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A remark on algebraic curves derived from convolution sums

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Abstract

Hahn (Rocky Mt. J. Math. 37:1593-1622, 2007) established three differential equations according to $\mathcal{P}(q)$, $\mathcal{E}(q)$, and $\mathcal{Q}(q)$, which allows us to obtain the values of the formulas for

$$\sum_{l+m+n=N} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(n), \qquad \sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\tilde{\sigma}(n),$$

etc. Finally, by using the above equations, we derive the algebraic curves.

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1 Introduction

Let $\mathbb{N}$ denote the sets of positive integers. In number theory, divisor functions are defined as

$$\sigma_s(N) = \sum_{d|N} d^s, \qquad \sigma(N) := \sigma_1(N) = \sum_{d|N} d,$$
$$\tilde{\sigma}_s(n) = \sum_{d|n} (-1)^{d-1} d^s, \qquad \hat{\sigma}_s(n) = \sum_{d|n} (-1)^{(n/d)-1} d^s$$

for $N, s, d \in \mathbb{N}$. The convolution sum is special one of divisor functions begun by Ramanujan's effort and expanded by many authors; *e.g.*, see [1]. For example,

$$\sum_{m=1}^{n-1} \sigma(m)\sigma(n-m) = \frac{1}{12}\big(5\sigma_3(n) + (1-6n)\sigma(n)\big) \tag{1}$$

is the result of Ramanujan and also Huard, Ou, Spearman, and Williams [1, (3.10)]. We can see

$$\sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m) = \frac{n}{24}\big(5\sigma_3(n) + (1-6n)\sigma(n)\big), \tag{2}$$

$$\sum_{m=1}^{n-1} \sigma(m)\sigma_3(n-m) = \frac{1}{240}\big(21\sigma_5(n) + (10-30n)\sigma_3(n) - \sigma(n)\big) \tag{3}$$

in [1, (3.11), (3.12)], respectively. Moreover, there are

$$\sum_{m<n/2} \sigma(m)\sigma(n-2m) = \frac{1}{24}\left(2\sigma_3(n) + (1-3n)\sigma(n) + 8\sigma_3\left(\frac{n}{2}\right) + (1-6n)\sigma\left(\frac{n}{2}\right)\right) \tag{4}$$

in [1, (4.4)] and

$$\begin{aligned}
\sum_{m<n/2} \sigma_3(m)\sigma(n-2m) &= \frac{1}{240}\left(\sigma_5(n) - \sigma(n) + 20\sigma_5\left(\frac{n}{2}\right) + (10-30n)\sigma_3\left(\frac{n}{2}\right)\right), \\
\sum_{m<n/2} \sigma(m)\sigma_3(n-2m) &= \frac{1}{240}\left(5\sigma_5(n) + (10-15n)\sigma_3(n) + 16\sigma_5\left(\frac{n}{2}\right) - \sigma\left(\frac{n}{2}\right)\right)
\end{aligned} \tag{5}$$

in [1, Theorem 6]. In addition, Lahiri [2] gave the value of the sum

$$\sum_{m_1+\cdots+m_r=n} m_1^{a_1}\cdots m_r^{a_r}\sigma_{b_1}(m_1)\cdots\sigma_{b_r}(m_r) \quad (r \geq 3),$$

where the sum is over all positive integers $m_1,\dots,m_r$ satisfying $m_1 + \cdots + m_r = n$, $a_i \in \mathbb{N} \cup \{0\}$, and $b_i \in \mathbb{N}$. In this paper we substitute $\hat{\sigma}_{b_i}(m_i)$ for $\sigma_{b_i}(m_i)$ and obtain the formulas as Lahiri's evaluation. So, we refer to Hahn's paper [3]. Officially, we look into Hanh's definition of three functions for $|q| < 1$,

$$\mathcal{P}(q) := 1 + 8\sum_{n=1}^{\infty} \tilde{\sigma}(n)q^n, \tag{6}$$

$$\mathcal{E}(q) := 1 + 24\sum_{n=1}^{\infty} \hat{\sigma}(n)q^n, \tag{7}$$

$$\mathcal{Q}(q) := 1 - 16\sum_{n=1}^{\infty} \tilde{\sigma}_3(n)q^n. \tag{8}$$

Three functions (6), (7), and (8) satisfy the following differential equations:

$$q\frac{d\mathcal{P}(q)}{dq} = \frac{\mathcal{P}^2(q) - \mathcal{Q}(q)}{4}, \tag{9}$$

$$q\frac{d\mathcal{E}(q)}{dq} = \frac{\mathcal{E}(q)\mathcal{P}(q) - \mathcal{Q}(q)}{2}, \tag{10}$$

$$q\frac{d\mathcal{Q}(q)}{dq} = \mathcal{P}(q)\mathcal{Q}(q) - \mathcal{E}(q)\mathcal{Q}(q). \tag{11}$$

The paper is organized as follows. In Section 2, we obtain the values of the formulas for

$$\sum_{l+m+n=N} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(n),$$

$$\sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\tilde{\sigma}(n),$$

etc. and insist on the following. Let

$$\begin{aligned} f(N) \in \Biggl\{ & \sum_{l+m+n=N} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(n), \sum_{m=1}^{N-1} \hat{\sigma}(m)\hat{\sigma}(N-m), \\ & \sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(n), \sum_{m=1}^{N-1} \hat{\sigma}(m)\tilde{\sigma}_3(N-m), \\ & \sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\tilde{\sigma}(n), \sum_{m=1}^{N-1} m\hat{\sigma}(m)\hat{\sigma}(N-m) \Biggr\}. \end{aligned}$$

If N is an odd integer and $n \in \mathbb{N} \cup \{0\}$, then there exist $u, a, b, c, d, e, g \in \mathbb{Z}$ satisfying

$$f(N) = \frac{1}{u}\left[a\sigma_5(N) + (bN + c)\sigma_3(N) + \left(dN^2 + eN + g\right)\sigma(N)\right]$$

with $a + b + c + d + e + g = 0$ (see Theorem 2.9).

In Section 2, we derive the convolution sums of the restricted divisor functions. In Section 3, we obtain the algebraic curves by using the convolution sums in Section 2.

2 Convolution sum $\sum_{l+m+n=N} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(n)$

Theorem 2.1 *Let N (≥ 3) be any positive integer. Then we have*

$$\sum_{l+m+n=N} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(n) = \frac{1}{64}\left\{\tilde{\sigma}_5(N) + 6(1-N)\tilde{\sigma}_3(N) + \left(8N^2 - 12N + 3\right)\tilde{\sigma}(N)\right\}.$$

Proof Multiplying $\mathcal{P}(q)$ on both sides in (9), we obtain

$$\mathcal{P}^3(q) = 4\mathcal{P}(q) \cdot q\frac{d\mathcal{P}(q)}{dq} + \mathcal{P}(q)\mathcal{Q}(q). \tag{12}$$

Employing the definition of $\mathcal{P}(q)$ and $\mathcal{Q}(q)$, we can rewrite (12) as

$$\begin{aligned} \left(1 + 8\sum_{n=1}^{\infty} \tilde{\sigma}(n)q^n\right)^3 = {} & 4\left(1 + 8\sum_{n=1}^{\infty} \tilde{\sigma}(n)q^n\right) q \left(8\sum_{m=1}^{\infty} m\tilde{\sigma}(m)q^{m-1}\right) \\ & + \left(1 + 8\sum_{n=1}^{\infty} \tilde{\sigma}(n)q^n\right)\left(1 - 16\sum_{m=1}^{\infty} \tilde{\sigma}_3(m)q^m\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} & 512\sum_{N=3}^{\infty}\sum_{L=2}^{N-1}\sum_{l=1}^{L-1} \tilde{\sigma}(N-L)\tilde{\sigma}(L-l)\tilde{\sigma}(l)q^N \\ & = -16\sum_{N=1}^{\infty}\left\{\tilde{\sigma}(N) + \tilde{\sigma}_3(N) - 2N\tilde{\sigma}(N)\right\}q^N - 192\sum_{N=2}^{\infty}\sum_{l=1}^{N-1} \tilde{\sigma}(l)\tilde{\sigma}(N-l)q^N \\ & \quad - 128\sum_{N=2}^{\infty}\sum_{l=1}^{N-1} \tilde{\sigma}(l)\tilde{\sigma}_3(N-l)q^N + 256\sum_{N=2}^{\infty}\sum_{l=1}^{N-1} l\tilde{\sigma}(l)\tilde{\sigma}(N-l)q^N. \end{aligned}$$

