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An algorithm for a common minimum-norm zero of a finite family of monotone mappings in Banach spaces

Habtu Zegeye[1] and Naseer Shahzad[2*]

*Correspondence:
nshahzad@kau.edu.sa
[2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
We introduce an iterative process which converges strongly to a common minimum-norm point of solutions of a finite family of monotone mappings in Banach spaces. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.
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1 Introduction

In many problems, it is quite often to seek a particular solution of the minimum-norm solution of a given nonlinear problem. In an abstract way, we may formulate such problems as finding a point x^* with the property

$$x^* \in C \quad \text{such that} \quad \|x^*\| = \min_{x\in C}\{\|x\|\}, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H. In other words, x^* is the (nearest point or metric) projection of the origin onto C,

$$x^* = P_C(0), \tag{1.2}$$

where P_C is the metric (or nearest point) projection from H onto C. For instance, the split feasibility problem (SFP), introduced in [1, 2], is to find a point

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{1.3}$$

where C and Q are closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and A is a linear bounded operator from H_1 to H_2. We note that problem (1.3) can be extended to a problem of finding

$$x \in D(A)\cap D(B) \quad \text{such that} \quad x \in A^{-1}(0)\cap B^{-1}(0), \tag{1.4}$$

where $A : D(A) \to E^*$ and $B : D(B) \to E^*$ are *monotone mappings* on a subset of a Banach space E. The problem has been addressed by many authors in view of the applications in image recovery and signal processing; see, for example, [3–5] and the references therein.

A mapping $A : C \to E^*$ is said to be *monotone* if for each $x, y \in C$, the following inequality holds:

$$\langle x - y, Ax - Ay \rangle \geq 0, \tag{1.5}$$

where C is a nonempty subset of a real Banach space E with E^* as its dual. A is said to be *maximal monotone* if its graph is not properly contained in the graph of any other monotone mapping. A mapping $A : C \to E^*$ is said to be γ*-inverse strongly monotone* if there exists a positive real number γ such that

$$\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2 \quad \text{for all } x, y \in C, \tag{1.6}$$

and it is called *strongly monotone* if there exists $k > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq k\|x - y\|^2 \quad \text{for all } x, y \in C. \tag{1.7}$$

An operator $A : C \to E$ is called *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\bigl\langle Ax - Ay, j(x - y) \bigr\rangle \geq 0 \quad \text{for all } x, y \in C, \tag{1.8}$$

where J is the normalized duality mapping from E into 2^{E^*} defined for each $x \in E$ by

$$Jx := \bigl\{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\bigr\}.$$

It is well known that E is smooth if and only if J is single-valued, and if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E (see [6]). A is called *m-accretive* if it is accretive and $R(I + rA)$, the range of $(I + rA)$, is E for all $r > 0$; and an accretive mapping A is said to satisfy *range condition* if

$$D(A) \subseteq C \subseteq \bigcap_{r>0} R(I + rA) \tag{1.9}$$

for some nonempty closed convex subset C of a real Banach space H.

Clearly, the class of monotone mappings includes the class of strongly monotone and the class of γ-inverse strongly monotone mappings. However, we observe that accretive mappings and monotone mappings have different natures in Banach spaces more general than Hilbert spaces.

When A and B are *maximal monotone* mappings in Hilbert spaces, Bauschke *et al.* [7] proved that sequences generated from the method of alternating resolvents given by

$$\begin{cases} x_{2n+1} = J^A_\beta(x_{2n}), & n \geq 0, \\ x_{2n} = J^B_\mu(x_{2n-1}), & n \geq 0, \end{cases} \tag{1.10}$$

where $J^A_\mu := (I + \mu A)^{-1}$ is the *resolvent of* A, converge weakly to a point of $A^{-1}(0) \cap B^{-1}(0)$ provided that $A^{-1}(0) \cap B^{-1}(0)$ is nonempty. Note that strong convergence of these methods fails in general (see a counter example by Hundal [8]).

