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# Some common fixed point results in ordered partial $b$-metric spaces

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**Abstract**
In this paper, we introduce a modified version of ordered partial $b$-metric spaces. We demonstrate a fundamental lemma for the convergence of sequences in such spaces. Using this lemma, we prove some fixed point and common fixed point results for $(\psi,\varphi)$-weakly contractive mappings in the setup of ordered partial $b$-metric spaces. Finally, examples are presented to verify the effectiveness and applicability of our main results.
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**Keywords:** ordered metric space; partial metric space; $b$-metric space; common fixed point; altering distance function

## 1 Introduction

Fixed points theorems in partially ordered metric spaces were firstly obtained in 2004 by Ran and Reurings [1], and then by Nieto and Lopez [2]. In this direction several authors obtained further results under weak contractive conditions (see, *e.g.*, [3–8]).

The concept of $b$-metric space was introduced by Bakhtin [9] and extensively used by Czerwik in [10, 11]. After that, several interesting results about the existence of a fixed point for single-valued and multi-valued operators in (ordered) $b$-metric spaces have been obtained (see, *e.g.*, [12–26]).

**Definition 1** [10] Let $X$ be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a $b$-metric on $X$ if, for all $x, y, z \in X$, the following conditions hold:

$(b_1)$ $d(x,y) = 0$ if and only if $x = y$,
$(b_2)$ $d(x,y) = d(y,x)$,
$(b_3)$ $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair $(X,d)$ is called a $b$-metric space.

On the other hand, Matthews [27] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. In partial metric spaces, self-distance of an arbitrary point need not be equal to zero. Several authors obtained many useful fixed point results in these spaces - we mention just [28–33].

**Definition 2** [27] A partial metric on a nonempty set $X$ is a mapping $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

($p_1$) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,
($p_2$) $p(x,x) \leq p(x,y)$,
($p_3$) $p(x,y) = p(y,x)$,
($p_4$) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

In this case, $(X,p)$ is called a partial metric space.

It is clear that if $p(x,y) = 0$, then from ($p_1$) and ($p_2$), $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$.

Each partial metric $p$ on a set $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

**Definition 3** [27] Let $(X,p)$ be a partial metric space, and let $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then:

(i) The sequence $\{x_n\}$ is said to converge to $x$ with respect to $\tau_p$ if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.
(ii) The sequence $\{x_n\}$ is said to be Cauchy in $(X,p)$ if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(iii) $(X,p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x)$.

The following example shows that a convergent sequence $\{x_n\}$ in a partial metric space $(X,p)$ may not be Cauchy. In particular, it shows that the limit may not be unique.

**Example 1** [32] Let $X = [0,\infty)$ and $p(x,y) = \max\{x,y\}$. Let

$$x_n = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k+1. \end{cases}$$

Then, clearly, $\{x_n\}$ is a convergent sequence and for every $x \geq 1$, we have $\lim_{n\to\infty} p(x_n,x) = p(x,x)$. But $\lim_{n,m\to\infty} p(x_n,x_m)$ does not exist, that is, $\{x_n\}$ is not a Cauchy sequence.

As a generalization and unification of partial metric and $b$-metric spaces, Shukla [34] introduced the concept of partial $b$-metric space as follows.

**Definition 4** [34] A partial $b$-metric on a nonempty set $X$ is a mapping $p_b : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

($p_{b1}$) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$,
($p_{b2}$) $p_b(x,x) \leq p_b(x,y)$,
($p_{b3}$) $p_b(x,y) = p_b(y,x)$,
($p_{b4}$) $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

A partial $b$-metric space is a pair $(X,p_b)$ such that $X$ is a nonempty set and $p_b$ is a partial $b$-metric on $X$. The number $s \geq 1$ is called the coefficient of $(X,p_b)$.

In a partial $b$-metric space $(X, p_b)$, if $x, y \in X$ and $p_b(x, y) = 0$, then $x = y$, but the converse may not be true. It is clear that every partial metric space is a partial $b$-metric space with the coefficient $s = 1$ and every $b$-metric space is a partial $b$-metric space with the same coefficient and zero self-distance. However, the converse of these facts need not hold.

**Example 2** [34] Let $X = \mathbb{R}^+$, $q > 1$ be a constant and $p_b : X \times X \to \mathbb{R}^+$ be defined by

$$p_b(x, y) = \left[\max\{x, y\}\right]^q + |x - y|^q \quad \text{for all } x, y \in X.$$

Then $(X, p_b)$ is a partial $b$-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a $b$-metric nor a partial metric space.

Note that in a partial $b$-metric space the limit of a convergent sequence may not be unique (see [34, Example 2]).

Some more examples of partial $b$-metrics can be constructed with the help of the following propositions.

**Proposition 1** [34] *Let $X$ be a nonempty set, and let $p$ be a partial metric and $d$ be a $b$-metric with the coefficient $s \geq 1$ on $X$. Then the function $p_b : X \times X \to \mathbb{R}^+$, defined by $p_b(x, y) = p(x, y) + d(x, y)$ for all $x, y \in X$, is a partial $b$-metric on $X$ with the coefficient $s$.*

**Proposition 2** [34] *Let $(X, p)$ be a partial metric space and $q \geq 1$. Then $(X, p_b)$ is a partial $b$-metric space with the coefficient $s = 2^{q-1}$, where $p_b$ is defined by $p_b(x, y) = [p(x, y)]^q$.*

Altering distance functions were introduced by Khan *et al.* in [35].

**Definition 5** [35] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

1. $\psi$ is continuous and nondecreasing;
2. $\psi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems which are based on altering distance functions (see, *e.g.*, [12, 28, 36–41]).

In this paper, we introduce a modified version of ordered partial $b$-metric spaces. We demonstrate a fundamental lemma for the convergence of sequences in such spaces. Using this lemma, we prove some fixed point and common fixed point results for $(\psi, \varphi)$-weakly contractive mappings in the setup of ordered partial $b$-metric spaces. Finally, examples are presented to verify the effectiveness and applicability of our main results.

## 2 Definition and basic properties of partial *b*-metric spaces

In the following definition, we modify Definition 4 in order to obtain that each partial $b$-metric $p_b$ generates a $b$-metric $d_{p_b}$.