Then we refer to

$$4\sum_{m<n}\tilde{\sigma}(m)\tilde{\sigma}(n-m) = -\tilde{\sigma}_3(n) + (2n-1)\tilde{\sigma}(n)$$

in [3, (4.4)],

$$16\sum_{m<n}\tilde{\sigma}(m)\tilde{\sigma}_3(n-m) = -\tilde{\sigma}_5(n) + 2(n-1)\tilde{\sigma}_3(n) + \tilde{\sigma}(n)$$

in [3, (4.9)] and by direct calculation we get

$$\sum_{l=1}^{N-1} l\tilde{\sigma}(l)\tilde{\sigma}(N-l) = \frac{N}{2}\sum_{l=1}^{N-1}\tilde{\sigma}(l)\tilde{\sigma}(N-l).$$

This completes the proof of the theorem. □

Corollary 2.2 *We obtain*

$$\ln\bigl|\mathcal{Q}(q)\bigr| = 8\sum_{n=1}^{\infty}\frac{\tilde{\sigma}(n)-3\hat{\sigma}(n)}{n}q^n,$$

where $\mathcal{Q}(q) = 1 - 16\sum_{n=1}^{\infty}\tilde{\sigma}_3(n)q^n$ *as* (8).

Proof By separating the variables in (11), we can have

$$-\int_0^q \frac{d\mathcal{Q}(x)}{\mathcal{Q}(x)} = \int_0^q \frac{\mathcal{E}(x)-\mathcal{P}(x)}{x}\,dx.$$

Thus,

$$-\ln\bigl|\mathcal{Q}(q)\bigr| = \int_0^q 8\sum_{n=1}^{\infty}\bigl(3\hat{\sigma}(n)-\tilde{\sigma}(n)\bigr)x^{n-1}\,dx.$$

We note that $3\hat{\sigma}(n)-\tilde{\sigma}(n)$ is positive because

$$3\hat{\sigma}(n)-\tilde{\sigma}(n) = 2\bigl(\sigma(n)-\sigma(n/2)\bigr)$$

according to

$$\tilde{\sigma}_s(n) = \sigma_s(n) - 2^{s+1}\sigma_s(n/2) \quad \text{and} \quad \hat{\sigma}_s(n) = \sigma_s(n) - 2\sigma_s(n/2)$$

in [3, (1.12), (1.13)]. Therefore, we can exchange the summation and the integral. This completes the proof of the corollary. □

We introduce the following Lemma 2.3 to deduce Theorem 2.6.

Lemma 2.3 *Let* N (≥ 2) *be any positive integer. Then we have*

(a) $\sum_{m=1}^{N-1}\hat{\sigma}(m)\hat{\sigma}(N-m) = \frac{1}{12}(\sigma_3(N) + 4\sigma_3(\frac{N}{2}) - \hat{\sigma}(N))$.

(b) $\sum_{m=1}^{N-1}\hat{\sigma}(m)\tilde{\sigma}_3(N-m) = -\frac{1}{48}(\tilde{\sigma}_5(N) + 2\tilde{\sigma}_3(N) - 3\hat{\sigma}(N))$.

(c) $\sum_{m=1}^{N-1}m\hat{\sigma}(m)\hat{\sigma}(N-m) = \frac{N}{24}(\sigma_3(N) + 4\sigma_3(\frac{N}{2}) - \hat{\sigma}(N))$.

Proof

(a) The proof is the same as Remark in [3, p.13]. Also, Hahn showed that

$$36\sum_{m<n}\hat{\sigma}(m)\hat{\sigma}(n-m)=\begin{cases}-3\hat{\sigma}(n)+3\tilde{\sigma}_3(n), & \text{if } n \text{ is odd},\\ -3\hat{\sigma}(n)-5\tilde{\sigma}_3(n)+4\tilde{\sigma}_3(n/2), & \text{if } n \text{ is even}\end{cases}$$

in [3, Theorem 4.2] derived by the identity $\mathcal{E}^2(q)=z^4(1+x)^2$, which is the same result (a).

(b) The convolution sum can be written as

$$\begin{aligned}\sum_{m=1}^{N-1}\hat{\sigma}(m)\tilde{\sigma}_3(N-m)&=\sum_{m=1}^{N-1}\left\{\sigma(m)-2\sigma\left(\frac{m}{2}\right)\right\}\left\{\sigma_3(N-m)-16\sigma_3\left(\frac{N-m}{2}\right)\right\}\\&=\sum_{m=1}^{N-1}\sigma(m)\sigma_3(N-m)-16\sum_{m=1}^{N-1}\sigma(m)\sigma_3\left(\frac{N-m}{2}\right)\\&\quad-2\sum_{m=1}^{N-1}\sigma\left(\frac{m}{2}\right)\sigma_3(N-m)+32\sum_{m=1}^{N-1}\sigma\left(\frac{m}{2}\right)\sigma_3\left(\frac{N-m}{2}\right).\end{aligned}$$

Then we obtain

$$\sum_{m=1}^{N-1}\sigma(m)\sigma_3\left(\frac{N-m}{2}\right)=\sum_{t<N/2}\sigma(N-2t)\sigma_3(t),$$

$$\sum_{m=1}^{N-1}\sigma\left(\frac{m}{2}\right)\sigma_3(N-m)=\sum_{t<N/2}\sigma(t)\sigma_3(N-2t),$$

$$\sum_{m=1}^{N-1}\sigma\left(\frac{m}{2}\right)\sigma_3\left(\frac{N-m}{2}\right)=\sum_{t<N/2}\sigma(t)\sigma_3\left(\frac{N}{2}-t\right).$$

Therefore, we have proved (b) by using (3) and (5).

(c) We obtain the proof by direct calculation of the index m. □

Corollary 2.4 *Let $N=2q+1$ be an odd prime in Lemma* 2.3 *and let $T_i(n)=\sum_{k=1}^{n}k^i$ $(i\geq 1)$. Then we have*

(a) $\sum_{m=1}^{2q}\hat{\sigma}(m)\hat{\sigma}(2q+1-m)=2T_2(q)$.
(b) $\sum_{m=1}^{2q}\hat{\sigma}(m)\tilde{\sigma}_3(2q+1-m)=-2(q^2+q+1)T_2(q)$.
(c) $\sum_{m=1}^{2q}m\hat{\sigma}(m)\hat{\sigma}(2q+1-m)=NT_2(q)$.

Proof From Lemma 2.3(a), (b), and (c), we obtain

$$\sum_{m=1}^{2q}\hat{\sigma}(m)\hat{\sigma}(2q+1-m)=\frac{1}{12}\left\{(2q+1)^3-(2q+1)\right\},$$

$$\sum_{m=1}^{2q}\hat{\sigma}(m)\tilde{\sigma}_3(2q+1-m)=-\frac{1}{48}\left\{(2q+1)^5+2(2q+1)^3-3(2q+1)\right\},$$

$$\sum_{m=1}^{2q}m\hat{\sigma}(m)\hat{\sigma}(2q+1-m)=\frac{2q+1}{24}\left\{(2q+1)^3-(2q+1)\right\}.$$

This completes the proof of the corollary. □

Table 1 Examples for $\sum_{m=1}^{2q}\hat{\sigma}(m)\hat{\sigma}(2q+1-m)$ and $2T_2(q)$ $(1\leq q\leq 11)$

q	1	2	3	4	5	6	7	8	9	10	11
$\sum_{m=1}^{2q}\hat{\sigma}(m)\hat{\sigma}(2q+1-m)$	2	10	28	62	110	182	292	408	570	800	1,012
$2T_2(q)$	2	10	28	60	110	182	280	408	570	770	1,012

Example 2.5 The first eleven values of $\sum_{m=1}^{2q}\hat{\sigma}(m)\hat{\sigma}(2q+1-m)$ and $2T_2(q)$ are listed in Table 1.

