have

$$\lambda \geq \frac{1}{2}\left\{b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 + 4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}.$$

That is,

$$\begin{aligned}\tau\left(B\circ A^{-1}\right) &\geq \frac{1}{2}\left\{b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 \right.\right.\\ &\qquad \left.\left. + 4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}\\ &\geq \min_{i\neq j}\frac{1}{2}\left\{b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 \right.\right.\\ &\qquad \left.\left. + 4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}.\end{aligned}$$

Now, assume that $B\circ A^{-1}$ is reducible. It is known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [14]). If we denote by $D = (d_{ij})$ the $n\times n$ permutation matrix with $d_{12} = d_{23} = \cdots = d_{n-1,n} = d_{n1} = 1$, then both $A - tD$ and $B - tD$ are irreducible nonsingular M-matrices for any chosen positive real number t, sufficiently small such that all the leading principal minors of both $A - tD$ and $B - tD$ are positive. Now we substitute $A - tD$ and $B - tD$ for A and B, respectively in the previous case, and then letting $t \longrightarrow 0$, the result follows by continuity. □

Theorem 2.2 *Let $A, B = (b_{ij}) \in \mathbb{R}^{n\times n}$ be two nonsingular M-matrices, and let $A^{-1} = (\beta_{ij})$. Then*

$$\begin{aligned}&\min_{i\neq j}\frac{1}{2}\left\{b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 + 4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}\\ &\quad\geq \min_{1\leq i\leq n}\left\{\frac{b_{ii} - s_i\sum_{k\neq i}|b_{ki}|}{a_{ii}}\right\}.\end{aligned}$$

Proof Since $T_{ji} = \min\{m_{ji}, s_{ji}\}$, $j\neq i$, $T_i = \max_{j\neq i}\{T_{ij}\}$, so $T_i \leq s_i$, $i\in N$. Without loss of generality, for $i\neq j$, assume that

$$b_{ii}\beta_{ii} - T_i\sum_{k\neq i}|b_{ki}|\beta_{ii} \leq b_{jj}\beta_{jj} - T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}. \tag{2.2}$$

Thus, (2.2) is equivalent to

$$T_j\sum_{k\neq j}|b_{kj}|\beta_{jj} \leq T_i\sum_{k\neq i}|b_{ki}|\beta_{ii} + b_{jj}\beta_{jj} - b_{ii}\beta_{ii}. \tag{2.3}$$

From (2.1) and (2.3), we have

$$\begin{aligned}
&\frac{1}{2}\left\{b_{ii}\beta_{ii}+b_{jj}\beta_{jj}-\left[(b_{ii}\beta_{ii}-b_{jj}\beta_{jj})^2+4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}\\
&\geq\frac{1}{2}\left\{b_{ii}\beta_{ii}+b_{jj}\beta_{jj}-\left[(b_{ii}\beta_{ii}-b_{jj}\beta_{jj})^2\right.\right.\\
&\quad\left.\left.+4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}+b_{jj}\beta_{jj}-b_{ii}\beta_{ii}\right)\right]^{\frac{1}{2}}\right\}\\
&=\frac{1}{2}\left\{b_{ii}\beta_{ii}+b_{jj}\beta_{jj}-\left[(b_{ii}\beta_{ii}-b_{jj}\beta_{jj})^2\right.\right.\\
&\quad\left.\left.+4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)^2+4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)(b_{jj}\beta_{jj}-b_{ii}\beta_{ii})\right]^{\frac{1}{2}}\right\}\\
&=\frac{1}{2}\left\{b_{ii}\beta_{ii}+b_{jj}\beta_{jj}-\left[\left(b_{jj}\beta_{jj}-b_{ii}\beta_{ii}+2T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)^2\right]^{\frac{1}{2}}\right\}\\
&=\frac{1}{2}\left\{b_{ii}\beta_{ii}+b_{jj}\beta_{jj}-\left(b_{jj}\beta_{jj}-b_{ii}\beta_{ii}+2T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\right\}\\
&=b_{ii}\beta_{ii}-T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\\
&=\beta_{ii}\left(b_{ii}-T_i\sum_{k\neq i}|b_{ki}|\right)\\
&\geq\beta_{ii}\left(b_{ii}-s_i\sum_{k\neq i}|b_{ki}|\right)\\
&\geq\frac{b_{ii}-s_i\sum_{k\neq i}|b_{ki}|}{a_{ii}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\tau\left(B\circ A^{-1}\right)&\geq\min_{i\neq j}\frac{1}{2}\left\{b_{ii}\beta_{ii}+b_{jj}\beta_{jj}-\left[(b_{ii}\beta_{ii}-b_{jj}\beta_{jj})^2\right.\right.\\
&\quad\left.\left.+4\left(T_i\sum_{k\neq i}|b_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|b_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}\\
&\geq\min_{1\leq i\leq n}\left\{\frac{b_{ii}-s_i\sum_{k\neq i}|b_{ki}|}{a_{ii}}\right\}.
\end{aligned}$$

