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Existence of positive solutions of higher-order nonlinear neutral equations

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Abstract
In this work, we consider the existence of positive solutions of higher-order nonlinear neutral differential equations. In the special case, our results include some well-known results. In order to obtain new sufficient conditions for the existence of a positive solution, we use Schauder's fixed point theorem.

Keywords: neutral equations; fixed point; higher-order; positive solution

1 Introduction

The purpose of this article is to study higher-order neutral nonlinear differential equations of the form

$$\left[r(t)\left[x(t)-P_1(t)x(t-\tau)\right]^{(n-1)}\right]'+(-1)^nQ_1(t)f\bigl(x(t-\sigma)\bigr)=0, \tag{1}$$

$$\left[r(t)\left[x(t)-P_1(t)x(t-\tau)\right]^{(n-1)}\right]'+(-1)^n\int_c^d Q_2(t,\xi)f\bigl(x(t-\xi)\bigr)\,d\xi=0 \tag{2}$$

and

$$\left[r(t)\left[x(t)-\int_a^b P_2(t,\xi)x(t-\xi)\,d\xi\right]^{(n-1)}\right]'+(-1)^n\int_c^d Q_2(t,\xi)f\bigl(x(t-\xi)\bigr)\,d\xi=0, \tag{3}$$

where $n\geq 2$ is an integer, $\tau>0$, $\sigma\geq 0$, $d>c\geq 0$, $b>a\geq 0$, r, $P_1\in C([t_0,\infty),(0,\infty))$, $P_2\in C([t_0,\infty)\times[a,b],(0,\infty))$, $Q_1\in C([t_0,\infty),(0,\infty))$, $Q_2\in C([t_0,\infty)\times[c,d],(0,\infty))$, $f\in C(\mathbb{R},\mathbb{R})$, f is a nondecreasing function with $xf(x)>0$, $x\neq 0$.

The motivation for the present work was the recent work of Culáková *et al.* [1] in which the second-order neutral nonlinear differential equation of the form

$$\left[r(t)\left[x(t)-P(t)x(t-\tau)\right]'\right]'+Q(t)f\bigl(x(t-\sigma)\bigr)=0 \tag{4}$$

was considered. Note that when $n=2$ in (1), we obtain (4). Thus, our results contain the results established in [1] for (1). The results for (2) and (3) are completely new.

Existence of nonoscillatory or positive solutions of higher-order neutral differential equations was investigated in [2–5], but in this work our results contain not only existence of solutions but also behavior of solutions. For books, we refer the reader to [6–11].

Let $\rho_1=\max\{\tau,\sigma\}$. By a solution of (1) we understand a function $x\in C([t_1-\rho_1,\infty),\mathbb{R})$, for some $t_1\geq t_0$, such that $x(t)-P_1(t)x(t-\tau)$ is $n-1$ times continuously differentiable,

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$r(t)(x(t) - P_1(t)x(t-\tau))^{(n-1)}$ is continuously differentiable on $[t_1,\infty)$ and (1) is satisfied for $t \geq t_1$. Similarly, let $\rho_2 = \max\{\tau, d\}$. By a solution of (2) we understand a function $x \in C([t_1 - \rho_2, \infty), \mathbb{R})$, for some $t_1 \geq t_0$, such that $x(t) - P_1(t)x(t-\tau)$ is $n-1$ times continuously differentiable, $r(t)(x(t) - P_1(t)x(t-\tau))^{(n-1)}$ is continuously differentiable on $[t_1,\infty)$ and (2) is satisfied for $t \geq t_1$. Finally, let $\rho_3 = \max\{b, d\}$. By a solution of (3) we understand a function $x \in C([t_1 - \rho_3, \infty), \mathbb{R})$, for some $t_1 \geq t_0$, such that $x(t) - \int_a^b P_2(t,\xi)x(t-\xi)\,d\xi$ is $n-1$ times continuously differentiable, $r(t)[x(t) - \int_a^b P_2(t,\xi)x(t-\xi)\,d\xi]^{(n-1)}$ is continuously differentiable on $[t_1,\infty)$ and (3) is satisfied for $t \geq t_1$.