Theorem 2.6 *Let N (≥ 3) be any positive integer. Then we have*

$$\sum_{l+m+n=N}\hat{\sigma}(l)\hat{\sigma}(m)\tilde{\sigma}(n)=\frac{1}{576}\biggl\{\tilde{\sigma}_5(N)+4\tilde{\sigma}_3(N)+\tilde{\sigma}(N)-6(2N-1)\hat{\sigma}(N)+6(N-1)\biggl(\sigma_3(N)+4\sigma_3\biggl(\frac{N}{2}\biggr)\biggr)\biggr\}.$$

Proof Multiplying $\mathcal{E}(q)$ in (10), we obtain

$$\mathcal{E}^2(q)\mathcal{P}(q)=\mathcal{E}(q)\mathcal{Q}(q)+2\mathcal{E}(q)\cdot q\frac{d\mathcal{E}(q)}{dq}.$$

So, we have

$$\begin{aligned}&\left(1+24\sum_{n=1}^{\infty}\hat{\sigma}(n)q^n\right)^2\left(1+8\sum_{m=1}^{\infty}\tilde{\sigma}(m)q^m\right)\\&=\left(1+24\sum_{n=1}^{\infty}\hat{\sigma}(n)q^n\right)\left(1-16\sum_{m=1}^{\infty}\tilde{\sigma}_3(m)q^m\right)\\&\quad+2\left(1+24\sum_{n=1}^{\infty}\hat{\sigma}(n)q^n\right)q\left(24\sum_{m=1}^{\infty}m\hat{\sigma}(m)q^{m-1}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&4{,}608\sum_{N=3}^{\infty}\sum_{L=2}^{N-1}\sum_{l=1}^{L-1}\hat{\sigma}(N-L)\hat{\sigma}(L-l)\tilde{\sigma}(l)q^N\\&=-8\sum_{N=1}^{\infty}\bigl\{\tilde{\sigma}(N)+2\tilde{\sigma}_3(N)+3\hat{\sigma}(N)-6N\hat{\sigma}(N)\bigr\}q^N\\&\quad-384\sum_{N=2}^{\infty}\sum_{l=1}^{N-1}\hat{\sigma}(l)\tilde{\sigma}(N-l)q^N-384\sum_{N=2}^{\infty}\sum_{l=1}^{N-1}\hat{\sigma}(l)\tilde{\sigma}_3(N-l)q^N\\&\quad-576\sum_{N=2}^{\infty}\sum_{l=1}^{N-1}\hat{\sigma}(l)\hat{\sigma}(N-l)q^N+1{,}152\sum_{N=2}^{\infty}\sum_{l=1}^{N-1}l\hat{\sigma}(l)\hat{\sigma}(N-l)q^N.\end{aligned}$$

Lastly, we use Lemma 2.3. □

Theorem 2.7 *Let N (≥ 3) be any positive integer. Then we have*

$$\sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(n) = \frac{1}{192}\left\{\sigma_5(N) - 8\sigma_5\left(\frac{N}{2}\right) - 2\sigma_3(N) - 8\sigma_3\left(\frac{N}{2}\right) + \hat{\sigma}(N)\right\}.$$

Proof Let us consider

$$\begin{aligned}
&\sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(n) \\
&= \sum_{l=1}^{N-2} \hat{\sigma}(l) \sum_{m+n=N-l} \hat{\sigma}(m)\hat{\sigma}(n) \\
&= \sum_{l=1}^{N-2} \hat{\sigma}(l)\frac{1}{12}\left\{\sigma_3(N-l) + 4\sigma_3\left(\frac{N-l}{2}\right) - \hat{\sigma}(N-l)\right\} \\
&= \sum_{l=1}^{N-1} \hat{\sigma}(l)\frac{1}{12}\left\{\sigma_3(N-l) + 4\sigma_3\left(\frac{N-l}{2}\right) - \hat{\sigma}(N-l)\right\} \qquad (13)
\end{aligned}$$

by Lemma 2.3(a). Then (13) becomes

$$\begin{aligned}
\sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(n) = \frac{1}{12}\Bigg\{&\sum_{l=1}^{N-1} \sigma(l)\sigma_3(N-l) - 2\sum_{l<N/2} \sigma(l)\sigma_3(N-2l) \\
&+ 4\sum_{t<N/2} \sigma_3(t)\sigma(N-2t) - 8\sum_{t<N/2} \sigma(t)\sigma_3\left(\frac{N}{2}-t\right) \\
&- \sum_{l=1}^{N-1} \hat{\sigma}(l)\hat{\sigma}(N-l)\Bigg\}
\end{aligned}$$

by $\hat{\sigma}(n) = \sigma(n) - 2\sigma(\frac{n}{2})$. □

Remark 2.8 Let $N = 2q+1$ be an odd prime in Theorem 2.7. Then we have

$$\sum_{l+m+n=2q+1} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(n) = T_1(q) \cdot T_2(q).$$

Proof It is obvious. □

Importantly, we propose the following theorem.

Theorem 2.9 *Let*

$$\begin{aligned}
f \in \Bigg\{&\sum_{l+m+n=N} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(n), \sum_{m=1}^{N-1} \hat{\sigma}(m)\hat{\sigma}(N-m), \\
&\sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(n), \sum_{m=1}^{N-1} \hat{\sigma}(m)\tilde{\sigma}_3(N-m), \\
&\sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\tilde{\sigma}(n), \sum_{m=1}^{N-1} m\hat{\sigma}(m)\hat{\sigma}(N-m)\Bigg\}.
\end{aligned}$$

Table 2 Formulas for $f(N)$ with odd N

	$f(N)$
$\sum_{l+m+n=N} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(n)$	$\frac{1}{64}(\sigma_5(N) + 6(1-N)\sigma_3(N) + (8N^2 - 12N + 3)\sigma(N))$
$\sum_{m=1}^{N-1} \hat{\sigma}(m)\hat{\sigma}(N-m)$	$\frac{1}{12}(\sigma_3(N) - \sigma(N))$
$\sum_{m=1}^{N-1} \hat{\sigma}(m)\tilde{\sigma}_3(N-m)$	$-\frac{1}{48}(\sigma_5(N) + 2\sigma_3(N) - 3\sigma(N))$
$\sum_{m=1}^{N-1} m\hat{\sigma}(m)\hat{\sigma}(N-m)$	$\frac{N}{24}(\sigma_3(N) - \sigma(N))$
$\sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\tilde{\sigma}(n)$	$\frac{1}{576}(\sigma_5(N) + (6N-2)\sigma_3(N) + (7-12N)\sigma(N))$
$\sum_{l+m+n=N} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(n)$	$\frac{1}{192}(\sigma_5(N) - 2\sigma_3(N) + \sigma(N))$

If N is an odd integer and $n \in \mathbb{N} \cup \{0\}$, then there exist $u, a, b, c, d, e, g \in \mathbb{Z}$ satisfying

$$f(N) = \frac{1}{u}\big[a\sigma_5(N) + (bN + c)\sigma_3(N) + \big(dN^2 + eN + g\big)\sigma(N)\big]$$

with $a + b + c + d + e + g = 0$.

Proof It is satisfied by Theorem 2.1, Lemma 2.3, Theorem 2.6, and Theorem 2.7. □

The expressions are shown in Table 2.