With regard to a finite family of m-accretive mappings, Zegeye and Shahzad [9] proved that under appropriate conditions, an iterative process of Halpern type defined by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)S_{r_n}x_n, \quad n \geq 0, \tag{1.11}$$

where $\alpha_n \in (0,1)$ for all $n \geq 0$, $u,x_0 \in H$, $S_r := a_0 I + a_1 J_r^1 + a_2 J_r^2 + \cdots + a_N J_r^N$ with $J_r^i = (I + rA_i)^{-1}$ for $a_i \in (0,1)$, $i = 0,1,\ldots,N$, and $\sum_{i=1} a_i^N = 1$, converges strongly to a point in $\bigcap_{i=1}^N A^{-1}(0)$ nearest to u, where $\{A_i : i = 1,2,\ldots,N\}$ is the set of a finite family of m-accretive mappings in a strictly convex and reflexive (real) Banach space E which has a uniformly Gâteaux differentiable norm.

In 2009, Hu and Liu [10] also proved that under appropriate conditions, an iterative process of Halpern type defined by

$$x_{n+1} = \alpha_n u + \delta_n x_n + \gamma_n S_{r_n}x_n, \quad n \geq 0, \tag{1.12}$$

where $\alpha_n, \delta_n, \gamma_n \in (0,1)$ with $\alpha_n + \delta_n + \gamma_n = 1$, for all $n \geq 0$, $u = x_0 \in H$, $S_{r_n} := a_0 I + a_1 J_{r_n}^1 + a_2 J_{r_n}^2 + \cdots + a_N J_{r_n}^N$ with $J_r^i = (I + rA_i)^{-1}$, for $a_i \in (0,1)$, $i = 0,1,\ldots,N$, and $\sum_{i=1} a_i^N = 1$, and $\{r_n\} \subset (0,\infty)$, for A_i, $i = 1,2,\ldots,N$, accretive mappings satisfying range condition (1.9), converges strongly to a point in $\bigcap_{i=1}^N A^{-1}(0)$ nearest to u in a strictly convex and reflexive (real) Banach space E which has a uniformly Gâteaux differentiable norm.

A natural question arises whether we can have the results of Zegeye and Shahzad [9] *and Hu and Liu* [10] *for the class of monotone mappings or not, in Banach spaces more general than Hilbert spaces?*

Let C be a nonempty, closed, and convex subset of a smooth and uniformly convex real Banach space E. Let $A_i : C \to E^*$ for $i = 1,2,\ldots,N$ be continuous monotone mappings satisfying range condition (2.1) with $F := \bigcap_{i=1}^N A_i^{-1}(0) \neq \emptyset$.

It is our purpose in this paper to introduce an iterative scheme (see (3.1)) which converges strongly to the common minimum-norm zero of the family $\{A_i, i = 1,2,\ldots,N\}$. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear mappings.

2 Preliminaries

Let E be a normed linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\right\}.$$

The space E is said to be *smooth* if $\rho_E(\tau) > 0$, $\forall \tau > 0$, and E is called *uniformly smooth* if and only if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$.

The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\right\}.$$

E is called *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$.

Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E with dual E^*. A monotone mapping A is said to satisfy *range condition* if we have that

$$D(A) \subseteq C \subseteq \bigcap_{r>0} J^{-1}R(J + rA) \tag{2.1}$$

for some nonempty closed convex subset C of a smooth, strictly convex, and reflexive Banach space E. In the sequel, the resolvent of a monotone mapping $A : C \to E^*$ shall be denoted by $Q_r^A := (J + rA)^{-1}J$ for $r > 0$. We know the following lemma.

Lemma 2.1 [11] *Let E be a smooth and strictly convex Banach space, C be a nonempty, closed, and convex subset of E, and $A \subset E \times E^*$ be a monotone mapping satisfying* (2.1). *Let $Q_{r_n}^A$ be the resolvent of A for $\{r_n\} \subset (0, \infty)$ such that* $\lim_{n\to\infty} r_n = \infty$. *If $\{x_n\}$ is a bounded sequence of C such that $Q_{r_n}^A x_n \rightharpoonup z$, then $z \in A^{-1}(0)$.*

Let E be a smooth Banach space with dual E^*. Let the Lyapunov function $\phi : E \times E \to \mathbb{R}$, introduced by Alber [12], be defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \quad \text{for } x, y \in E, \tag{2.2}$$

where J is the normalized duality mapping. If $E = H$, a Hilbert space, then the duality mapping becomes the identity map on H. We observe that in a Hilbert space H, (2.2) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$.