**Definition 6** Let $X$ be a (nonempty) set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}^+$ is a partial $b$-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

($p_{b1}$) $x = y \Longleftrightarrow p_b(x, x) = p_b(x, y) = p_b(y, y)$,
($p_{b2}$) $p_b(x, x) \leq p_b(x, y)$,

($\mathrm{p}_{b3}$) $p_b(x,y) = p_b(y,x)$,
($\mathrm{p}_{b4'}$) $p_b(x,y) \leq s(p_b(x,z) + p_b(z,y) - p_b(z,z)) + (\frac{1-s}{2})(p_b(x,x) + p_b(y,y))$.

The pair $(X, p_b)$ is called a partial $b$-metric space.

Since $s \geq 1$, from ($\mathrm{p}_{b4'}$) we have

$$p_b(x,y) \leq s\big(p_b(x,z) + p_b(z,y) - p_b(z,z)\big) \leq s\big(p_b(x,z) + p_b(z,y)\big) - p_b(z,z).$$

Hence, a partial $b$-metric in the sense of Definition 6 is also a partial $b$-metric in the sense of Definition 4.

It should be noted that the class of partial $b$-metric spaces is larger than the class of partial metric spaces, since a partial $b$-metric is a partial metric when $s = 1$. We present an example which shows that a partial $b$-metric on $X$ (in the sense of Definition 6) might be neither a partial metric, nor a $b$-metric on $X$.

**Example 3** Let $(X,d)$ be a metric space and $p_b(x,y) = d(x,y)^q + a$, where $q > 1$ and $a \geq 0$ are real numbers. We will show that $p_b$ is a partial $b$-metric with $s = 2^{q-1}$.

Obviously, conditions ($\mathrm{p}_{b1}$)-($\mathrm{p}_{b3}$) of Definition 6 are satisfied.

Since $q > 1$, the convexity of the function $f(x) = x^q$ ($x > 0$) implies that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$ holds for $a, b \geq 0$. Thus, for each $x, y, z \in X$, we obtain

$$\begin{aligned} p_b(x,y) &= d(x,y)^q + a \leq \big(d(x,z) + d(z,y)\big)^q + a \\ &\leq 2^{q-1}\big(d(x,z)^q + d(z,y)^q\big) + a \\ &= 2^{q-1}\big(d(x,z)^q + a + d(z,y)^q + a - a\big) + a - 2^{q-1}a \\ &= 2^{q-1}\big(p_b(x,z) + p_b(z,y) - p_b(z,z)\big) + \left(\frac{1-2^{q-1}}{2}\right)\big(p_b(x,x) + p_b(y,y)\big). \end{aligned}$$

Hence, condition ($\mathrm{p}_{b4'}$) of Definition 6 is fulfilled and $p_b$ is a partial $b$-metric on $X$.

Note that $(X, p_b)$ is not necessarily a partial metric space. For example, if $X = \mathbb{R}$ is the set of real numbers, $d(x,y) = |x-y|$, $q = 2$ and $a = 3$, then $p_b(x,y) = (x-y)^2 + 3$ is a partial $b$-metric on $X$ with $s = 2^{2-1} = 2$, but it is not a partial metric on $X$. Indeed, the ordinary (partial) triangle inequality does not hold. To see this, let $x = 2$, $y = 5$ and $z = \frac{5}{2}$. Then $p_b(2,5) = 12$, $p_b(2,\frac{5}{2}) = \frac{13}{4}$ and $p_b(\frac{5}{2},5) = \frac{37}{4}$, hence $p_b(2,5) = 12 \nleq \frac{38}{4} = p_b(2,\frac{5}{2}) + p_b(\frac{5}{2},5) - p_b(\frac{5}{2},\frac{5}{2})$.

Also, $p_b$ is not a $b$-metric since $p_b(x,x) \neq 0$ for $x \in X$.

**Proposition 3** *Every partial b-metric $p_b$ defines a b-metric $d_{p_b}$, where*

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y)$$

*for all $x, y \in X$.*

*Proof* Let $x, y, z \in X$. Then we have

$$\begin{aligned} &d_{p_b}(x,y) \\ &\quad = 2p_b(x,y) - p_b(x,x) - p_b(y,y) \end{aligned}$$

$$
\begin{aligned}
&\leq 2\left[s\bigl(p_b(x,z)+p_b(z,y)-p_b(z,z)\bigr)+\left(\frac{1-s}{2}\right)\bigl(p_b(x,x)+p_b(y,y)\bigr)\right] \\
&\quad - p_b(x,x)-p_b(y,y) \\
&= 2sp_b(x,z)+2sp_b(z,y)-2sp_b(z,z)+(1-s)\bigl(p_b(x,x)+p_b(y,y)\bigr) \\
&\quad - p_b(x,x)-p_b(y,y) \\
&= 2sp_b(x,z)+2sp_b(z,y)-2sp_b(z,z)-sp_b(x,x)-sp_b(y,y) \\
&= s\bigl[2p_b(x,z)-p_b(x,x)-p_b(z,z)+2p_b(z,y)-p_b(z,z)-p_b(y,y)\bigr] \\
&= s\bigl[d_{p_b}(x,z)+d_{p_b}(z,y)\bigr].
\end{aligned}
$$
□

Hence, the advantage of our definition of partial $b$-metric is that by using it we can define a dependent $b$-metric which we call the $b$-metric associated with $p_b$. This allows us to readily transport many concepts and results from $b$-metric spaces into a partial $b$-metric space.

Now, we present some definitions and propositions in a partial $b$-metric space.

**Definition 7** Let $(X,p_b)$ be a partial $b$-metric space. Then, for $x \in X$ and $\epsilon > 0$, the $p_b$-ball with center $x$ and radius $\epsilon$ is

$$B_{p_b}(x,\epsilon) = \bigl\{y \in X \mid p_b(x,y) < p_b(x,x)+\epsilon\bigr\}.$$

For example, let $(X,p_b)$ be the partial $b$-metric space from Example 3 (with $X = \mathbb{R}$, $q = 2$ and $a = 3$). Then

$$
\begin{aligned}
B_{p_b}(1,4) &= \bigl\{y \in X \mid p_b(1,y) < p_b(1,1)+4\bigr\} = \bigl\{y \in X \mid (y-1)^2+3 < 3+4\bigr\} \\
&= \bigl\{y \in X \mid (y-1)^2 < 4\bigr\} = (-1,3).
\end{aligned}
$$

**Proposition 4** *Let $(X,p_b)$ be a partial $b$-metric space, $x \in X$ and $r > 0$. If $y \in B_{p_b}(x,r)$, then there exists $\delta > 0$ such that $B_{p_b}(y,\delta) \subseteq B_{p_b}(x,r)$.*

*Proof* Let $y \in B_{p_b}(x,r)$. If $y = x$, then we choose $\delta = r$. Suppose that $y \neq x$. Then we have $p_b(x,y) \neq 0$. Now, we consider two cases.