3 Algebraic curves derived from convolution sums

To obtain the result of this section, we need a general theory and it is this that we describe. Suppose that the two polynomials

$$f_1(x) = c_0x^n + \cdots + c_{n-1}x + c_n,$$
$$f_2(x) = d_0x^m + \cdots + d_{m-1}x + d_m$$

have common zero, say x_0. Then each of the equations

$$f_1(x) = xf_1(x) = \cdots = x^{m-1}f_1(x) = 0 = f_2(x) = xf_2(x) = \cdots = x^{n-1}f_2(x)$$

is of the form $p(x) = 0$, where p is a polynomial of degree at most $m + n - 1$, and as each of these equations is satisfied when $x = x_0$, the determinant of the coefficients must vanish. This $(m + n) \times (m + n)$ determinant is the resultant $R(f_1, f_2)$ of f_1 and g_1 and

$$R(f_1,f_2) = \begin{vmatrix} c_0 & \cdots & \cdots & c_n & & & \\ & \ddots & & & & \ddots & \\ & & & c_0 & \cdots & \cdots & c_n \\ d_0 & \cdots & \cdots & d_m & & & \\ & \ddots & & & & \ddots & \\ & & & d_0 & \cdots & \cdots & d_m \end{vmatrix},$$

where the omitted elements are zero, and the diagonal of $R(f_1, f_2)$ contains m occurrences of c_0 and n of d_m [4, p.206]. We can obtain Table 3 from the results of Section 2.

Table 3 Formulas for $a_1 \sim a_{12}$

	$x = n$	$x = p$
$\sum_{m=1}^{x-1} \sigma(m)\sigma(x-m)$	$\frac{1}{12}(5\sigma_3(n) + (1-6n)\sigma(n))$	$\frac{1}{12}(p-1)(p+1)(5p-6)$
$\sum_{m=1}^{x-1} m\sigma(m)\sigma(x-m)$	$\frac{n}{24}(5\sigma_3(n) + (1-6n)\sigma(n))$	$\frac{1}{24}p(p-1)(p+1)(5p-6)$
$\sum_{m=1}^{x-1} \sigma(m)\sigma_3(n-m)$	$\frac{1}{240}(21\sigma_5(n) + (10-30n)\sigma_3(n) - \sigma(n))$	$\frac{1}{240}(p-1)(p+1) \times (21p^3 - 30p^2 + 31p - 30)$
$\sum_{m<x/2} \sigma(m)\sigma(x-2m)$	$\frac{1}{24}(2\sigma_3(n) + (1-3n)\sigma(n) + 8\sigma_3(n/2) + (1-6n)\sigma(n/2))$	$\frac{1}{24}(p-1)(p+1)(2p-3)$
$\sum_{m<x/2} \sigma_3(m)\sigma(x-2m)$	$\frac{1}{240}(\sigma_5(n) - \sigma(n) + 20\sigma_5(n/2) + (10-30n)\sigma_3(n/2))$	$\frac{1}{240}(p-1)p(p+1)(p^2+1)$
$\sum_{m<x/2} \sigma(m)\sigma_3(x-2m)$	$\frac{1}{240}(5\sigma_5(n) + (10-15n)\sigma_3(n) + 16\sigma_5(n/2) - \sigma(n/2))$	$\frac{1}{48}(p-1)(p+1)(p^3 - 3p^2 + 3p - 3)$
$\sum_{l+m+s=x} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(s)$	$\frac{1}{64}\{\tilde{\sigma}_5(n) + 6(1-n)\tilde{\sigma}_3(n) + (8n^2 - 12n + 3)\tilde{\sigma}(n)\}$	$\frac{1}{64}(p+1)(p-1)^2(p^2 - 5p + 10)$
$\sum_{m=1}^{x-1} \hat{\sigma}(m)\hat{\sigma}(x-m)$	$\frac{1}{12}(\sigma_3(n) + 4\sigma_3(\frac{n}{2}) - \hat{\sigma}(n))$	$\frac{1}{12}(p-1)p(p+1)$
$\sum_{m=1}^{x-1} \hat{\sigma}(m)\tilde{\sigma}_3(x-m)$	$-\frac{1}{48}(\tilde{\sigma}_5(n) + 2\tilde{\sigma}_3(n) - 3\hat{\sigma}(n))$	$-\frac{1}{48}(p-1)p(p+1)(p^2+3)$
$\sum_{m=1}^{x-1} m\hat{\sigma}(m)\hat{\sigma}(x-m)$	$\frac{n}{24}(\sigma_3(n) + 4\sigma_3(\frac{n}{2}) - \hat{\sigma}(n))$	$\frac{1}{24}(p-1)p^2(p+1)$
$\sum_{l+m+s=x} \hat{\sigma}(l)\hat{\sigma}(m)\tilde{\sigma}(s)$	$\frac{1}{576}\{\tilde{\sigma}_5(n) + 4\tilde{\sigma}_3(n) + \tilde{\sigma}(n) - 6(2n-1)\hat{\sigma}(n) + 6(n-1)(\sigma_3(n) + 4\sigma_3(\frac{n}{2}))\}$	$\frac{1}{576}(p-1)^2(p+1)^2(p+6)$
$\sum_{l+m+s=x} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(s)$	$\frac{1}{192}(\sigma_5(n) - 8\sigma_5(\frac{n}{2}) - 2\sigma_3(n) - 8\sigma_3(\frac{n}{2}) + \hat{\sigma}(n))$	$\frac{1}{192}(p-1)^2p(p+1)^2$

Corollary 3.1 *Let*

$$a_1(n) := \sum_{m=1}^{n-1} \sigma(m)\sigma(n-m), \qquad a_2(n) := \sum_{m=1}^{n-1} m\sigma(m)\sigma(n-m),$$

$$a_3(n) := \sum_{m=1}^{n-1} \sigma(m)\sigma_3(n-m), \qquad a_4(n) := \sum_{m<n/2} \sigma(m)\sigma(n-2m),$$

$$a_5(n) := \sum_{m<n/2} \sigma_3(m)\sigma(n-2m), \qquad a_6(n) := \sum_{m<n/2} \sigma(m)\sigma_3(n-2m),$$

$$a_7(n) := \sum_{l+m+s=n} \tilde{\sigma}(l)\tilde{\sigma}(m)\tilde{\sigma}(s), \qquad a_8(n) := \sum_{m=1}^{n-1} \hat{\sigma}(m)\hat{\sigma}(n-m),$$

$$a_9(n) := \sum_{m=1}^{n-1} \hat{\sigma}(m)\tilde{\sigma}_3(n-m), \qquad a_{10}(n) := \sum_{m=1}^{n-1} m\hat{\sigma}(m)\hat{\sigma}(n-m),$$

$$a_{11}(n) := \sum_{l+m+s=n} \hat{\sigma}(l)\hat{\sigma}(m)\tilde{\sigma}(s), \qquad a_{12}(n) := \sum_{l+m+s=n} \hat{\sigma}(l)\hat{\sigma}(m)\hat{\sigma}(s).$$

There exists a polynomial $T(x,y) \in \mathbb{Z}[x,y]$ such that $T(a_i(p), a_j(p)) = 0$ with $i,j \in \{1,2,\ldots,12\}$ where p is an odd prime. We abbreviate $a_1(n)$ to a_1 and it is also applied to the other values. Then we get Table 4.