In the sequel, we shall make use of the following lemmas.

Lemma 2.2 [13] *Let E be a smooth and strictly convex Banach space, and C be a nonempty, closed, and convex subset of E. Let $A \subset E \times E^*$ be a monotone mapping satisfying* (2.1), *$A^{-1}(0)$ be nonempty and Q_r^A be the resolvent of A for some $r > 0$. Then, for each $r > 0$, we have that*

$$\phi\left(p, Q_r^A x\right) + \phi\left(Q_r^A x, x\right) \leq \phi(p, x)$$

for all $p \in A^{-1}(0)$ and $x \in C$.

Lemma 2.3 [14] *Let E be a smooth and strictly convex Banach space, C be a nonempty, closed, and convex subset of E, and T be a mapping from C into itself such that $F(T)$ is nonempty and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. Then $F(T)$ is closed and convex.*

Lemma 2.4 [15] *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V\left(x, x^*\right) = \|x\|^2 - 2\left\langle x, x^*\right\rangle + \left\|x^*\right\|^2 \quad \text{for all } x \in E \text{ and } x^* \in E,$$

studied by Alber [12]. That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.5 [12] *Let E be a reflexive, strictly convex, and smooth Banach space with E^* as its dual. Then*

$$V(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x,x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Let E be a reflexive, strictly convex, and smooth Banach space, and let C be a nonempty, closed, and convex subset of E. The *generalized projection mapping*, introduced by Alber [12], is a mapping $\Pi_C : E \to C$ that assigns an arbitrary point $x \in E$ to the minimizer, $\bar{x}$, of $\phi(\cdot, x)$ over C, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y,x), y \in C\}. \tag{2.3}$$

Lemma 2.6 [12] *Let C be a nonempty, closed, and convex subset of a real reflexive, strictly convex, and smooth Banach space E, and let $x \in E$. Then, $\forall y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y,x).$$

Lemma 2.7 [12] *Let C be a convex subset of a real smooth Banach space E. Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx - Jx_0\rangle \leq 0, \quad \forall z \in C.$$

Lemma 2.8 [16] *Let E be a uniformly convex Banach space and $B_R(0)$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_N x_N\|^2 \leq \sum_{i=0}^{N} \alpha_i \|x_i\|^2 - \alpha_i \alpha_j g(\|x_i - x_j\|)$$

for $\alpha_i \in (0,1)$ such that $\sum_{i=0}^{N} \alpha_i = 1$ and $x_i \in B_R(0) := \{x \in E : \|x\| \leq R\}$ for some $R > 0$.

Lemma 2.9 [17] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n \delta_n, \quad \forall n \geq n_0,$$

where $\{\beta_n\} \subset (0,1)$ and $\{\delta_n\} \subset R$ satisfy the following conditions: $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10 [18] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1,2,\dots,k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

3 Main result

We now prove the following theorem.

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a smooth and uniformly convex real Banach space E. Let $A_i : C \to E^*$, for $i = 1,2,\dots,N$, be continuous monotone mappings satisfying* (2.1). *Assume that $\mathcal{F} := \bigcap_{i=1}^N A_i^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad \textit{chosen arbitrarily}, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}(\beta_0 Jy_n + \sum_{i=1}^N \beta_i JQ_{r_n}^{A_i} y_n), \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=1}^N \subset [c,d] \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{i=0}^N \beta_i = 1$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then $\{x_n\}$ converges strongly to the minimum-norm point of $\mathcal{F}$.*