The following fixed point theorem will be used in proofs.

Theorem 1 (Schauder's fixed point theorem [9]) *Let A be a closed, convex and nonempty subset of a Banach space Ω. Let $S : A \to A$ be a continuous mapping such that SA is a relatively compact subset of Ω. Then S has at least one fixed point in A. That is, there exists $x \in A$ such that $Sx = x$.*

2 Main results

Theorem 2 *Let*

$$\int_{t_0}^{\infty} Q_1(t)\,dt = \infty. \tag{5}$$

Assume that $0 < k_1 \leq k_2$ and there exists $\gamma \geq 0$ such that

$$\frac{k_1}{k_2}\exp\left((k_2 - k_1)\int_{t_0-\gamma}^{t_0} Q_1(t)\,dt\right) \geq 1, \tag{6}$$

$$\begin{aligned}
&\exp\left(-k_2\int_{t-\tau}^{t} Q_1(s)\,ds\right) + \exp\left(k_2\int_{t_0-\gamma}^{t-\tau} Q_1(s)\,ds\right)\\
&\quad\times \frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_s^{\infty} Q_1(u) f\left(\exp\left(-k_1\int_{t_0-\gamma}^{u-\sigma} Q_1(z)\,dz\right)\right)du\,ds\\
&\leq P_1(t) \leq \exp\left(-k_1\int_{t-\tau}^{t} Q_1(s)\,ds\right) + \exp\left(k_1\int_{t_0-\gamma}^{t-\tau} Q_1(s)\,ds\right)\\
&\quad\times \frac{1}{(n-2)!}\int_t^{\infty}\frac{(s-t)^{n-2}}{r(s)}\int_s^{\infty} Q_1(u) f\left(\exp\left(-k_2\int_{t_0-\gamma}^{u-\sigma} Q_1(z)\,dz\right)\right)du\,ds,\\
&\quad t \geq t_1 \geq t_0 + \max\{\tau,\sigma\}.
\end{aligned} \tag{7}$$

Then (1) *has a positive solution which tends to zero.*

Proof Let Ω be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Then Ω is a Banach space. Define a subset A of Ω by

$$A = \{x \in \Omega : v_1(t) \leq x(t) \leq v_2(t), t \geq t_0\},$$

where $v_1(t)$ and $v_2(t)$ are nonnegative functions such that

$$v_1(t) = \exp\left(-k_2\int_{t_0-\gamma}^{t} Q_1(s)\,ds\right), \qquad v_2(t) = \exp\left(-k_1\int_{t_0-\gamma}^{t} Q_1(s)\,ds\right), \quad t \geq t_0. \tag{8}$$

It is clear that A is a bounded, closed and convex subset of Ω. We define the operator $S : A \longrightarrow \Omega$ as

$$(Sx)(t) = \begin{cases} P_1(t)x(t-\tau) - \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty Q_1(u)f(x(u-\sigma))\,du\,ds, & t \geq t_1, \\ (Sx)(t_1) + v_2(t) - v_2(t_1), & t_0 \leq t \leq t_1. \end{cases}$$

We show that S satisfies the assumptions of Schauder's fixed point theorem.

First, S maps A into A. For $t \geq t_1$ and $x \in A$, using (7) and (8), we have