Proof We illustrate the proof for the first $T(x(p), y(p)) = 0$ in Table 4. In Table 3 we consider $\sum_{m=1}^{x-1} \sigma(m)\sigma(x-m)$ and $\sum_{m=1}^{x-1} m\sigma(m)\sigma(x-m)$ with an odd prime x put by $2q+1$. As

$$\sum_{m=1}^{2q} \sigma(m)\sigma(2q+1-m) = \frac{1}{3}\left(10q^3 + 9q^2 - q\right),$$

$$\sum_{m=1}^{2q} m\sigma(m)\sigma(2q+1-m) = \frac{1}{6}\left(20q^4 + 28q^3 + 7q^2 - q\right),$$

Table 4 Formulas of $T(x(p), y(p)) = 0$

(x,y)	$T(x,y)$
(a_1,a_2)	$-3a_1^3+6a_1^4+5a_1^2a_2+12a_1a_2^2-20a_2^3$
(a_1,a_3)	$168a_1^2+162{,}358a_1^3-143{,}775a_1^4+83{,}349a_1^5-750a_1a_3-1{,}045{,}050a_1^2a_3$ $+283{,}500a_1^3a_3+825a_3^2+2{,}221{,}875a_1a_3^2-1{,}562{,}500a_3^3$
(a_1,a_4)	$-5a_1^2+32a_1^3+32a_1a_4-480a_1^2a_4-44a_4^2+2{,}400a_1a_4^2-4{,}000a_4^3$
(a_1,a_5)	$-61a_1^2+194a_1^3-180a_1^4+72a_1^5+6{,}100a_1a_5-28{,}000a_1^2a_5+18{,}000a_1^3a_5$ $+67{,}100a_5^2+1{,}525{,}000a_1a_5^2-12{,}500{,}000a_5^3$
(a_1,a_6)	$-415a_1^2+408a_1^3-360a_1^4+144a_1^5+2{,}656a_1a_6+304a_1^2a_6$ $-3{,}600a_1^3a_6-3{,}652a_6^2+38{,}000a_1a_6^2-200{,}000a_6^3$
(a_1,a_7)	$22{,}032a_1^3+3{,}402a_1^4+243a_1^5-3{,}264a_1a_7-217{,}152a_1^2a_7$ $-32{,}400a_1^3a_7+5{,}984a_7^2+762{,}000a_1a_7^2-800{,}000a_7^3$
(a_1,a_8)	$-a_1^2+2a_1^3+10a_1a_8-30a_1^2a_8+11a_8^2+150a_1a_8^2-250a_8^3$
(a_1,a_9)	$-111a_1^2+258a_1^3-81a_1^4+18a_1^5-1{,}110a_1a_9+3{,}940a_1^2a_9$ $-900a_1^3a_9+1{,}221a_9^2+22{,}750a_1a_9^2+25{,}000a_9^3$
(a_1,a_{10})	$3a_1^2-12a_1^3+12a_1^4+72a_1a_{10}-94a_1^2a_{10}+132a_{10}^2$ $+2{,}400a_1a_{10}^2-5{,}000a_{10}^3$
(a_1,a_{11})	$105a_1^4+a_1^5-10{,}224a_1^2a_{11}+2{,}400a_1^3a_{11}+5{,}808a_{11}^2+350{,}000a_1a_{11}^2-2{,}400{,}000a_{11}^3$
(a_1,a_{12})	$-9a_1^4+18a_1^5-1{,}440a_1^2a_{12}+3{,}600a_1^3a_{12}+1{,}936a_{12}^2+124{,}000a_1a_{12}^2-1{,}600{,}000a_{12}^3$
(a_2,a_3)	$10{,}080a_2^2+16{,}307{,}536a_2^3-36{,}244{,}800a_2^4+28{,}005{,}264a_2^5-2{,}700a_2a_3$ $-23{,}500{,}980a_2^2a_3-37{,}184{,}400a_2^3a_3-12{,}375a_3^2-14{,}849{,}125a_2a_3^2$ $+208{,}935{,}000a_2^2a_3^2+23{,}536{,}500a_3^3+22{,}500{,}000a_2a_3^3-187{,}500{,}000a_3^4$
(a_2,a_4)	$-15a_2^2+128a_2^3-18a_2a_4-768a_2^2a_4+33a_4^2-944a_2a_4^2+2{,}736a_4^3-24{,}000a_4^4$
(a_2,a_5)	$61a_2^3-120a_2^4+144a_2^5+7{,}320a_2^2a_5-18{,}000a_2^3a_5+236{,}375a_2a_5^2$ $-165{,}000a_2^2a_5^2+1{,}006{,}500a_5^3+22{,}500{,}000a_2a_5^3-187{,}500{,}000a_5^4$
(a_2,a_6)	$-415a_2^2+1{,}344a_2^3-1{,}344a_2^4+384a_2^5-498a_2a_6+5{,}192a_2^2a_6-6{,}528a_2^3a_6$ $+913a_6^2+21{,}816a_2a_6^2-68{,}800a_2^2a_6^2+35{,}392a_6^3-480{,}000a_2a_6^3-800{,}000a_6^4$
(a_2,a_7)	$55{,}080a_2^3+7{,}290a_2^4+243a_2^5-2{,}040a_2a_7-121{,}140a_2^2a_7$ $-16{,}956a_2^3a_7-1{,}870a_7^2+127{,}401a_2a_7^2-747{,}000a_2^2a_7^2$ $+261{,}968a_7^3-2{,}160{,}000a_2a_7^3-1{,}600{,}000a_7^4$
(a_2,a_8)	$4a_2^3+36a_2^2a_8+83a_2a_8^2+33a_8^3-750a_8^4$
(a_2,a_9)	$444a_2^3-240a_2^4+48a_2^5-4{,}440a_2^2a_9+2{,}880a_2^3a_9+11{,}877a_2a_9^2$ $+5{,}800a_2^2a_9^2-4{,}884a_9^3-60{,}000a_2a_9^3-100{,}000a_9^4$
(a_2,a_{10})	$2a_2^4+3a_2^2a_{10}-40a_2^3a_{10}-30a_2a_{10}^2+300a_2^2a_{10}^2-33a_{10}^3$ $-1{,}000a_2a_{10}^3+1{,}250a_{10}^4$
(a_2,a_{11})	$-315a_2^4+a_2^5+7{,}668a_2^2a_{11}+30{,}228a_2^3a_{11}-1{,}089a_{11}^2-110{,}101a_2a_{11}^2$ $-916{,}200a_2^2a_{11}^2+554{,}544a_{11}^3+10{,}800{,}000a_2a_{11}^3$ $-43{,}200{,}000a_{11}^4$
(a_2,a_{12})	$9a_2^5+360a_2^3a_{12}-121a_2a_{12}^2-30{,}600a_2^2a_{12}^2+5{,}808a_{12}^3$ $+720{,}000a_2a_{12}^3-4{,}800{,}000a_{12}^4$
(a_3,a_4)	$-6{,}625a_3^2+64{,}000a_3^3+20{,}140a_3a_4-612{,}000a_3^2a_4-11{,}872a_4^2$ $+1{,}492{,}080a_3a_4^2-1{,}040{,}816a_4^3-2{,}721{,}600a_3a_4^3$ $+4{,}723{,}920a_4^4-10{,}668{,}672a_4^5$
(a_3,a_5)	$-a_3^2+8a_3^3-144a_3^4+1{,}152a_3^5+52a_3a_5+56a_3^2a_5$ $+4{,}896a_3^3a_5-120{,}960a_3^4a_5+224a_5^2-22{,}736a_3a_5^2+144{,}576a_3^2a_5^2$ $+5{,}080{,}320a_3^3a_5^2-181{,}328a_5^3-735{,}264a_3a_5^3-106{,}686{,}720a_3^2a_5^3$ $+50{,}985{,}936a_5^4+1{,}120{,}210{,}560a_3a_5^4-4{,}704{,}884{,}352a_5^5$
(a_3,a_6)	$-341{,}375a_3^2+4{,}749{,}000a_3^3-18{,}000{,}000a_3^4+7{,}200{,}000a_3^5+1{,}037{,}780a_3a_6$ $-21{,}435{,}400a_3^2a_6+133{,}848{,}000a_3^3a_6-151{,}200{,}000a_3^4a_6-611{,}744a_6^2$ $+43{,}392{,}400a_3a_6^2-587{,}858{,}400a_3^2a_6^2+1{,}270{,}080{,}000a_3^3a_6^2$ $-43{,}483{,}216a_6^3+1{,}153{,}638{,}720a_3a_6^3-5{,}334{,}336{,}000a_3^2a_6^3$ $-669{,}437{,}136a_6^4+11{,}202{,}105{,}600a_3a_6^4-9{,}409{,}768{,}704a_6^5$
(a_3,a_7)	$1{,}155{,}060{,}000a_3^3+104{,}034{,}375a_3^4+3{,}037{,}500a_3^5-27{,}379{,}200a_3a_7$ $-4{,}160{,}682{,}000a_3^2a_7-566{,}676{,}000a_3^3a_7-85{,}050{,}000a_3^4a_7+25{,}553{,}920a_7^2$ $+6{,}748{,}222{,}080a_3a_7^2-20{,}313{,}741{,}600a_3^2a_7^2+952{,}560{,}000a_3^3a_7^2-3{,}624{,}210{,}688a_7^3$ $+20{,}040{,}791{,}040a_3a_7^3-5{,}334{,}336{,}000a_3^2a_7^3+176{,}788{,}224a_7^4$ $+14{,}936{,}140{,}800a_3a_7^4-16{,}728{,}477{,}696a_7^5$
(a_3,a_8)	$-125a_3^2+1{,}000a_3^3+650a_3a_8-6{,}750a_3^2a_8+280a_8^2+5{,}820a_3a_8^2$ $-5{,}978a_8^3+56{,}700a_3a_8^3-7{,}695a_8^4-166{,}698a_8^5$
(a_3,a_9)	$-52{,}125a_3^2+360{,}750a_3^3+421{,}875a_3^4+225{,}000a_3^5-271{,}050a_3a_9$ $+2{,}621{,}450a_3^2a_9+3{,}667{,}500a_3^3a_9+4{,}725{,}000a_3^4a_9+116{,}760a_9^2+4{,}007{,}030a_3a_9^2$ $-780{,}750a_3^2a_9^2+39{,}690{,}000a_3^3a_9^2+2{,}270{,}618a_9^3-43{,}218{,}900a_3a_9^3+166{,}698{,}000a_3^2a_9^3$ $+9{,}146{,}115a_9^4+350{,}065{,}800a_3a_9^4+294{,}055{,}272a_9^5$