*Case 1.* If $p_b(x,y) = p_b(x,x)$, then for $s = 1$ we choose $\delta = r$. If $s > 1$, then we consider the set

$$A = \left\{n \in \mathbb{N} \,\Big|\, \frac{r}{2s^{n+1}(s-1)} < p_b(x,x)\right\}.$$

By the Archimedean property, $A$ is a nonempty set; then by the well ordering principle, $A$ has the least element $m$. Since $m-1 \notin A$, we have $p_b(x,x) \leq r/(2s^m(s-1))$ and we choose $\delta = r/(2s^{m+1})$. Let $z \in B_{p_b}(y,\delta)$; by the property $(\mathrm{p}_{b4})$, we have

$$
\begin{aligned}
p_b(x,z) &\leq s\bigl(p_b(x,y)+p_b(y,z)-p_b(y,y)\bigr) \\
&\leq s\bigl(p_b(x,x)+\delta\bigr) \\
&\leq p_b(x,x)+\frac{r}{2s^m}+\frac{r}{2s^m}
\end{aligned}
$$

$$\begin{aligned}
&= p_b(x,x) + \frac{r}{s^m} \\
&< p_b(x,x) + r.
\end{aligned}$$

Hence, $B_{p_b}(y,\delta) \subseteq B_{p_b}(x,r)$.

*Case 2.* If $p_b(x,y) \neq p_b(x,x)$, then from the property ($\mathrm{p}_{b2}$) we have $p_b(x,x) < p_b(x,y)$ and for $s = 1$ we consider the set

$$B = \left\{ n \in \mathbb{N} \;\middle|\; \frac{r}{2^{n+3}} < p_b(x,y) - p_b(x,x) \right\}.$$

Similarly, by the well ordering principle, there exists an element $m$ such that $p_b(x,y) - p_b(x,x) \le r/(2^{m+2})$, and we choose $\delta = r/(2^{m+2})$. One can easily obtain that $B_{p_b}(y,\delta) \subseteq B_{p_b}(x,r)$.

For $s > 1$, we consider the set

$$C = \left\{ n \in \mathbb{N} \;\middle|\; \frac{r}{2s^{n+2}} < p_b(x,y) - \frac{1}{s} p_b(x,x) \right\}$$

and by the well ordering principle, there exists an element $m$ such that $p_b(x,y) - \frac{1}{s}p_b(x,x) \le \frac{r}{2s^{m+1}}$ and we choose $\delta = \frac{r}{2s^{m+1}}$. Let $z \in B_{p_b}(y,\delta)$. By the property ($\mathrm{p}_{b4}$), we have

$$\begin{aligned}
p_b(x,z) &\le s\bigl(p_b(x,y) + p_b(y,z) - p_b(y,y)\bigr) \\
&\le s\bigl(p_b(x,y) + \delta\bigr) \\
&\le p_b(x,x) + \frac{r}{2s^m} + \frac{r}{2s^m} \\
&= p_b(x,x) + \frac{r}{s^m} \\
&< p_b(x,x) + r.
\end{aligned}$$

Hence, $B_{p_b}(y,\delta) \subseteq B_{p_b}(x,r)$. □

Thus, from the above proposition the family of all $p_b$-balls

$$\Delta = \bigl\{ B_{p_b}(x,r) \mid x \in X, r > 0 \bigr\}$$

is a base of a $T_0$ topology $\tau_{p_b}$ on $X$ which we call the $p_b$-metric topology.

The topological space $(X,p_b)$ is $T_0$, but need not be $T_1$.

**Definition 8** A sequence $\{x_n\}$ in a partial $b$-metric space $(X,p_b)$ is said to be:

(i) $p_b$-convergent to a point $x \in X$ if $\lim_{n\to\infty} p_b(x,x_n) = p_b(x,x)$;

(ii) a $p_b$-Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n,x_m)$ exists (and is finite).

(iii) A partial $b$-metric space $(X,p_b)$ is said to be $p_b$-complete if every $p_b$-Cauchy sequence $\{x_n\}$ in $X$ $p_b$-converges to a point $x \in X$ such that $\lim_{n,m\to\infty} p_b(x_n,x_m) = \lim_{n,m\to\infty} p_b(x_n,x) = p_b(x,x)$.

The following lemma shows the relationship between the concepts of $p_b$-convergence, $p_b$-Cauchyness and $p_b$-completeness in two spaces $(X,p_b)$ and $(X,d_{p_b})$ which we state and prove according to Lemma 2.2 of [31].

**Lemma 1**

(1) *A sequence $\{x_n\}$ is a $p_b$-Cauchy sequence in a partial b-metric space $(X,p_b)$ if and only if it is a b-Cauchy sequence in the b-metric space $(X,d_{p_b})$.*

(2) *A partial b-metric space $(X,p_b)$ is $p_b$-complete if and only if the b-metric space $(X,d_{p_b})$ is b-complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x,x_n) = 0$ if and only if*

$$\lim_{n\to\infty} p_b(x,x_n) = \lim_{n,m\to\infty} p_b(x_n,x_m) = p_b(x,x).$$

*Proof* First, we show that every $p_b$-Cauchy sequence in $(X,p_b)$ is a $b$-Cauchy sequence in $(X,d_{p_b})$. Let $\{x_n\}$ be a $p_b$-Cauchy sequence in $(X,p_b)$. Then, there exists $\alpha \in \mathbb{R}$ such that, for arbitrary $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ with

$$\left|p_b(x_n,x_m) - \alpha\right| < \frac{\varepsilon}{4}$$

for all $n,m \geq n_\varepsilon$. Hence,

$$\begin{aligned}
&\left|d_{p_b}(x_n,x_m)\right| \\
&\quad = 2p_b(x_n,x_m) - p_b(x_n,x_n) - p_b(x_m,x_m) \\
&\quad = \left|p_b(x_n,x_m) - \alpha + \alpha - p_b(x_n,x_n) + p_b(x_m,x_n) - \alpha + \alpha - p_b(x_m,x_m)\right| \\
&\quad \leq \left|p_b(x_n,x_m) - \alpha\right| + \left|\alpha - p_b(x_n,x_n)\right| + \left|p_b(x_m,x_n) - \alpha\right| + \left|\alpha - p_b(x_m,x_m)\right| \\
&\quad < \varepsilon
\end{aligned}$$

for all $n,m \geq n_\varepsilon$. Hence, we conclude that $\{x_n\}$ is a $b$-Cauchy sequence in $(X,d_{p_b})$.