Proof From Lemmas 2.2 and 2.3 we get that $A_i^{-1}(0)$ is closed and convex. Thus, $\Pi_{\mathcal{F}}(0)$ is well defined. Let $p = \Pi_{\mathcal{F}}(0)$. Then from (3.1), Lemma 2.6 and the property of ϕ, we get that

$$\begin{aligned} \phi(p,y_n) &= \phi\bigl(p,\Pi_C(1-\alpha_n)x_n\bigr) \leq \phi\bigl(p,(1-\alpha_n)x_n\bigr) \\ &= \phi\bigl(p,J^{-1}\bigl(\alpha_n J0 + (1-\alpha_n)Jx_n\bigr)\bigr) \\ &= \|p\|^2 - 2\bigl\langle p,\alpha_n J0 + (1-\alpha_n)Jx_n\bigr\rangle + \bigl\|\alpha_n J0 + (1-\alpha_n)Jx_n\bigr\|^2 \\ &\leq \|p\|^2 - 2\alpha_n\langle p,J0\rangle - 2(1-\alpha_n)\langle p,Jx_n\rangle \\ &\quad + \alpha_n\|J0\|^2 + (1-\alpha_n)\|Jx_n\|^2 \\ &= \alpha_n\phi(p,0) + (1-\alpha_n)\phi(p,x_n). \end{aligned} \tag{3.2}$$

Moreover, from (3.1), Lemma 2.8, Lemma 2.2 and (3.2), we get that

$$\begin{aligned} \phi(p,x_{n+1}) &= \phi\left(p,J^{-1}\left(\beta_0 Jy_n + \sum_{i=1}^N \beta_i JQ_{r_n}^{A_i} y_n\right)\right) \\ &= \|p\|^2 - 2\left\langle p,\beta_0 Jy_n + \sum_{i=1}^N \beta_i JQ_{r_n}^{A_i} y_n\right\rangle + \left\|\beta_0 Jy_n + \sum_{i=1}^N \beta_i JQ_{r_n}^{A_i} y_n\right\|^2 \\ &\leq \|p\|^2 - 2\beta_0\langle p,Jy_n\rangle - 2\sum_{i=1}^N \beta_i\bigl\langle p,JQ_{r_n}^{A_i} y_n\bigr\rangle \\ &\quad + \beta_0\|y_n\|^2 + \sum_{i=1}^N \beta_i\bigl\|Q_{r_n}^{A_i} y_n\bigr\|^2 - \beta_0\beta_i g\bigl(\bigl\|Jy_n - JQ_{r_n}^{A_i} y_n\bigr\|\bigr) \\ &= \beta_0\phi(p,y_n) + \sum_{i=1}^N \beta_i\phi\bigl(p,Q_{r_n}^{A_i} y_n\bigr) - \beta_0\beta_i g\bigl(\bigl\|Jy_n - JQ_{r_n}^{A_i} y_n\bigr\|\bigr) \end{aligned}$$

$$\begin{aligned}
&\le \beta_0\phi(p,y_n) + (1-\beta_0)\phi(p,y_n) - \beta_0\beta_i g\bigl(\|Jy_n - JQ_{r_n}^{A_i}y_n\|\bigr) \\
&\le \phi(p,y_n) - \beta_0\beta_i g\bigl(\|Jy_n - JQ_{r_n}^{A_i}y_n\|\bigr) \le \phi(p,y_n) \qquad (3.3)\\
&\le \alpha_n\phi(p,0) + (1-\alpha_n)\phi(p,x_n) \qquad (3.4)
\end{aligned}$$

for each $i \in \{1,2,\dots,N\}$. Thus, by induction,

$$\phi(p,x_{n+1}) \le \max\bigl\{\phi(p,0),\phi(p,x_0)\bigr\}, \quad \forall n \ge 0,$$

which implies that $\{x_n\}$ and hence $\{y_n\}$ are bounded. Now let $z_n = (1-\alpha_n)x_n$. Then we note that $y_n = \Pi_C z_n$. Using Lemma 2.6, Lemma 2.5 and the property of ϕ, we obtain that

$$\begin{aligned}
\phi(p,y_n) &\le \phi(p,z_n) = V(p,Jz_n) \\
&\le V\bigl(p,Jz_n - \alpha_n(J0-Jp)\bigr) - 2\bigl\langle z_n - p, -\alpha_n(J0-Jp)\bigr\rangle \\
&= \phi\bigl(p,J^{-1}\bigl(\alpha_nJp + (1-\alpha_n)Jx_n\bigr)\bigr) - 2\alpha_n\langle z_n - p, Jp\rangle \\
&\le \alpha_n\phi(p,p) + (1-\alpha_n)\phi(p,x_n) - 2\alpha_n\langle z_n - p, Jp\rangle \\
&= (1-\alpha_n)\phi(p,x_n) - 2\alpha_n\langle z_n - p, Jp\rangle \\
&\le (1-\alpha_n)\phi(p,x_n) - 2\alpha_n\langle z_n - p, Jp\rangle. \qquad (3.5)
\end{aligned}$$