Table 4 (*Continued*)

(x,y)	$T(x,y)$
(a_3,a_{10})	$1{,}875a_3^2 - 30{,}000a_3^3 + 120{,}000a_3^4 + 22{,}500a_3a_{10} - 311{,}950a_3^2a_{10}$ $+ 1{,}440{,}000a_3^3a_{10} + 16{,}800a_{10}^2 + 37{,}800a_3a_{10}^2 + 2{,}826{,}000a_3^2a_{10}^2$ $- 955{,}472a_{10}^3 - 2{,}613{,}600a_3a_{10}^3 + 12{,}946{,}608a_{10}^4 - 56{,}010{,}528a_{10}^5$
(a_3,a_{11})	$5{,}315{,}625a_3^4 + 6{,}250a_3^5 - 133{,}844{,}400a_3^2a_{11} + 358{,}344{,}000a_3^3a_{11}$ $- 1{,}575{,}000a_3^4a_{11} + 12{,}192{,}768a_{11}^2 + 1{,}542{,}181{,}760a_3a_{11}^2 + 607{,}773{,}600a_3^2a_{11}^2$ $+ 158{,}760{,}000a_3^3a_{11}^2 - 5{,}851{,}104{,}000a_{11}^3 - 41{,}822{,}645{,}760a_3a_{11}^3 - 8{,}001{,}504{,}000a_3^2a_{11}^3$ $+ 213{,}353{,}968{,}896a_{11}^4 + 201{,}637{,}900{,}800a_3a_{11}^4 - 2{,}032{,}510{,}040{,}064a_{11}^5$
(a_3,a_{12})	$-28{,}125a_3^4 + 225{,}000a_3^5 - 1{,}170{,}000a_3^2a_{12} + 12{,}420{,}000a_3^3a_{12}$ $- 18{,}900{,}000a_3^4a_{12} + 250{,}880a_{12}^2 + 33{,}272{,}960a_3a_{12}^2 - 205{,}380{,}000a_3^2a_{12}^2$ $+ 635{,}040{,}000a_3^3a_{12}^2 - 228{,}392{,}192a_{12}^3 + 287{,}539{,}200a_3a_{12}^3 - 10{,}668{,}672{,}000a_3^2a_{12}^3$ $+ 5{,}816{,}344{,}320a_{12}^4 + 89{,}616{,}844{,}800a_3a_{12}^4 - 301{,}112{,}598{,}528a_{12}^5$
(a_4,a_5)	$-13a_4^2 + 176a_4^3 - 720a_4^4 + 1{,}152a_4^5 + 260a_4a_5 - 4{,}880a_4^2a_5 + 14{,}400a_4^3a_5$ $+ 1{,}625a_5^2 + 52{,}000a_4a_5^2 - 64{,}000a_5^3$
(a_4,a_6)	$-25a_4^2 + 168a_4^3 - 576a_4^4 + 1{,}152a_4^5 + 50a_4a_6 - 200a_4^2a_6 - 576a_4^3a_6$ $- 25a_6^2 + 544a_4a_6^2 - 512a_6^3$
(a_4,a_7)	$6{,}156a_4^3 + 5{,}589a_4^4 + 1{,}944a_4^5 - 228a_4a_7 - 10{,}980a_4^2a_7 - 9{,}072a_4^3a_7$ $+ 95a_7^2 + 8{,}544a_4a_7^2 - 2{,}048a_7^3$
(a_4,a_8)	$-4a_4^2 + 32a_4^3 + 8a_4a_8 - 96a_4^2a_8 + 5a_8^2 + 96a_4a_8^2 - 32a_8^3$
(a_4,a_9)	$-84a_4^2 + 816a_4^3 - 1{,}296a_4^4 + 1{,}152a_4^5 - 168a_4a_9 + 2{,}432a_4^2a_9$ $- 2{,}880a_4^3a_9 + 105a_9^2 + 2{,}848a_4a_9^2 + 512a_9^3$
(a_4,a_{10})	$3a_4^2 - 48a_4^3 + 192a_4^4 + 18a_4a_{10} - 112a_4^2a_{10} + 15a_{10}^2 + 384a_4a_{10}^2 - 128a_{10}^3$
(a_4,a_{11})	$175a_4^4 + 8a_4^5 - 900a_4^2a_{11} + 816a_4^3a_{11} + 125a_{11}^2 + 5{,}920a_4a_{11}^2 - 6{,}144a_{11}^3$
(a_4,a_{12})	$-9a_4^4 + 72a_4^5 - 72a_4^2a_{12} + 720a_4^3a_{12} + 25a_{12}^2 + 1{,}312a_4a_{12}^2 - 2{,}048a_{12}^3$
(a_5,a_6)	$-125a_5^2 + 24{,}000a_5^3 - 1{,}800{,}000a_5^4 + 7{,}200{,}000a_5^5 - 20a_5a_6 + 2{,}600a_5^2a_6$ $- 432{,}000a_5^3a_6 - 7{,}200{,}000a_5^4a_6 + a_6^2 + 200a_5a_6^2 - 158{,}400a_5^2a_6^2$ $+ 2{,}880{,}000a_5^3a_6^2 - 16a_6^3 + 4{,}320a_5a_6^3 - 576{,}000a_5^2a_6^3 + 144a_6^4$ $+ 57{,}600a_5a_6^4 - 2{,}304a_6^5$
(a_5,a_7)	$10{,}732{,}500a_5^3 + 15{,}946{,}875a_5^4 + 6{,}075{,}000a_5^5 - 15{,}900a_5a_7 - 4{,}972{,}500a_5^2a_7$ $- 10{,}827{,}000a_5^3a_7 - 8{,}100{,}000a_5^4a_7 + 265a_7^2 + 183{,}960a_5a_7^2 - 12{,}355{,}200a_5^2a_7^2$ $+ 4{,}320{,}000a_5^3a_7^2 - 2{,}016a_7^3 + 17{,}280a_5a_7^3 - 1{,}152{,}000a_5^2a_7^3 + 3{,}328a_7^4$ $+ 153{,}600a_5a_7^4 - 8{,}192a_7^5$
(a_5,a_8)	$2{,}000a_5^3 - 500a_5^2a_8 + 40a_5a_8^2 - a_8^3 - 36a_8^5$
(a_5,a_9)	$1{,}500a_5^3 + 450{,}000a_5^5 + 400a_5^2a_9 + 450{,}000a_5^4a_9 + 35a_5a_9^2 + 180{,}000a_5^3a_9^2$ $+ a_9^3 + 36{,}000a_5^2a_9^3 + 3{,}600a_5a_9^4 + 144a_9^5$
(a_5,a_{10})	$60{,}000a_5^4 + 25a_5^2a_{10} - 3{,}000a_5^2a_{10}^2 - a_{10}^3 + 24a_{10}^4 - 144a_{10}^5$
(a_5,a_{11})	$809{,}375a_5^4 + 25{,}000a_5^5 - 11{,}100a_5^2a_{11} + 561{,}000a_5^3a_{11} - 300{,}000a_5^4a_{11}$ $+ 37a_{11}^2 - 760a_5a_{11}^2 - 1{,}353{,}600a_5^2a_{11}^2 + 1{,}440{,}000a_5^3a_{11}^2 - 5{,}280a_{11}^3$ $- 51{,}840a_5a_{11}^3 - 3{,}456{,}000a_5^2a_{11}^3 + 186{,}624a_{11}^4 + 4{,}147{,}200a_5a_{11}^4 - 1{,}990{,}656a_{11}^5$
(a_5,a_{12})	$28{,}125a_5^5 - 112{,}500a_5^4a_{12} - 5a_5a_{12}^2 + 180{,}000a_5^3a_{12}^2$ $- 4a_{12}^3 - 144{,}000a_5^2a_{12}^3 + 57{,}600a_5a_{12}^4 - 9{,}216a_{12}^5$
(a_6,a_7)	$44{,}388a_6^3 + 12{,}879a_6^4 + 3{,}888a_6^5 - 1{,}644a_6a_7 + 756a_6^2a_7 + 7{,}776a_6^3a_7$ $- 25{,}920a_6^4a_7 + 685a_7^2 - 6{,}840a_6a_7^2 - 101{,}088a_6^2a_7^2 + 69{,}120a_6^3a_7^2$ $- 672a_7^3 + 43{,}776a_6a_7^3 - 92{,}160a_6^2a_7^3 + 61{,}440a_6a_7^4 - 16{,}384a_7^5$
(a_6,a_8)	$-4a_6^2 + 64a_6^3 + 8a_6a_8 - 176a_6^2a_8 + 5a_8^2 + 16a_6a_8^2 + 24a_8^3$ $+ 432a_6a_8^3 + 72a_8^4 - 144a_8^5$
(a_6,a_9)	$-4a_6^2 + 48a_6^3 + 240a_6^4 + 256a_6^5 - 8a_6a_9 + 144a_6^2a_9 + 960a_6^3a_9$ $+ 1{,}280a_6^4a_9 + 5a_9^2 + 72a_6a_9^2 + 288a_6^2a_9^2 + 2{,}560a_6^3a_9^2$ $- 24a_9^3 - 1{,}344a_6a_9^3 + 2{,}560a_6^2a_9^3 + 384a_9^4 + 1{,}280a_6a_9^4 + 256a_9^5$
(a_6,a_{10})	$a_6^2 - 32a_6^3 + 256a_6^4 + 6a_6a_{10} - 168a_6^2a_{10} + 1{,}536a_6^3a_{10} + 5a_{10}^2$ $- 88a_6a_{10}^2 + 2{,}624a_6^2a_{10}^2 - 48a_{10}^3 + 1{,}344a_6a_{10}^3 + 384a_{10}^4 - 384a_{10}^5$
(a_6,a_{11})	$4{,}025a_6^4 + 16a_6^5 - 20{,}700a_6^2a_{11} + 124{,}512a_6^3a_{11} - 960a_6^4a_{11}$ $+ 2{,}875a_{11}^2 - 13{,}000a_6a_{11}^2 + 705{,}888a_6^2a_{11}^2 + 23{,}040a_6^3a_{11}^2 - 99{,}936a_{11}^3$ $+ 546{,}048a_6a_{11}^3 - 276{,}480a_6^2a_{11}^3 + 1{,}327{,}104a_{11}^4 + 1{,}658{,}880a_6a_{11}^4 - 3{,}981{,}312a_{11}^5$
(a_6,a_{12})	$-9a_6^4 + 144a_6^5 - 72a_6^2a_{12} + 1{,}440a_6^3a_{12} - 2{,}880a_6^4a_{12} + 25a_{12}^2 - 184a_6a_{12}^2$ $+ 5{,}472a_6^2a_{12}^2 + 23{,}040a_6^3a_{12}^2 - 1{,}632a_{12}^3 - 6{,}912a_6a_{12}^3 - 92{,}160a_6^2a_{12}^3$ $+ 36{,}864a_{12}^4 + 184{,}320a_6a_{12}^4 - 147{,}456a_{12}^5$
(a_7,a_8)	$-80a_7^2 + 512a_7^3 + 480a_7a_8 - 4{,}512a_7^2a_8 + 9{,}360a_7a_8^2 - 3{,}240a_8^3$ $+ 3{,}888a_7a_8^3 - 1{,}215a_8^4 - 486a_8^5$
(a_7,a_9)	$-2{,}480a_7^2 + 14{,}592a_7^3 + 6{,}912a_7^4 + 8{,}192a_7^5 - 14{,}880a_7a_9 + 139{,}392a_7^2a_9$ $+ 129{,}024a_7^3a_9 + 30{,}720a_7^4a_9 + 378{,}000a_7a_9^2 + 505{,}440a_7^2a_9^2 + 46{,}080a_7^3a_9^2$ $+ 100{,}440a_9^3 - 147{,}744a_7a_9^3 + 34{,}560a_7^2a_9^3 - 27{,}945a_9^4 + 12{,}960a_7a_9^4 + 1{,}944a_9^5$