$$\begin{aligned} (Sx)(t) &\leq P_1(t)v_2(t-\tau) - \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty Q_1(u)f\bigl(v_1(u-\sigma)\bigr)\,du\,ds \\ &= P_1(t)\exp\biggl(-k_1\int_{t_0-\gamma}^{t-\tau} Q_1(s)\,ds\biggr) \\ &\quad - \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty Q_1(u)f\biggl(\exp\biggl(-k_2\int_{t_0-\gamma}^{u-\sigma} Q_1(z)\,dz\biggr)\biggr)\,du\,ds \\ &\leq v_2(t) \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) &\geq P_1(t)v_1(t-\tau) - \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty Q_1(u)f\bigl(v_2(u-\sigma)\bigr)\,du\,ds \\ &= P_1(t)\exp\biggl(-k_2\int_{t_0-\gamma}^{t-\tau} Q_1(s)\,ds\biggr) \\ &\quad - \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty Q_1(u)f\biggl(\exp\biggl(-k_1\int_{t_0-\gamma}^{u-\sigma} Q_1(z)\,dz\biggr)\biggr)\,du\,ds \\ &\geq v_1(t). \end{aligned}$$

For $t \in [t_0, t_1]$ and $x \in A$, we obtain

$$(Sx)(t) = (Sx)(t_1) + v_2(t) - v_2(t_1) \leq v_2(t)$$

and in order to show $(Sx)(t) \geq v_1(t)$, consider

$$H(t) = v_2(t) - v_2(t_1) - v_1(t) + v_1(t_1).$$

By making use of (6), it follows that

$$\begin{aligned} H'(t) &= v_2'(t) - v_1'(t) = -k_1Q_1(t)v_2(t) + k_2Q_1(t)v_1(t) \\ &= Q_1(t)v_2(t)\biggl[-k_1 + k_2v_1(t)\exp\biggl(k_1\int_{t_0-\gamma}^{t} Q_1(s)\,ds\biggr)\biggr] \\ &= Q_1(t)v_2(t)\biggl[-k_1 + k_2\exp\biggl((k_1-k_2)\int_{t_0-\gamma}^{t} Q_1(s)\,ds\biggr)\biggr] \\ &\leq Q_1(t)v_2(t)\biggl[-k_1 + k_2\exp\biggl((k_1-k_2)\int_{t_0-\gamma}^{t_0} Q_1(s)\,ds\biggr)\biggr] \leq 0, \quad t_0 \leq t \leq t_1. \end{aligned}$$

Since $H(t_1) = 0$ and $H'(t) \leq 0$ for $t \in [t_0, t_1]$, we conclude that

$$H(t) = v_2(t) - v_2(t_1) - v_1(t) + v_1(t_1) \geq 0, \quad t_0 \leq t \leq t_1.$$

Then $t \in [t_0, t_1]$ and for any $x \in A$,

$$(Sx)(t) = (Sx)(t_1) + v_2(t) - v_2(t_1) \geq v_1(t_1) + v_2(t) - v_2(t_1) \geq v_1(t), \quad t_0 \leq t \leq t_1.$$

Hence, S maps A into A.

Second, we show that S is continuous. Let $\{x_i\}$ be a convergent sequence of functions in A such that $x_i(t) \to x(t)$ as $i \to \infty$. Since A is closed, we have $x \in A$. It is obvious that for $t \in [t_0, t_1]$ and $x \in A$, S is continuous. For $t \geq t_1$,

$$\begin{aligned}
&\bigl|(Sx_i)(t) - (Sx)(t)\bigr| \\
&\quad \leq P_1(t)\bigl|x_i(t-\tau) - x(t-\tau)\bigr| \\
&\qquad + \left|\frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty Q_1(u)\bigl[f\bigl(x_i(u-\sigma)\bigr) - f\bigl(x(u-\sigma)\bigr)\bigr]\,du\,ds\right| \\
&\quad \leq P_1(t)\bigl|x_i(t-\tau) - x(t-\tau)\bigr| \\
&\qquad + \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty Q_1(u)\bigl|f\bigl(x_i(u-\sigma)\bigr) - f\bigl(x(u-\sigma)\bigr)\bigr|\,du\,ds.
\end{aligned}$$

Since $|f(x_i(t-\sigma)) - f(x(t-\sigma))| \to 0$ as $i \to \infty$, by making use of the Lebesgue dominated convergence theorem, we see that

$$\lim_{t\to\infty} \bigl\|(Sx_i)(t) - (Sx)(t)\bigr\| = 0$$

and therefore S is continuous.