Furthermore, from (3.3) and (3.5) we have that

$$\begin{aligned}
\phi(p,x_{n+1}) &\le (1-\alpha_n)\phi(p,x_n) - 2\alpha_n\langle z_n - p, Jp\rangle \\
&\quad - \beta_0\beta_i g\bigl(\|Jy_n - JQ_{r_n}^{A_i}y_n\|\bigr) \qquad (3.6)\\
&\le (1-\alpha_n)\phi(p,x_n) - 2\alpha_n\langle z_n - p, Jp\rangle. \qquad (3.7)
\end{aligned}$$

Now, following the method of proof of Lemma 3.2 of Maingé [18], we consider two cases as follows.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(p,x_n)\}$ is nonincreasing for all $n \ge n_0$. In this situation, $\{\phi(p,x_n)\}$ is convergent. Then from (3.6) we have that

$$\beta_0\beta_i g\bigl(\|Jy_n - JQ_{r_n}^{A_i}y_n\|\bigr) \to 0, \qquad (3.8)$$

which implies, by the property of g, that

$$Jy_n - JQ_{r_n}^{A_i}y_n \to 0 \quad \text{as } n \to \infty, \qquad (3.9)$$

and hence, since J^{-1} is uniformly continuous on bounded sets, we obtain that

$$y_n - Q_{r_n}^{A_i}y_n \to 0 \quad \text{as } n \to \infty, \qquad (3.10)$$

for each $i \in \{1,2,\dots,N\}$.

Furthermore, Lemma 2.6, the property of ϕ and the fact that $\alpha_n \to 0$, as $n \to \infty$, imply that

$$\begin{aligned}\phi(x_n, y_n) &= \phi(x_n, \Pi_C z_n) \\ &\leq \phi(x_n, z_n) \\ &= \phi\bigl(x_n, J^{-1}\bigl(\alpha_n J0 + (1-\alpha_n)Jx_n\bigr)\bigr) \\ &\leq \alpha_n \phi(x_n, 0) + (1-\alpha_n)\phi(x_n, x_n) \\ &\leq \alpha_n \phi(x_n, 0) + (1-\alpha_n)\phi(x_n, x_n) \to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{3.11}$$

and hence from Lemma 2.4 we get that

$$x_n - y_n \to 0, \qquad x_n - z_n \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Since $\{z_n\}$ is bounded and E is reflexive, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle z_n - p, Jp\rangle = \lim_{i\to\infty}\langle z_{n_i} - p, Jp\rangle$. Then from (3.12) we get that

$$y_{n_i} \rightharpoonup z \quad \text{as } i \to \infty. \tag{3.13}$$

Thus, from (3.10) and Lemma 2.1, we obtain that $z \in A_i^{-1}(0)$ for each $i \in \{1, 2, \ldots, N\}$ and hence $z \in \bigcap_{i=1}^N A_i^{-1}(0)$.

Therefore, by Lemma 2.7, we immediately obtain that $\limsup_{n\to\infty}\langle z_n - p, Jp\rangle = \lim_{i\to\infty}\langle z_{n_i} - p, Jp\rangle = \langle z - p, Jp\rangle \geq 0$. It follows from Lemma 2.9 and (3.7) that $\phi(p, x_n) \to 0$ as $n \to \infty$. Consequently, from Lemma 2.4 we obtain that $x_n \to p$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\phi(p, x_{n_i}) < \phi(p, x_{n_i+1})$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\phi(p, x_{m_k}) \leq \phi(p, x_{m_k+1})$, and $\phi(p, x_k) \leq \phi(p, x_{m_k+1})$ for all $k \in \mathbb{N}$. Then, from (3.6) and the fact that $\alpha_n \to 0$, we obtain that

$$g\bigl(\bigl\|Jy_{m_k} - JQ_{r_{m_k}}^{A_i} y_{m_k}\bigr\|\bigr) \to 0 \quad \text{as } k \to \infty,$$