Next, we prove that $b$-completeness of $(X,d_{p_b})$ implies $p_b$-completeness of $(X,p_b)$. Indeed, if $\{x_n\}$ is a $p_b$-Cauchy sequence in $(X,p_b)$, then according to the above discussion, it is also a $b$-Cauchy sequence in $(X,d_{p_b})$. Since the $b$-metric space $(X,d_{p_b})$ is $b$-complete, we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} d_{p_b}(y,x_n) = 0$. Hence,

$$\lim_{n\to\infty}\left[p_b(x_n,y) - p_b(y,y) + p_b(y,x_n) - p_b(x_n,x_n)\right] = 0,$$

therefore, $\lim_{n\to\infty}[p_b(x_n,y) - p_b(y,y)] = 0$. Further, we have

$$\lim_{n\to\infty}\left[p_b(y,x_n) - p_b(x_n,x_n)\right] = 0.$$

Consequently,

$$\lim_{n\to\infty} p_b(x_n,y) = p_b(y,y) = \lim_{n\to\infty} p_b(x_n,x_n).$$

On the other hand,

$$\begin{aligned}
\lim_{n,m\to\infty} p_b(x_n,x_m) &\leq \lim_{n,m\to\infty} sp_b(x_n,y) + \lim_{n,m\to\infty} sp_b(x_m,y) - sp_b(y,y) \\
&\quad + \left(\frac{1-s}{2}\right)\left(p_b(x_n,x_n) + p_b(x_m,x_m)\right) \\
&= p_b(y,y).
\end{aligned}$$

Also, from $(\mathrm{p}_{b2})$,

$$p_b(y,y) \le \lim_{n,m\to\infty} p_b(x_n,y) = \lim_{n,m\to\infty} p_b(x_n,x_n) \le \lim_{n,m\to\infty} p_b(x_n,x_m).$$

Hence, we obtain that $\{x_n\}$ is a $p_b$-convergent sequence in $(X,p_b)$.

Now, we prove that every $b$-Cauchy sequence $\{x_n\}$ in $(X,d_{p_b})$ is a $p_b$-Cauchy sequence in $(X,p_b)$. Let $\varepsilon = \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that $d_{p_b}(x_n,x_m) < \frac{1}{2}$ for all $n,m \ge n_0$. Since

$$p_b(x_n,x_{n_0}) - p_b(x_{n_0},x_{n_0}) \le d_{p_b}(x_n,x_{n_0}) < \frac{1}{2},$$

hence

$$p_b(x_n,x_n) \le p_b(x_n,x_{n_0}) \le d_{p_b}(x_n,x_{n_0}) + p_b(x_{n_0},x_{n_0}) < \frac{1}{2} + p_b(x_{n_0},x_{n_0}).$$

Consequently, the sequence $\{p_b(x_n,x_n)\}$ is bounded in $\mathbb{R}$, and so there exists $a \in \mathbb{R}$ such that a subsequence $\{p_b(x_{n_k},x_{n_k})\}$ of $\{p_b(x_n,x_n)\}$ is convergent to $a$, *i.e.*,

$$\lim_{k\to\infty} p_b(x_{n_k},x_{n_k}) = a.$$

Now, we prove that $\{p_b(x_n,x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a $b$-Cauchy sequence in $(X,d_{p_b})$ for given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d_{p_b}(x_n,x_m) < \varepsilon$ for all $n,m \ge n_\varepsilon$. Thus, for all $n,m \ge n_\varepsilon$,

$$\begin{aligned} p_b(x_n,x_n) - p_b(x_m,x_m) &\le p_b(x_n,x_m) - p_b(x_m,x_m) \\ &\le d_{p_b}(x_m,x_n) < \varepsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} p_b(x_n,x_n) = a$.

On the other hand,

$$\begin{aligned} \bigl|p_b(x_n,x_m) - a\bigr| &= \bigl|p_b(x_n,x_m) - p_b(x_n,x_n) + p_b(x_n,x_n) - a\bigr| \\ &\le d_{p_b}(x_m,x_n) + \bigl|p_b(x_n,x_n) - a\bigr| \end{aligned}$$

for all $n,m \ge n_\varepsilon$. Hence, $\lim_{n,m\to\infty} p_b(x_n,x_m) = a$, and consequently, $\{x_n\}$ is a $p_b$-Cauchy sequence in $(X,p_b)$.

Conversely, let $\{x_n\}$ be a $b$-Cauchy sequence in $(X,d_{p_b})$. Then $\{x_n\}$ is a $p_b$-Cauchy sequence in $(X,p_b)$, and so it is convergent to a point $x \in X$ with

$$\lim_{n\to\infty} p_b(x,x_n) = \lim_{n,m\to\infty} p_b(x_m,x_n) = p_b(x,x).$$

Then, for given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$p_b(x,x_n) - p_b(x,x) < \frac{\varepsilon}{4}$$

and

$$p_b(x_n,x_n) - p_b(x,x) \le p_b(x_m,x_n) - p_b(x,x) < \frac{\varepsilon}{4}.$$

Therefore,

$$\begin{aligned}\left|d_{p_b}(x_n,x)\right| &= \left|p_b(x_n,x) - p_b(x_n,x_n) + p_b(x_n,x) - p_b(x,x)\right| \\ &\leq \left|p_b(x_n,x) - p_b(x,x)\right| + \left|p_b(x,x) - p_b(x_n,x_n)\right| + \left|p_b(x_n,x) - p_b(x,x)\right| \\ &< \varepsilon,\end{aligned}$$

whenever $n \geq n_\varepsilon$. Therefore, $(X, d_{p_b})$ is complete.