Third, we show that SA is relatively compact. In order to prove that SA is relatively compact, it suffices to show that the family of functions $\{Sx : x \in A\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. Since uniform boundedness of $\{Sx : x \in A\}$ is obvious, we need only to show equicontinuity. For $x \in A$ and any $\epsilon > 0$, we take $T \geq t_1$ large enough such that $(Sx)(T) \leq \frac{\epsilon}{2}$. For $x \in A$ and $T_2 > T_1 \geq T$, we have

$$\bigl|(Sx)(T_2) - (Sx)(T_1)\bigr| \leq \bigl|(Sx)(T_2)\bigr| + \bigl|(Sx)(T_1)\bigr| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Note that

$$\begin{aligned}
X^n - Y^n &= (X-Y)\bigl(X^{n-1} + X^{n-2}Y + \cdots + XY^{n-2} + Y^{n-1}\bigr) \\
&\leq n(X-Y)X^{n-1}, \quad X > Y > 0.
\end{aligned} \tag{9}$$

For $x \in A$ and $t_1 \leq T_1 < T_2 \leq T$, by using (9) we obtain

$$\begin{aligned}
&\bigl|(Sx)(T_2) - (Sx)(T_1)\bigr| \\
&\quad \leq \bigl|P_1(T_2)x(T_2-\tau) - P_1(T_1)x(T_1-\tau)\bigr|
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{(n-2)!}\int_{T_1}^{T_2} \frac{(s-T_1)^{n-2}}{r(s)} \int_s^\infty Q_1(u) f\bigl(x(u-\sigma)\bigr)\,du\,ds \\
&+ \frac{1}{(n-2)!}\int_{T_2}^{\infty} \frac{(s-T_1)^{n-2}-(s-T_2)^{n-2}}{r(s)} \int_s^\infty Q_1(u) f\bigl(x(u-\sigma)\bigr)\,du\,ds \\
\le{}& \bigl|P_1(T_2)x(T_2-\tau) - P_1(T_1)x(T_1-\tau)\bigr| \\
&+ \max_{T_1\le s\le T_2}\left\{\frac{1}{(n-2)!}\frac{s^{n-2}}{r(s)}\int_s^\infty Q_1(u) f\bigl(x(u-\sigma)\bigr)\,du\right\}(T_2-T_1) \\
&+ \frac{1}{(n-3)!}\int_{T_2}^{\infty} \frac{(s-T_1)^{n-3}}{r(s)} \int_s^\infty Q_1(u) f\bigl(x(u-\sigma)\bigr)\,du\,ds(T_2-T_1).
\end{aligned}
$$

Thus there exits $\delta > 0$ such that

$$
\bigl|(Sx)(T_2) - (Sx)(T_1)\bigr| < \epsilon \quad \text{if } 0 < T_2 - T_1 < \delta.
$$

Finally, for $x \in A$ and $t_0 \le T_1 < T_2 \le t_1$, there exits $\delta > 0$ such that

$$
\bigl|(Sx)(T_2) - (Sx)(T_1)\bigr| = \bigl|v_2(T_1) - v_2(T_2)\bigr| < \epsilon \quad \text{if } 0 < T_2 - T_1 < \delta.
$$

Therefore SA is relatively compact. In view of Schauder's fixed point theorem, we can conclude that there exists $x \in A$ such that $Sx = x$. That is, x is a positive solution of (1) which tends to zero. The proof is complete. □