for each $i \in \{1, 2, \ldots, N\}$. Thus, following the method of proof of Case 1, we obtain that $y_{m_k} - Q_{r_{m_k}}^{A_i} y_{m_k} \to 0$, $x_{m_k} - y_{m_k} \to 0$, $x_{m_k} - z_{m_k} \to 0$ as $k \to \infty$, and hence we obtain that

$$\limsup_{k\to\infty}\langle z_{m_k} - p, Jp\rangle \geq 0. \tag{3.14}$$

Then from (3.7) we have that

$$\phi(p, x_{m_k+1}) \leq (1 - \alpha_{m_k})\phi(p, x_{m_k}) - 2\alpha_{m_k}\langle z_{m_k} - p, Jp\rangle. \tag{3.15}$$

Now, since $\phi(p, x_{m_k}) \leq \phi(p, x_{m_k+1})$, inequality (3.15) implies that

$$\begin{aligned}\alpha_{m_k}\phi(p, x_{m_k}) &\leq \phi(p, x_{m_k}) - \phi(p, x_{m_k+1}) - 2\alpha_{m_k}\langle z_{m_k} - p, Jp\rangle \\ &\leq -2\alpha_{m_k}\langle z_{m_k} - p, Jp\rangle.\end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\phi(p, x_{m_k}) \leq -2\langle z_{m_k} - p, Jp\rangle.$$

Then from (3.14) we obtain $\phi(p, x_{m_k}) \to 0$ as $k \to \infty$. This together with (3.15) gives $\phi(p, x_{m_k+1}) \to 0$ as $k \to \infty$. But $\phi(p, x_k) \leq \phi(p, x_{m_k+1})$ for all $k \in \mathbb{N}$, thus we obtain that $x_k \to p$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to p, which is the common minimum-norm zero of the family $\{A_i, i = 1, 2, \ldots, N\}$, and the proof is complete. □

We would like to mention that the method of proof of Theorem 3.1 provides the following theorem.

Theorem 3.2 *Let C be a nonempty, closed, and convex subset of a smooth and uniformly convex real Banach space E. Let $A_i : C \to E^*$, for $i = 1, 2, \ldots, N$, be continuous monotone mappings satisfying* (2.1). *Assume that $\mathcal{F} := \bigcap_{i=1}^N A_i^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u = x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)Jx_n), \\ x_{n+1} = J^{-1}(\beta_0 Jy_n + \sum_{i=1}^N \beta_i JQ_{r_n}^{A_i} y_n), \quad \forall n \geq 0, \end{cases} \tag{3.16}$$

where $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=1}^N \subset [c,d] \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{i=0}^N \beta_i = 1$, and $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}(u)$.

If in Theorem 3.1, $N = 1$, then we get the following corollary.

Corollary 3.3 *Let C be a nonempty, closed, and convex subset of a smooth and uniformly convex real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping satisfying* (2.1). *Assume that $A^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}(\beta Jy_n + (1-\beta)JQ_{r_n}^A y_n), \quad \forall n \geq 0, \end{cases} \tag{3.17}$$

where $\alpha_n \in (0,1)$, $\beta \in (0,1)$, and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $A^{-1}(0)$.

We remark that if A is a maximal monotone mapping, then $A^{-1}(0)$ is closed and convex (see [6] for more details). The following lemma is well known.

Lemma 3.4 [19] *Let E be a smooth, strictly convex, and reflexive Banach space, let C be a nonempty closed convex subset of E, and let $A \subset E \times E^*$ be a monotone mapping. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

We note from the above lemma that if A is maximal, then it satisfies condition (2.1) and hence we have the following corollary.

Corollary 3.5 *Let C be a nonempty, closed, and convex subset of a smooth and uniformly convex real Banach space E. Let $A_i : C \to E^*$, $i = 1, 2, \ldots, N$, be maximal monotone mappings. Assume that $\mathcal{F} := \bigcap_{i=1}^N A_i^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}(\beta_0 Jy_n + \sum_{i=1}^N \beta_i JQ_{r_n}^{A_i} y_n), \quad \forall n \geq 0, \end{cases} \tag{3.18}$$

where $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=1}^N \subset [c,d] \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{i=0}^N \beta_i = 1$ and $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $\mathcal{F}$.