Table 4 (*Continued*)

(x,y)	$T(x,y)$
(a_7,a_{10})	$25a_7^2-320a_7^3+1{,}024a_7^4+300a_7a_{10}-2{,}010a_7^2a_{10}+9{,}216a_7^3a_{10}+17{,}460a_7a_{10}^2$ $+22{,}320a_7^2a_{10}^2-8{,}100a_{10}^3+9{,}288a_7a_{10}^3-3{,}645a_{10}^4-486a_{10}^5$
(a_7,a_{11})	$4{,}655a_7^4+16a_7^5-134{,}064a_7^2a_{11}+241{,}812a_7^3a_{11}-720a_7^4a_{11}+1{,}034{,}208a_7a_{11}^2$ $+2{,}320{,}650a_7^2a_{11}^2+12{,}960a_7^3a_{11}^2-1{,}994{,}544a_{11}^3+1{,}787{,}508a_7a_{11}^3-116{,}640a_7^2a_{11}^3$ $-2{,}735{,}937a_{11}^4+524{,}880a_7a_{11}^4-944{,}784a_{11}^5$
(a_7,a_{12})	$-5a_7^4+32a_7^5-240a_7^2a_{12}+1{,}836a_7^3a_{12}-480a_7^4a_{12}+4{,}320a_7a_{12}^2$ $+29{,}322a_7^2a_{12}^2+2{,}880a_7^3a_{12}^2-19{,}440a_{12}^3+8{,}316a_7a_{12}^3-8{,}640a_7^2a_{12}^3$ $-23{,}085a_{12}^4+12{,}960a_7a_{12}^4-7{,}776a_{12}^5$
(a_8,a_9)	$3a_8^3+9a_8^5+10a_8^2a_9+11a_8a_9^2+4a_9^3$
(a_8,a_{10})	$6a_8^4+a_8^2a_{10}-4a_{10}^3$
(a_8,a_{11})	$35a_8^4+2a_8^5-48a_8^2a_{11}+144a_8^3a_{11}+16a_{11}^2-32a_8a_{11}^2-1{,}536a_{11}^3$
(a_8,a_{12})	$9a_8^5-16a_8a_{12}^2-256a_{12}^3$
(a_9,a_{10})	$32a_9^4+3a_9^2a_{10}-104a_9^2a_{10}^2-12a_{10}^3+48a_{10}^4-48a_{10}^5$
(a_9,a_{11})	$-1{,}365a_9^4+8a_9^5+1{,}872a_9^2a_{11}+7{,}584a_9^3a_{11}+480a_9^4a_{11}-624a_{11}^2$ $-1{,}280a_9a_{11}^2+103{,}392a_9^2a_{11}^2+11{,}520a_9^3a_{11}^2+66{,}816a_{11}^3-27{,}648a_9a_{11}^3$ $+138{,}240a_9^2a_{11}^3-684{,}288a_{11}^4+829{,}440a_9a_{11}^4+1{,}990{,}656a_{11}^5$
(a_9,a_{12})	$9a_9^5+180a_9^4a_{12}-16a_9a_{12}^2+1{,}440a_9^3a_{12}^2+192a_{12}^3$ $+5{,}760a_9^2a_{12}^3+11{,}520a_9a_{12}^4+9{,}216a_{12}^5$
(a_{10},a_{11})	$-105a_{10}^4+2a_{10}^5+36a_{10}^2a_{11}+1{,}752a_{10}^3a_{11}-3a_{11}^2-290a_{10}a_{11}^2$ $-10{,}512a_{10}^2a_{11}^2+576a_{11}^3+27{,}648a_{10}a_{11}^3-27{,}648a_{11}^4$
(a_{10},a_{12})	$9a_{10}^5-a_{10}a_{12}^2-72a_{10}^2a_{12}^2-1{,}536a_{12}^4$
(a_{11},a_{12})	$-81a_{11}^4+7{,}776a_{11}^5+108a_{11}^3a_{12}-12{,}960a_{11}^4a_{12}+594a_{11}^2a_{12}^2+8{,}640a_{11}^3a_{12}^2$ $-420a_{11}a_{12}^3-2{,}880a_{11}^2a_{12}^3-1{,}225a_{12}^4+480a_{11}a_{12}^4-32a_{12}^5$