Theorem 3 *Let*

$$
\int_{t_0}^\infty \tilde{Q}_2(t)\,dt = \infty, \tag{10}
$$

where $\tilde{Q}_2(t) = \int_c^d Q_2(t,\xi)\,d\xi$. *Assume that* $0 < k_1 \le k_2$ *and there exists* $\gamma \ge 0$ *such that*

$$
\begin{aligned}
&\frac{k_1}{k_2}\exp\left((k_2-k_1)\int_{t_0-\gamma}^{t_0} \tilde{Q}_2(t)\,dt\right) \ge 1, \qquad (11)\\
&\exp\left(-k_2\int_{t-\tau}^{t} \tilde{Q}_2(s)\,ds\right) + \exp\left(k_2\int_{t_0-\gamma}^{t-\tau} \tilde{Q}_2(s)\,ds\right) \\
&\quad\times \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty\int_c^d Q_2(u,\xi) f\left(\exp\left(-k_1\int_{t_0-\gamma}^{u-\xi}\tilde{Q}_2(z)\,dz\right)\right)d\xi\,du\,ds \\
&\le P_1(t) \le \exp\left(-k_1\int_{t-\tau}^{t} \tilde{Q}_2(s)\,ds\right) + \exp\left(k_1\int_{t_0-\gamma}^{t-\tau} \tilde{Q}_2(s)\,ds\right) \\
&\quad\times \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty\int_c^d Q_2(u,\xi) f\left(\exp\left(-k_2\int_{t_0-\gamma}^{u-\xi}\tilde{Q}_2(z)\,dz\right)\right)d\xi\,du\,ds, \\
&\quad t \ge t_1 \ge t_0 + \max\{\tau, d\}.
\end{aligned}
$$

Then (2) *has a positive solution which tends to zero.*

Proof Let Ω be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Then Ω is a Banach space. Define a subset A of Ω by

$$A = \{x \in \Omega : v_1(t) \le x(t) \le v_2(t), t \ge t_0\},$$

where $v_1(t)$ and $v_2(t)$ are nonnegative functions such that

$$v_1(t) = \exp\biggl(-k_2 \int_{t_0-\gamma}^{t} \tilde{Q}_2(s)\,ds\biggr), \qquad v_2(t) = \exp\biggl(-k_1 \int_{t_0-\gamma}^{t} \tilde{Q}_2(s)\,ds\biggr), \quad t \ge t_0.$$

It is clear that A is a bounded, closed and convex subset of Ω. We define the operator $S : A \longrightarrow \Omega$ as follows:

$$(Sx)(t) = \begin{cases} P_1(t)x(t-\tau) - \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \int_s^\infty \int_c^d Q_2(u,\xi)f(x(u-\xi))\,d\xi\,du\,ds, & t \ge t_1, \\ (Sx)(t_1) + v_2(t) - v_2(t_1), \quad t_0 \le t \le t_1. \end{cases}$$

Since the remaining part of the proof is similar to those in the proof of Theorem 2, it is omitted. Thus the theorem is proved. □

Theorem 4 *Suppose that* (10) *and* (11) *hold. In addition, assume that*

$$\begin{aligned} &\exp\biggl(-k_2 \int_{t-a}^{t} \tilde{Q}_2(s)\,ds\biggr) + \exp\biggl(k_2 \int_{t_0-\gamma}^{t-a} \tilde{Q}_2(s)\,ds\biggr) \\ &\quad \times \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \int_s^\infty \int_c^d Q_2(u,\xi)f\biggl(\exp\biggl(-k_1\int_{t_0-\gamma}^{u-\xi}\tilde{Q}_2(z)\,dz\biggr)\biggr)\,d\xi\,du\,ds \\ &\le \tilde{P}_2(t) \le \exp\biggl(-k_1 \int_{t-b}^{t} \tilde{Q}_2(s)\,ds\biggr) + \exp\biggl(k_1 \int_{t_0-\gamma}^{t-b} \tilde{Q}_2(s)\,ds\biggr) \qquad (12) \\ &\quad \times \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)} \int_s^\infty \int_c^d Q_2(u,\xi)f\biggl(\exp\biggl(-k_2\int_{t_0-\gamma}^{u-\xi}\tilde{Q}_2(z)\,dz\biggr)\biggr)\,d\xi\,du\,ds, \\ &t \ge t_1 \ge t_0 + \max\{b,d\}, \end{aligned}$$

where $\tilde{P}_2(t) = \int_a^b P_2(t,\xi)\,d\xi$. *Then* (3) *has a positive solution which tends to zero.*