Finally, let $\lim_{n\to\infty} d_{p_b}(x_n,x) = 0$. So,

$$\lim_{n\to\infty}\left[p_b(x_n,x) - p_b(x_n,x_n)\right] + \lim_{n\to\infty}\left[p_b(x_n,x) - p_b(x,x)\right] = 0.$$

On the other hand,

$$\begin{aligned}&\lim_{n,m\to\infty}\left[p_b(x_n,x_m) - p_b(x,x)\right] \\ &\quad\leq \lim_{n\to\infty}\Bigg[sp_b(x_n,x) + sp_b(x,x_m) - sp_b(x,x) \\ &\qquad + \left(\frac{1-s}{2}\right)\left(p_b(x_n,x_n) + p_b(x_m,x_m)\right) - p_b(x,x)\Bigg] \\ &\quad= 0.\end{aligned}$$

□

**Definition 9** Let $(X,p_b)$ and $(X',p'_b)$ be two partial $b$-metric spaces, and let $f:(X,p_b)\to(X',p'_b)$ be a mapping. Then $f$ is said to be $p_b$-continuous at a point $a\in X$ if for a given $\varepsilon>0$, there exists $\delta>0$ such that $x\in X$ and $p_b(a,x)<\delta+p_b(a,a)$ imply that $p'_b(f(a),f(x))<\varepsilon+p'_b(f(a),f(a))$. The mapping $f$ is $p_b$-continuous on $X$ if it is $p_b$-continuous at all $a\in X$.

**Proposition 5** *Let $(X,p_b)$ and $(X',p'_b)$ be two partial b-metric spaces. Then a mapping $f:X\to X'$ is $p_b$-continuous at a point $x\in X$ if and only if it is $p_b$-sequentially continuous at $x$; that is, whenever $\{x_n\}$ is $p_b$-convergent to $x$, $\{f(x_n)\}$ is $p'_b$-convergent to $f(x)$.*

**Definition 10** A triple $(X,\preceq,p_b)$ is called an ordered partial $b$-metric space if $(X,\preceq)$ is a partially ordered set and $p_b$ is a partial $b$-metric on $X$.

## 3 Fixed point results in partial *b*-metric spaces

The following crucial lemma is useful in proving our main results.

**Lemma 2** *Let $(X,p_b)$ be a partial b-metric space with the coefficient $s>1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to $x$ and $y$, respectively. Then we have*

$$\begin{aligned}\frac{1}{s^2}p_b(x,y) - \frac{1}{s}p_b(x,x) - p_b(y,y) &\leq \liminf_{n\to\infty} p_b(x_n,y_n) \leq \limsup_{n\to\infty} p_b(x_n,y_n) \\ &\leq sp_b(x,x) + s^2p_b(y,y) + s^2p_b(x,y).\end{aligned}$$

*In particular, if $p_b(x,y)=0$, then we have $\lim_{n\to\infty} p_b(x_n,y_n)=0$.*

*Moreover, for each $z \in X$, we have*

$$\begin{aligned}\frac{1}{s}p_b(x,z) - p_b(x,x) &\leq \liminf_{n\to\infty} p_b(x_n,z) \leq \limsup_{n\to\infty} p_b(x_n,z)\\ &\leq sp_b(x,z) + sp_b(x,x).\end{aligned}$$

*In particular, if $p_b(x,x) = 0$, then we have*

$$\frac{1}{s}p_b(x,z) \leq \liminf_{n\to\infty} p_b(x_n,z) \leq \limsup_{n\to\infty} p_b(x_n,z) \leq sp_b(x,z).$$

*Proof* Using the triangle inequality in a partial $b$-metric space, it is easy to see that

$$p_b(x,y) \leq sp_b(x,x_n) + s^2 p_b(x_n,y_n) + s^2 p_b(y_n,y)$$

and

$$p_b(x_n,y_n) \leq sp_b(x_n,x) + s^2 p_b(x,y) + s^2 p_b(y,y_n).$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality, we obtain the first desired result. If $p_b(x,y) = 0$, then by the triangle inequality we get $p_b(x,x) = 0$ and $p_b(y,y) = 0$. Therefore, we have $\lim_{n\to\infty} p_b(x_n,y_n) = 0$. Similarly, using again the triangle inequality, the other assertions follow. □

Let $(X,\preceq,p_b)$ be an ordered partial $b$-metric space, and let $f : X \to X$ be a mapping. Set

$$M_s^f(x,y) = \max\left\{p_b(x,y), p_b(x,fx), p_b(y,fy), \frac{p_b(x,fy) + p_b(y,fx)}{2s}\right\}.$$

**Definition 11** Let $(X,p_b)$ be an ordered partial $b$-metric space. We say that a mapping $f : X \to X$ is a generalized $(\psi,\varphi)_s$-weakly contractive mapping if there exist two altering distance functions $\psi$ and $\varphi$ such that

$$\psi\left(sp_b(fx,fy)\right) \leq \psi\left(M_s^f(x,y)\right) - \varphi\left(M_s^f(x,y)\right) \tag{3.1}$$

for all comparable $x, y \in X$.

First, we prove the following result.

**Theorem 1** *Let $(X,\preceq,p_b)$ be a $p_b$-complete ordered partial b-metric space. Let $f : X \to X$ be a nondecreasing, with respect to $\preceq$, continuous mapping. Suppose that $f$ is a generalized $(\psi,\varphi)_s$-weakly contractive mapping. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then $f$ has a fixed point.*

*Proof* Let $x_0 \in X$ be such that $x_0 \preceq fx_0$. Then we define a sequence $(x_n)$ in $X$ such that $x_{n+1} = fx_n$ for all $n \geq 0$. Since $x_0 \preceq fx_0 = x_1$ and $f$ is nondecreasing, we have $x_1 = fx_0 \preceq x_2 =$

$fx_1$. Again, as $x_1 \preceq x_2$ and $f$ is nondecreasing, we have $x_2 = fx_1 \preceq x_3 = fx_2$. By induction, we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = fx_n$ and hence $x_n$ is a fixed point of $f$. So, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By (3.1), we have

$$\begin{aligned}\psi\left(p_b(x_n,x_{n+1})\right) &\leq \psi\left(sp_b(x_n,x_{n+1})\right)\\ &= \psi\left(sp_b(fx_{n-1},fx_n)\right)\\ &\leq \psi\left(M_s^f(x_{n-1},x_n)\right) - \varphi\left(M_s^f(x_{n-1},x_n)\right),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}M_s^f(x_{n-1},x_n) &= \max\bigg\{p_b(x_{n-1},x_n), p_b(x_{n-1},fx_{n-1}), p_b(x_n,fx_n),\\ &\qquad \frac{p_b(x_{n-1},fx_n)+p_b(x_n,fx_{n-1})}{2s}\bigg\}\\ &= \max\left\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1}), \frac{p_b(x_{n-1},x_{n+1})+p_b(x_n,x_n)}{2s}\right\}\\ &\leq \max\bigg\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1}),\\ &\qquad \frac{sp_b(x_{n-1},x_n)+sp_b(x_n,x_{n+1})+(1-s)p_b(x_n,x_n)}{2s}\bigg\}\\ &= \max\left\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1})\right\}.\end{aligned}$$