This proof is completed. □

Remark 2.1 Theorem 2.2 shows that the result of Theorem 2.1 is better than the result of Theorem 2.1 in [4].

If $A = B$, according to Theorem 2.1, we can obtain the following corollary.

Corollary 2.1 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix, and let $A^{-1} = (\beta_{ij})$ be a doubly stochastic matrix. Then*

$$\tau\left(A \circ A^{-1}\right) \geq \min_{i\neq j} \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4\left(T_i \sum_{k\neq i} |a_{ki}|\beta_{ii}\right)\left(T_j \sum_{k\neq j} |a_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}. \tag{2.4}$$

Theorem 2.3 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix, and let $A^{-1} = (\beta_{ij})$ be a doubly stochastic matrix. Then*

$$\begin{aligned}&\min_{i\neq j} \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4\left(T_i \sum_{k\neq i} |a_{ki}|\beta_{ii}\right)\left(T_j \sum_{k\neq j} |a_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}\\ &\quad\geq \min_i \left\{\frac{a_{ii} - T_i R_i}{1 + \sum_{j\neq i} T_{ji}}\right\}.\end{aligned}$$

Proof Since A is an irreducible M-matrix and A^{-1} is a doubly stochastic matrix by Lemma 2.4, we have

$$a_{ii} = \sum_{k\neq i} |a_{ik}| + 1 = \sum_{k\neq i} |a_{ki}| + 1, \quad i \in N.$$

Without loss of generality, for $i \neq j$, assume that

$$a_{ii}\beta_{ii} - T_i \sum_{k\neq i} |a_{ki}|\beta_{ii} \leq a_{jj}\beta_{jj} - T_j \sum_{k\neq j} |a_{kj}|\beta_{jj}. \tag{2.5}$$

Thus, (2.5) is equivalent to

$$T_j \sum_{k\neq j} |a_{kj}|\beta_{jj} \leq a_{jj}\beta_{jj} - a_{ii}\beta_{ii} + T_i \sum_{k\neq i} |a_{ki}|\beta_{ii}. \tag{2.6}$$

From (2.4) and (2.6), we have

$$\begin{aligned}&\frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4\left(T_i \sum_{k\neq i} |a_{ki}|\beta_{ii}\right)\left(T_j \sum_{k\neq j} |a_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}\\ &\quad\geq \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2\right.\right.\\ &\qquad\left.\left. + 4\left(T_i \sum_{k\neq i} |a_{ki}|\beta_{ii}\right)\left(a_{jj}\beta_{jj} - a_{ii}\beta_{ii} + T_i \sum_{k\neq i} |a_{ki}|\beta_{ii}\right)\right]^{\frac{1}{2}}\right\}\\ &\quad= \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2\right.\right.\\ &\qquad\left.\left. + 4\left(T_i \sum_{k\neq i} |a_{ki}|\beta_{ii}\right)^2 + 4\left(T_i \sum_{k\neq i} |a_{ki}|\beta_{ii}\right)(a_{jj}\beta_{jj} - a_{ii}\beta_{ii})\right]^{\frac{1}{2}}\right\}\end{aligned}$$

$$= \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[\left(a_{jj}\beta_{jj} - a_{ii}\beta_{ii} + 2T_i\sum_{k\neq i}|a_{ki}|\beta_{ii}\right)^2\right]^{\frac{1}{2}}\right\}$$