Proof Let Ω be the set of all continuous and bounded functions on $[t_0,\infty)$ with the sup norm. Then Ω is a Banach space. Define a subset A of Ω by

$$A = \{x \in \Omega : v_1(t) \le x(t) \le v_2(t), t \ge t_0\},$$

where $v_1(t)$ and $v_2(t)$ are nonnegative functions such that

$$v_1(t) = \exp\biggl(-k_2 \int_{t_0-\gamma}^{t} \tilde{Q}_2(s)\,ds\biggr), \qquad v_2(t) = \exp\biggl(-k_1 \int_{t_0-\gamma}^{t} \tilde{Q}_2(s)\,ds\biggr), \quad t \ge t_0. \qquad (13)$$

It is clear that A is a bounded, closed and convex subset of Ω. We define the operator $S : A \longrightarrow \Omega$ as

$$(Sx)(t) = \begin{cases} \int_a^b P_2(t,\xi)x(t-\xi)\,d\xi - \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty \int_c^d Q_2(u,\xi)f(x(u-\xi))\,d\xi\,du\,ds, \\ \quad t \geq t_1, \\ (Sx)(t_1) + v_2(t) - v_2(t_1), \quad t_0 \leq t \leq t_1. \end{cases}$$

We show that S satisfies the assumptions of Schauder's fixed point theorem.

First of all, S maps A into A. For $t \geq t_1$ and $x \in A$, using (12), (13), the decreasing nature of v_2 and v_1, we have

$$\begin{aligned} (Sx)(t) &\leq \int_a^b P_2(t,\xi)v_2(t-\xi)\,d\xi - \frac{1}{(n-2)!} \\ &\quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty \int_c^d Q_2(u,\xi)f\bigl(v_1(u-\xi)\bigr)\,d\xi\,du\,ds \\ &\leq \tilde{P}_2(t)\exp\left(-k_1\int_{t_0-\gamma}^{t-b}\tilde{Q}_2(s)\,ds\right) - \frac{1}{(n-2)!} \\ &\quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty \int_c^d Q_2(u,\xi)f\left(\exp\left(-k_2\int_{t_0-\gamma}^{u-\xi}\tilde{Q}_2(z)\,dz\right)\right)d\xi\,du\,ds \\ &\leq v_2(t) \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) &\geq \int_a^b P_2(t,\xi)v_1(t-\xi)\,d\xi - \frac{1}{(n-2)!} \\ &\quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty \int_c^d Q_2(u,\xi)f\bigl(v_2(u-\xi)\bigr)\,d\xi\,du\,ds \\ &\geq \tilde{P}_2(t)\exp\left(-k_2\int_{t_0-\gamma}^{t-a}\tilde{Q}_2(s)\,ds\right) - \frac{1}{(n-2)!} \\ &\quad \times \int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty \int_c^d Q_2(u,\xi)f\left(\exp\left(-k_1\int_{t_0-\gamma}^{u-\xi}\tilde{Q}_2(z)\,dz\right)\right)d\xi\,du\,ds \\ &\geq v_1(t). \end{aligned}$$

For $t \in [t_0, t_1]$ and $x \in A$, we obtain

$$(Sx)(t) = (Sx)(t_1) + v_2(t) - v_2(t_1) \leq v_2(t)$$

and to show $(Sx)(t) \geq v_1(t)$, consider

$$H(t) = v_2(t) - v_2(t_1) - v_1(t) + v_1(t_1).$$

By making use of (11), it follows that

$$\begin{aligned} H'(t) &= v_2'(t) - v_1'(t) \\ &= -k_1\tilde{Q}_2(t)v_2(t) + k_2\tilde{Q}_2(t)v_1(t) \end{aligned}$$

$$= \tilde{Q}_2(t)v_2(t)\left[-k_1 + k_2 v_1(t)\exp\left(k_1 \int_{t_0-\gamma}^{t} \tilde{Q}_2(s)\,ds\right)\right]$$
$$\leq \tilde{Q}_2(t)v_2(t)\left[-k_1 + k_2 \exp\left((k_1 - k_2)\int_{t_0-\gamma}^{t_0} \tilde{Q}_2(s)\,ds\right)\right] \leq 0, \quad t_0 \leq t \leq t_1.$$