So, we have

$$M_s^f(x_{n-1},x_n) = \max\left\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1})\right\}. \tag{3.3}$$

From (3.2), (3.3) we get

$$\begin{aligned}\psi\left(p_b(x_n,x_{n+1})\right) &\leq \psi\left(\max\left\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1})\right\}\right)\\ &\quad - \varphi\left(\max\left\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1})\right\}\right).\end{aligned} \tag{3.4}$$

If

$$\max\left\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1})\right\} = p_b(x_n,x_{n+1}),$$

then by (3.4) and properties of $\varphi$, we have

$$\begin{aligned}\psi\left(p_b(x_n,x_{n+1})\right) &\leq \psi\left(p_b(x_n,x_{n+1})\right) - \varphi\left(p_b(x_n,x_{n+1})\right)\\ &< \psi\left(p_b(x_n,x_{n+1})\right),\end{aligned}$$

which gives a contradiction. Thus,

$$\psi\big(p_b(x_n,x_{n+1})\big) \le \psi\big(p_b(x_{n-1},x_n)\big) - \varphi\big(p_b(x_{n-1},x_n)\big). \tag{3.5}$$

Therefore, $\{p_b(x_n,x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$ is a nonincreasing sequence of positive numbers. So, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} p_b(x_n,x_{n+1}) = r.$$

Letting $n \to \infty$ in (3.5), we get

$$\psi(r) \le \psi(r) - \varphi(r) \le \psi(r).$$

Therefore, $\varphi(r) = 0$, and hence $r = 0$. Thus, we have

$$\lim_{n\to\infty} p_b(x_n,x_{n+1}) = 0. \tag{3.6}$$

Next, we show that $\{x_n\}$ is a $p_b$-Cauchy sequence in $X$. For this, we have to show that $\{x_n\}$ is a $b$-Cauchy sequence in $(X, d_{p_b})$ (see Lemma 1). Suppose the contrary; that is, $\{x_n\}$ is not a $b$-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that $n_i$ is the smallest index for which

$$n_i > m_i > i, \quad d_{p_b}(x_{m_i},x_{n_i}) \ge \varepsilon. \tag{3.7}$$

This means that

$$d_{p_b}(x_{m_i},x_{n_i-1}) < \varepsilon. \tag{3.8}$$

From (3.7) and using the triangular inequality, we get

$$\varepsilon \le d_{p_b}(x_{m_i},x_{n_i}) \le s d_{p_b}(x_{m_i},x_{n_i-1}) + s d_{p_b}(x_{n_i-1},x_{n_i}). \tag{3.9}$$

Taking the upper limit as $i \to \infty$ and using (3.8), we get

$$\frac{\varepsilon}{s} \le \liminf_{i\to\infty} d_{p_b}(x_{m_i},x_{n_i-1}) \le \limsup_{i\to\infty} d_{p_b}(x_{m_i},x_{n_i-1}) \le \varepsilon. \tag{3.10}$$

Also, from (3.9) and (3.10),

$$\varepsilon \le \limsup_{i\to\infty} d_{p_b}(x_{m_i},x_{n_i}) \le s\varepsilon.$$

Further,

$$d_{p_b}(x_{m_i+1},x_{n_i}) \le s d_{p_b}(x_{m_i+1},x_{m_i}) + s d_{p_b}(x_{m_i},x_{n_i}),$$

and hence

$$\limsup_{i\to\infty} d_{p_b}(x_{m_i+1},x_{n_i}) \le s^2\varepsilon.$$

Finally,

$$d_{p_b}(x_{m_i+1}, x_{n_i-1}) \leq s d_{p_b}(x_{m_i+1}, x_{m_i}) + s d_{p_b}(x_{m_i}, x_{n_i-1}),$$

and hence

$$\limsup_{i\to\infty} d_{p_b}(x_{m_i+1}, x_{n_i-1}) \leq s\varepsilon.$$

On the other hand, by the definition of $d_{p_b}$ and (3.6),

$$\limsup_{i\to\infty} d_{p_b}(x_{m_i}, x_{n_i-1}) = 2 \limsup_{i\to\infty} p_b(x_{m_i}, x_{n_i-1}).$$

Hence, by (3.10),

$$\frac{\varepsilon}{2s} \leq \liminf_{i\to\infty} p_b(x_{m_i}, x_{n_i-1}) \leq \limsup_{i\to\infty} p_b(x_{m_i}, x_{n_i-1}) \leq \frac{\varepsilon}{2}. \tag{3.11}$$

Similarly,

$$\limsup_{i\to\infty} p_b(x_{m_i}, x_{n_i}) \leq \frac{s\varepsilon}{2}, \tag{3.12}$$

$$\frac{\varepsilon}{2s} \leq \limsup_{i\to\infty} p_b(x_{m_i+1}, x_{n_i}), \tag{3.13}$$

$$\limsup_{i\to\infty} p_b(x_{m_i+1}, x_{n_i-1}) \leq \frac{s\varepsilon}{2}. \tag{3.14}$$

From (3.1), we have

$$\begin{aligned}\psi\left(sp_b(x_{m_i+1}, x_{n_i})\right) &= \psi\left(sp_b(fx_{m_i}, fx_{n_i-1})\right) \\ &\leq \psi\left(M_s^f(x_{m_i}, x_{n_i-1})\right) - \varphi\left(M_s^f(x_{m_i}, x_{n_i-1})\right),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}&M_s^f(x_{m_i}, x_{n_i-1}) \\ &\quad= \max\left\{p_b(x_{m_i}, x_{n_i-1}), p_b(x_{m_i}, fx_{m_i}), p_b(x_{n_i-1}, fx_{n_i-1}),\right. \\ &\quad\quad \left.\frac{p_b(x_{m_i}, fx_{n_i-1}) + p_b(fx_{m_i}, x_{n_i-1})}{2s}\right\} \\ &\quad= \max\left\{p_b(x_{m_i}, x_{n_i-1}), p_b(x_{m_i}, x_{m_i+1}), p_b(x_{n_i-1}, x_{n_i}),\right. \\ &\quad\quad \left.\frac{p_b(x_{m_i}, x_{n_i}) + p_b(x_{m_i+1}, x_{n_i-1})}{2s}\right\}.\end{aligned} \tag{3.16}$$