$$= \frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left(a_{jj}\beta_{jj} - a_{ii}\beta_{ii} + 2T_i\sum_{k\neq i}|a_{ki}|\beta_{ii}\right)\right\}$$

$$= a_{ii}\beta_{ii} - T_i\sum_{k\neq i}|a_{ki}|\beta_{ii}$$

$$= \beta_{ii}\left(a_{ii} - T_i\sum_{k\neq i}|a_{ki}|\right)$$

$$\geq \frac{a_{ii} - T_iR_i}{1 + \sum_{j\neq i}T_{ji}}.$$

Thus, we have

$$\tau\left(A\circ A^{-1}\right) \geq \min_{i\neq j}\frac{1}{2}\left\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4\left(T_i\sum_{k\neq i}|a_{ki}|\beta_{ii}\right)\left(T_j\sum_{k\neq j}|a_{kj}|\beta_{jj}\right)\right]^{\frac{1}{2}}\right\}$$

$$\geq \min_i\left\{\frac{a_{ii} - T_iR_i}{1 + \sum_{j\neq i}T_{ji}}\right\}.$$

This proof is completed. □

Remark 2.2 Theorem 2.3 shows that the result of Corollary 2.1 is better than the result of Theorem 3.2 in [10].

3 Example

For convenience, we consider that the M-matrices A and B are the same as the matrices of [4].

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ 0 & 0 & -0.5 & 1 \end{bmatrix}.$$

(1) We consider the lower bound for $\tau(B\circ A^{-1})$.

If we apply (1.1), we have

$$\tau\left(B\circ A^{-1}\right) \geq \tau(B)\min_{1\leq i\leq n}\beta_{ii} = 0.07.$$

If we apply (1.2), we have

$$\tau\left(B\circ A^{-1}\right) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_A)}\min_{1\leq i\leq n}\frac{b_{ii}}{a_{ii}} = 0.048.$$

If we apply (1.3), we have

$$\tau\left(B \circ A^{-1}\right) \geq \min_i \left\{ \frac{b_{ii} - s_i \sum_{k \neq i} |b_{ki}|}{a_{ii}} \right\} = 0.08.$$

If we apply Theorem 2.1, we have

$$\tau\left(B \circ A^{-1}\right) \geq \min_{i \neq j} \frac{1}{2} \left\{ b_{ii}\beta_{ii} + b_{jj}\beta_{jj} - \left[(b_{ii}\beta_{ii} - b_{jj}\beta_{jj})^2 + 4\left(T_i \sum_{k \neq i} |b_{ki}|\beta_{ii} \right)\left(T_j \sum_{k \neq j} |b_{kj}|\beta_{jj} \right) \right]^{\frac{1}{2}} \right\} = 0.1753.$$

In fact, $\tau(B \circ A^{-1}) = 0.2148$.

(2) We consider the lower bound for $\tau(A \circ A^{-1})$.

If we apply (1.5), we have

$$\tau\left(A \circ A^{-1}\right) \geq \frac{2}{n} = \frac{1}{2} = 0.5.$$

If we apply (1.6), we have

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - t_i R_i}{1 + \sum_{j \neq i} t_{ji}} \right\} = 0.6624.$$

If we apply (1.7), we have

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - m_i R_i}{1 + \sum_{j \neq i} m_{ji}} \right\} = 0.7999.$$

If we apply (1.8), we have

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - T_i R_i}{1 + \sum_{j \neq i} T_{ji}} \right\} = 0.85.$$

If we apply Corollary 2.1, we have

$$\tau\left(A \circ A^{-1}\right) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii}\beta_{ii} + a_{jj}\beta_{jj} - \left[(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4\left(T_i \sum_{k \neq i} |a_{ki}|\beta_{ii} \right)\left(T_j \sum_{k \neq j} |a_{kj}|\beta_{jj} \right) \right]^{\frac{1}{2}} \right\} = 0.9755.$$

In fact, $\tau(A \circ A^{-1}) = 0.9755$.

Remark 3.1 The numerical example shows that the bounds of Theorem 2.1 and Corollary 2.1 are sharper than those of Theorem 2.1 in [4] and Theorem 3.2 in [10].

Competing interests
The author declares that he has no competing interests.

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