Since $H(t_1) = 0$ and $H'(t) \leq 0$ for $t \in [t_0, t_1]$, we conclude that

$$H(t) = v_2(t) - v_2(t_1) - v_1(t) + v_1(t_1) \geq 0, \quad t_0 \leq t \leq t_1.$$

Then $t \in [t_0, t_1]$ and for any $x \in A$,

$$(Sx)(t) = (Sx)(t_1) + v_2(t) - v_2(t_1) \geq v_1(t_1) + v_2(t) - v_2(t_1) \geq v_1(t), \quad t_0 \leq t \leq t_1.$$

Hence, S maps A into A.

Next, we show that S is continuous. Let $\{x_i\}$ be a convergent sequence of functions in A such that $x_i(t) \to x(t)$ as $i \to \infty$. Since A is closed, we have $x \in A$. It is obvious that for $t \in [t_0, t_1]$ and $x \in A$, S is continuous. For $t \geq t_1$,

$$\begin{aligned}
&\bigl|(Sx_i)(t) - (Sx)(t)\bigr| \\
&\quad\leq \int_a^b P_2(t,\xi)\bigl|x_i(t-\xi) - x(t-\xi)\bigr|\,d\xi \\
&\qquad + \frac{1}{(n-2)!}\int_t^\infty \frac{(s-t)^{n-2}}{r(s)}\int_s^\infty \int_c^d Q_2(u,\xi)\bigl|f\bigl(x_i(u-\xi)\bigr) - f\bigl(x(u-\xi)\bigr)\bigr|\,d\xi\,du\,ds.
\end{aligned}$$

Since $|f(x_i(t-\xi)) - f(x(t-\xi))| \to 0$ as $i \to \infty$ and $\xi \in [c,d]$, by making use of the Lebesgue dominated convergence theorem, we see that

$$\lim_{t\to\infty} \bigl\|(Sx_i)(t) - (Sx)(t)\bigr\| = 0.$$

Thus S is continuous.

Finally, we show that SA is relatively compact. In order to prove that SA is relatively compact, it suffices to show that the family of functions $\{Sx : x \in A\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. Since uniform boundedness of $\{Sx : x \in A\}$ is obvious, we need only to show equicontinuity. For $x \in A$ and any $\epsilon > 0$, we take $T \geq t_1$ large enough such that $(Sx)(T) \leq \frac{\epsilon}{2}$. For $x \in A$ and $T_2 > T_1 \geq T$, we have

$$\bigl|(Sx)(T_2) - (Sx)(T_1)\bigr| \leq \bigl|(Sx)(T_2)\bigr| + \bigl|(Sx)(T_1)\bigr| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