Taking the upper limit as $i \to \infty$ in (3.16) and using (3.6), (3.11), (3.12) and (3.14), we get

$$\begin{aligned}\limsup_{i\to\infty} M_s^f(x_{m_i}, x_{n_i-1}) &= \max\Big\{\limsup_{i\to\infty} p_b(x_{m_i}, x_{n_i-1}), 0, 0, \\ &\qquad \frac{\limsup_{i\to\infty} p_b(x_{m_i}, x_{n_i}) + \limsup_{i\to\infty} p_b(x_{m_i+1}, x_{n_i-1})}{2s}\Big\} \\ &\leq \max\left\{\frac{\varepsilon}{2}, \frac{\frac{\varepsilon s+\varepsilon s}{2}}{2s}\right\} = \frac{\varepsilon}{2}. \end{aligned} \tag{3.17}$$

Now, taking the upper limit as $i \to \infty$ in (3.15) and using (3.13) and (3.17), we have

$$\begin{aligned}\psi\left(s\frac{\varepsilon}{2s}\right) &\leq \psi\Big(s \limsup_{i\to\infty} p_b(x_{m_i+1}, x_{n_i})\Big) \\ &\leq \psi\Big(\limsup_{i\to\infty} M_s^f(x_{m_i}, x_{n_i-1})\Big) - \liminf_{i\to\infty} \varphi\big(M_s^f(x_{m_i}, x_{n_i-1})\big) \\ &\leq \psi\left(\frac{\varepsilon}{2}\right) - \varphi\Big(\liminf_{i\to\infty} M_s^f(x_{m_i}, x_{n_i-1})\Big),\end{aligned}$$

which further implies that

$$\varphi\Big(\liminf_{i\to\infty} M_s^f(x_{m_i}, x_{n_i-1})\Big) = 0,$$

so $\liminf_{i\to\infty} M_s^f(x_{m_i}, x_{n_i-1}) = 0$, and by (3.16) we get $\liminf_{i\to\infty} d_{p_b}(x_{m_i}, x_{n_i-1}) = 0$, a contradiction with (3.11).

Thus, we have proved that $\{x_n\}$ is a $b$-Cauchy sequence in the $b$-metric space $(X, d_{p_b})$. Since $(X, p_b)$ is $p_b$-complete, then from Lemma 1, $(X, d_{p_b})$ is a $b$-complete $b$-metric space. Therefore, the sequence $\{x_n\}$ converges to some $z \in X$, that is, $\lim_{n\to\infty} d_{p_b}(x_n, z) = 0$. Again, from Lemma 1,

$$\lim_{n\to\infty} p_b(z, x_n) = \lim_{n\to\infty} p_b(x_n, x_n) = p_b(z, z).$$

On the other hand, thanks to (3.6) and condition $(\mathrm{p}_{b2})$, $\lim_{n\to\infty} p_b(x_n, x_n) = 0$, which yields that

$$\lim_{n\to\infty} p_b(z, x_n) = \lim_{n\to\infty} p_b(x_n, x_n) = p_b(z, z) = 0.$$

Using the triangular inequality, we get

$$p_b(z, fz) \leq s p_b(z, fx_n) + s p_b(fx_n, fz).$$

Letting $n \to \infty$ and using the continuity of $f$, we get

$$p_b(z, fz) \leq s \lim_{n\to\infty} p_b(z, fx_n) + s \lim_{n\to\infty} p_b(fx_n, fz) = s p_b(fz, fz). \tag{3.18}$$

Note that from (3.1), we have

$$\psi\big(s p_b(fz, fz)\big) \leq \psi\big(M_s^f(z, z)\big) - \varphi\big(M_s^f(z, z)\big), \tag{3.19}$$

where

$$M_s^f(z,z) = \max\left\{p_b(z,z), p_b(z,fz), p_b(z,fz), \frac{p_b(z,fz)+p_b(z,fz)}{2s}\right\} = p_b(fz,z).$$

Hence, as $\psi$ is nondecreasing, we have $sp_b(fz,fz) \le p_b(fz,z)$. Thus, by (3.18) we obtain that $sp_b(fz,fz) = p_b(fz,z)$. But then, using (3.19), we get that $\varphi(M_s^f(z,z)) = 0$.

Hence, we have $p_b(fz,z) = 0$ and $fz = z$. Thus, $z$ is a fixed point of $f$. □

We will show now that the continuity of $f$ in Theorem 1 is not necessary and can be replaced by another assumption.

**Theorem 2** *Under the hypotheses of Theorem* 1, *without the continuity assumption on* $f$, *assume that whenever* $\{x_n\}$ *is a nondecreasing sequence in* $X$ *such that* $x_n \to x \in X$, *one has* $x_n \preceq x$ *for all* $n \in \mathbb{N}$. *Then* $f$ *has a fixed point in* $X$.

*Proof* Following similar arguments as those given in Theorem 1, we construct an increasing sequence $\{x_n\}$ in $X$ such that $x_n \to z$ for some $z \in X$. Using the assumption on $X$, we have $x_n \preceq z$ for all $n \in \mathbb{N}$. Now, we show that $fz = z$. By (3.1), we have

$$\begin{aligned}\psi\big(sp_b(x_{n+1},fz)\big) &= \psi\big(sp_b(fx_n,fz)\big)\\ &\le \psi\big(M_s^f(x_n,z)\big) - \varphi\big(M_s^f(x_n,z)\big),\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}M_s^f(x_n,z) &= \max\left\{p_b(x_n,z), p_b(x_n,fx_n), p_b(z,fz), \frac{p_b(x_n,fz)+p_b(fx_n,z)}{2s}\right\}\\ &= \max\left\{p_b(x_n,z), p_b(x_n,x_{n+1}), p_b(z,fz), \frac{p_b(x_n,fz)+p_b(x_{n+1},z)}{2s}\right\}.\end{aligned}\tag{3.21}$$