If in Corollary 3.5, $N = 1$, then we get the following corollary.

Corollary 3.6 *Let C be a nonempty, closed and convex subset of a smooth and uniformly convex real Banach space E. Let $A : C \to E^*$ be a maximal monotone mapping. Assume that $A^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}(\beta Jy_n + (1-\beta)JQ_{r_n}^{A} y_n), \quad \forall n \geq 0, \end{cases} \tag{3.19}$$

where $\alpha_n \in (0,1)$, $\beta \in (0,1)$, and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $A^{-1}(0)$.

If $E = H$, a real Hilbert space, then E is uniformly convex and smooth real Banach space. In this case, $J = I$, identity map on H, and $\Pi_C = P_C$, projection mapping from H onto C. Furthermore, (2.1) reduces to (1.9). Thus, the following corollaries hold.

Corollary 3.7 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A_i : C \to E^*$, for $i = 1, 2, \ldots, N$, be continuous monotone mappings satisfying* (1.9). *Assume that $\mathcal{F} := \bigcap_{i=1}^N A_i^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = P_C[(1-\alpha_n)x_n], \\ x_{n+1} = \beta_0 y_n + \sum_{i=1}^N \beta_i Q_{r_n}^{A_i} y_n, \quad \forall n \geq 0, \end{cases} \tag{3.20}$$

where $Q_r^A := (I + rA)^{-1}$, $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=1}^N \subset [c,d] \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{i=0}^N \beta_i = 1$, and $\lim_{n\to\infty} r_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm element of $\mathcal{F}$.

Corollary 3.8 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A_i : C \to H$, $i = 1, 2, \ldots, N$, be maximal monotone mappings. Assume that $\mathcal{F} := \bigcap_{i=1}^N A_i^{-1}(0)$*

is nonempty. Let $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = P_C[(1-\alpha_n)x_n], \\ x_{n+1} = \beta_0 y_n + \sum_{i=1}^{N} \beta_i Q_{r_n}^{A_i} y_n, \quad \forall n \geq 0, \end{cases} \tag{3.21}$$

where $Q_r^A := (I + rA)^{-1}$, $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=1}^N \subset [c,d] \subset (0,1)$, *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{N} \beta_i = 1$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm element of* $\mathcal{F}$.

4 Application

In this section, we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional which has minimum-norm in Banach spaces. The following is deduced from Corollary 3.6.

Theorem 4.1 *Let E be a uniformly convex and uniformly smooth real Banach space. Let* f_i *be a continuously Fréchet differentiable convex functional on E, and let* $\triangledown f_i$ *be maximal monotone with* $\mathcal{F} := \bigcap_{i=1}^{N} (\triangledown f_i)^{-1}(0) \neq \emptyset$, *where* $(\triangledown f_i)^{-1}(0) = \{z \in E : f_i(z) = \min_{y\in E} f_i(y)\}$, *for* $i = 1, 2, \ldots, N$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 \in C, \quad \text{chosen arbitrarily}, \\ y_n = \Pi_C[(1-\alpha_n)x_n], \\ x_{n+1} = J^{-1}(\beta_0 J y_n + \sum_{i=1}^{N} \beta_i J(J + r_n \triangledown f_i)^{-1} J y_n), \quad \forall n \geq 0, \end{cases} \tag{4.1}$$

where $\alpha_n \in (0,1)$, $\{\beta_i\}_{i=1}^N \subset [c,d] \subset (0,1)$, *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{i=0}^{N} \beta_i = 1$, *and* $\lim_{n\to\infty} r_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm element of* $\mathcal{F}$.

Remark 4.2 Theorem 3.1 provides convergence scheme to the common minimum-norm zero of a finite family of monotone mappings which improves the results of Bauschke *et al.* [7] to Banach spaces more general than Hilbert spaces. We also note that our results complement the results of Zegeye and Shahzad [9] and Hu and Liu [10] which are convergence results for accretive mappings.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved final manuscript.

Author details
[1]Department of Mathematics, University of Botswana, Pvt. Bag 00704, Gaborone, Botswana. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

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