the polynomials

$$f_1(X) = 10X^3 + 9X^2 - X - 3a_1(2q+1),$$
$$f_2(X) = 20X^4 + 28X^3 + 7X^2 - X - 6a_2(2q+1)$$

have common zero, namely, $X = q$. We deduce that for each odd prime p $(= 2q+1)$,

$$\begin{vmatrix} 10 & 9 & -1 & -3a_1 & 0 & 0 & 0 \\ 0 & 10 & 9 & -1 & -3a_1 & 0 & 0 \\ 0 & 0 & 10 & 9 & -1 & -3a_1 & 0 \\ 0 & 0 & 0 & 10 & 9 & -1 & -3a_1 \\ 20 & 28 & 7 & -1 & -6a_2 & 0 & 0 \\ 0 & 20 & 28 & 7 & -1 & -6a_2 & 0 \\ 0 & 0 & 20 & 28 & 7 & -1 & -6a_2 \end{vmatrix} = 0$$

and this simplifies to give $R(f_1,f_2) = 108{,}000(-3a_1^3 + 6a_1^4 + 5a_1^2a_2 + 12a_1a_2^2 - 20a_2^3) = 0$, so we can find the irreducible polynomial $T(a_1,a_2) = -3a_1^3 + 6a_1^4 + 5a_1^2a_2 + 12a_1a_2^2 - 20a_2^3 = 0$. Other results in Table 4 can also be obtained by using the resultant. $\square$

Remark 3.2 The plane curves $T(x,y) = 0$ in Corollary 3.1 all have zero-genus since x, y are polynomials of p, which leads to a morphism from the projective line $\mathbb{P}^1$ to plane curves $T(x,y) = 0$. Then it follows easily from the Riemann-Hurwitz theorem (*e.g.*, see Corollary 1 on page 91 of [5]).

Example 3.3 We suggest Figure 1, Figure 2 and Figure 3 for results of $Z_{i,j} = T(a_i,a_j)$ and $T(a_i,a_j) = 0$.

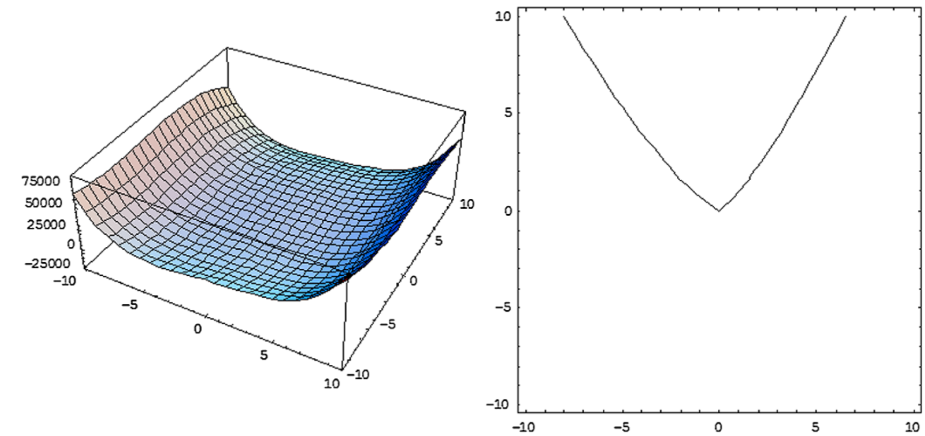

Figure 1 $Z_{1,2}$ and $T(a_1, a_2) = 0$.

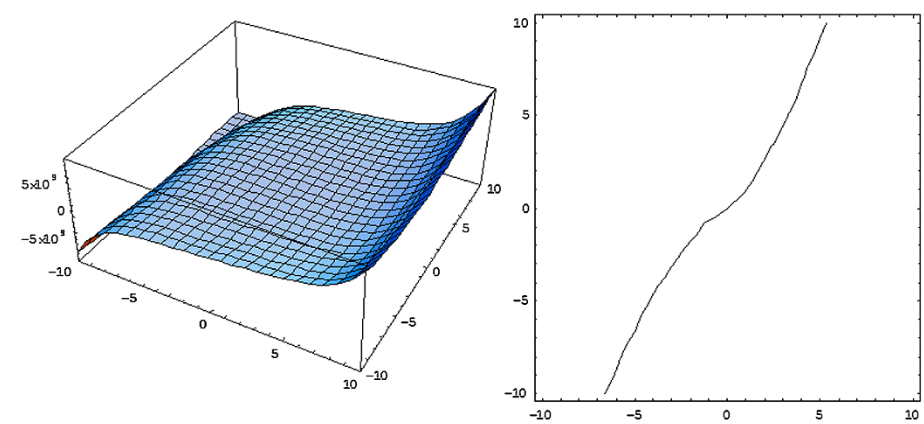

Figure 2 $Z_{1,3}$ and $T(a_1, a_3) = 0$.

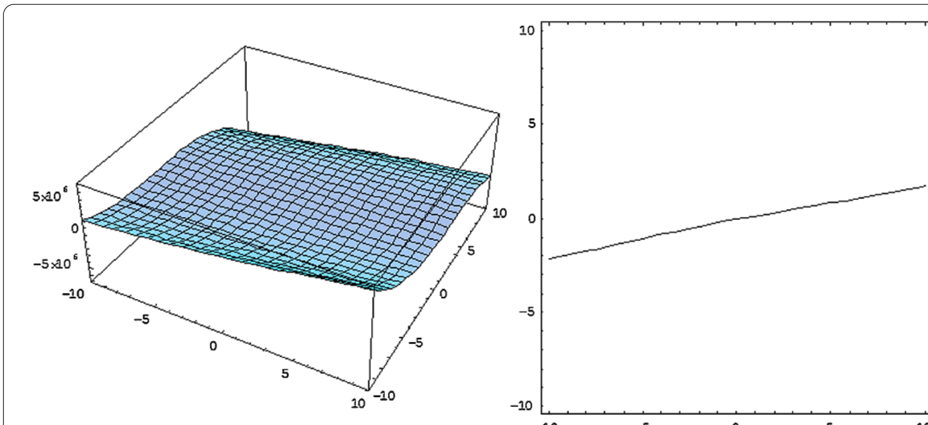

Figure 3 $Z_{1,4}$ and $T(a_1, a_4) = 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this paper. They read and approved the final manuscript.

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