For $x \in A$ and $t_1 \leq T_1 < T_2 \leq T$, by using (9) we obtain

$$\begin{aligned}
&\bigl|(Sx)(T_2) - (Sx)(T_1)\bigr| \\
&\quad\leq \int_a^b \bigl|P_2(T_2,\xi)x(T_2-\xi) - P_2(T_1,\xi)x(T_1-\xi)\bigr|\,d\xi \\
&\qquad + \frac{1}{(n-2)!}\int_{T_1}^{T_2} \frac{(s-T_1)^{n-2}}{r(s)}\int_s^\infty \int_c^d Q_2(u,\xi)f\bigl(x(u-\xi)\bigr)\,d\xi\,du\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{(n-2)!}\int_{T_2}^{\infty}\frac{(s-T_1)^{n-2}-(s-T_2)^{n-2}}{r(s)}\int_s^{\infty}\int_c^d Q_2(u,\xi)f\bigl(x(u-\xi)\bigr)\,d\xi\,du\,ds\\
&\le \int_a^b \bigl|P_2(T_2,\xi)x(T_2-\xi)-P_2(T_1,\xi)x(T_1-\xi)\bigr|\,d\xi\\
&+ \max_{T_1\le s\le T_2}\left\{\frac{1}{(n-2)!}\frac{s^{n-2}}{r(s)}\int_s^{\infty}\int_c^d Q_2(u,\xi)f\bigl(x(u-\xi)\bigr)\,d\xi\,du\right\}(T_2-T_1)\\
&+ \frac{1}{(n-3)!}\int_{T_2}^{\infty}\frac{(s-T_1)^{n-3}}{r(s)}\int_s^{\infty}\int_c^d Q_2(u,\xi)f\bigl(x(u-\xi)\bigr)\,d\xi\,du\,ds(T_2-T_1).
\end{aligned}
$$

Thus there exits $\delta > 0$ such that

$$
\bigl|(Sx)(T_2)-(Sx)(T_1)\bigr| < \epsilon \quad \text{if } 0 < T_2 - T_1 < \delta.
$$

For $x \in A$ and $t_0 \le T_1 < T_2 \le t_1$, there exits $\delta > 0$ such that

$$
\bigl|(Sx)(T_2)-(Sx)(T_1)\bigr| = \bigl|v_2(T_1)-v_2(T_2)\bigr| < \epsilon \quad \text{if } 0 < T_2 - T_1 < \delta.
$$

Therefore SA is relatively compact. In view of Schauder's fixed point theorem, we can conclude that there exists $x \in A$ such that $Sx = x$. That is, x is a positive solution of (1) which tends to zero. The proof is complete. □

Example 1 Consider the neutral differential equation

$$
\left[e^{t/2}\left[x(t)-P_1(t)x\left(t-\frac{3}{2}\right)\right]^{(2)}\right]' - qx(t-1) = 0, \quad t \ge t_0, \tag{14}
$$

where $q \in (0,\infty)$ and

$$
\begin{aligned}
&\exp(-k_2 q\tau) + \frac{\exp(q[k_2(t+\gamma-\tau-t_0)-k_1(\gamma-\sigma-t_0)])}{k_1}\frac{\exp((-qk_1-\frac{1}{2})t)}{(k_1 q+\frac{1}{2})^2}\\
&\quad\le P_1(t) \le \exp(-k_1 q\tau) + \frac{\exp(q[k_1(t+\gamma-\tau-t_0)-k_2(\gamma-\sigma-t_0)])}{k_2}\\
&\quad\times \frac{\exp((-qk_2-\frac{1}{2})t)}{(k_2 q+\frac{1}{2})^2}.
\end{aligned}
$$

Note that for $k_1 = \frac{2}{3}$, $k_2 = 1$, $q = 1$ and $t_0 = \gamma = \frac{13}{2}$, we have

$$
\frac{k_1}{k_2}\exp\left((k_2-k_1)\int_{t_0-\gamma}^{t_0} Q_1(t)\,dt\right) = \frac{2}{3}\exp\left(\frac{1}{3}\int_0^{\frac{13}{2}} 1\,dt\right) = 5.8194 \ge 1
$$

and

$$
\exp\left(\frac{-3}{2}\right) + \frac{54}{49}\exp\left(\frac{-t-5}{6}\right) \le P_1(t) \le \exp(-1) + \frac{4}{9}\exp\left(\frac{-5t}{6}\right), \quad t \ge 8.
$$

If $P_1(t)$ fulfils the last inequality above, a straightforward verification yields that the conditions of Theorem 2 are satisfied and therefore (14) has a positive solution which tends to zero.

Competing interests
The author declares that they have no competing interests.

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