Letting $n \to \infty$ in (3.21) and using Lemma 2, we get

$$\begin{aligned}\frac{p_b(z,fz)}{2s^2} &= \min\left\{p_b(z,fz), \frac{\frac{p_b(z,fz)}{s}}{2s}\right\} \le \liminf_{i\to\infty} M_s^f(x_n,z) \le \limsup_{i\to\infty} M_s^f(x_n,z)\\ &\le \max\left\{p_b(z,fz), \frac{sp_b(z,fz)}{2s}\right\} = p_b(z,fz).\end{aligned}\tag{3.22}$$

Again, taking the upper limit as $n \to \infty$ in (3.20) and using Lemma 2 and (3.22), we get

$$\begin{aligned}\psi\big(p_b(z,fz)\big) &= \psi\left(s\frac{1}{s}p_b(z,fz)\right) \le \psi\left(s\limsup_{n\to\infty} p_b(x_{n+1},fz)\right)\\ &\le \psi\left(\limsup_{n\to\infty} M_s^f(x_n,z)\right) - \liminf_{n\to\infty}\varphi\big(M_s^f(x_n,z)\big)\\ &\le \psi\big(p_b(z,fz)\big) - \varphi\left(\liminf_{n\to\infty} M_s^f(x_n,z)\right).\end{aligned}$$

Therefore, $\varphi(\liminf_{n\to\infty} M_s^f(x_n,z)) \le 0$, equivalently, $\liminf_{n\to\infty} M_s^f(x_n,z) = 0$. Thus, from (3.22) we get $z = fz$, and hence $z$ is a fixed point of $f$. □

**Corollary 1** *Let $(X,\preceq,p_b)$ be a $p_b$-complete ordered partial b-metric space. Let $f:X\to X$ be a continuous mapping, nondecreasing with respect to $\preceq$. Suppose that there exists $k\in[0,1)$ such that*

$$p_b(fx,fy)\le\frac{k}{s}\max\left\{p_b(x,y),p_b(x,fx),p_b(y,fy),\frac{p_b(x,fy)+p_b(y,fx)}{2s}\right\}$$

*for all comparable elements $x,y\in X$. If there exists $x_0\in X$ such that $x_0\preceq fx_0$, then $f$ has a fixed point.*

*Proof* Follows from Theorem 1 by taking $\psi(t)=t$ and $\varphi(t)=(1-k)t$, for all $t\in[0,+\infty)$. □

**Corollary 2** *Under the hypotheses of Corollary 1, without the continuity assumption on $f$, for any nondecreasing sequence $\{x_n\}$ in $X$ such that $x_n\to x\in X$, let us have $x_n\preceq x$ for all $n\in\mathbb{N}$. Then $f$ has a fixed point in $X$.*

Now, in order to support the usability of our results, we present the following example.

**Example 4** Let $X=[0,+\infty)$ be equipped with the partial order $\preceq$ defined by

$$x\preceq y\quad\Longleftrightarrow\quad x=y\vee\bigl(x,y\in[0,1]\wedge x\le y\bigr),$$

and with the partial $b$-metric $p_b$ given by $p_b(x,y)=[\max\{x,y\}]^2$ (with $s=2$). Consider the mapping $f:X\to X$ given by

$$fx=\begin{cases}\frac{x}{\sqrt{2}\sqrt{1+x}}, & x\in[0,1],\\ \frac{x}{2}, & x>1.\end{cases}$$

Then $f$ is continuous and increasing, and $0\preceq f0$. Take altering distance functions

$$\psi(t)=t,\qquad\varphi(t)=\begin{cases}\frac{t\sqrt{t}}{1+\sqrt{t}}, & 0\le t\le 1,\\ \frac{t}{2}, & t>1.\end{cases}$$

In order to check the contractive condition (3.1) of Theorem 1, without loss of generality, we may take $x,y\in X$ such that $y\preceq x$. Consider the following two possible cases.

*Case* 1. $0\le y\le x\le 1$. Then

$$p_b(fx,fy)=\left[\max\left\{\frac{x}{\sqrt{2}\sqrt{1+x}},\frac{y}{\sqrt{2}\sqrt{1+y}}\right\}\right]^2=\frac{x^2}{2(1+x)}$$

and

$$M_s^f(x,y)=\max\left\{x^2,x^2,y^2,\frac{x^2+\max^2\{y,\frac{x}{\sqrt{2}\sqrt{1+x}}\}}{2s}\right\}=x^2.$$

Thus, (3.1) reduces to

$$\psi\left(2\cdot\frac{x^2}{2(1+x)}\right)=\frac{x^2}{1+x}\le x^2-\frac{x^3}{1+x}=\psi\bigl(x^2\bigr)-\varphi\bigl(x^2\bigr).$$

*Case* 2. $x = y \geq 1$. Then $p_b(fx,fy) = \frac{x^2}{4}$ and $M_s^f(x,y) = x^2$, so (3.1) reduces to

$$\psi\left(2 \cdot \frac{x^2}{4}\right) = \frac{x^2}{2} \leq x^2 - \frac{x^2}{2} = \psi\left(x^2\right) - \varphi\left(x^2\right).$$

Hence, all the conditions of Theorem 1 are satisfied and $f$ has a fixed point (which is $z = 0$).

## 4 Common fixed point results in partial *b*-metric spaces

Let $(X, \preceq, p_b)$ be an ordered partial $b$-metric space with the coefficient $s \geq 1$, and let $f, g : X \to X$ be two mappings. Set

$$M_s^{f,g}(x,y) = \max\left\{p_b(x,y), p_b(x,fx), p_b(y,gy), \frac{p_b(x,gy) + p_b(y,fx)}{2s}\right\}.$$

Now, we present the following definition.

**Definition 12** Let $(X, \preceq, p_b)$ be an ordered partial $b$-metric space, and let $\psi$ and $\varphi$ be altering distance functions. We say that a pair $(f, g)$ of self-mappings $f, g : X \to X$ is a generalized $(\psi, \varphi)_s$-contraction pair if

$$\psi\left(s^2 p_b(fx,gy)\right) \leq \psi\left(M_s^{f,g}(x,y)\right) - \varphi\left(M_s^{f,g}(x,y)\right) \tag{4.1}$$

for all comparable $x, y \in X$.

**Definition 13** [42] Let $(X, \preceq)$ be a partially ordered set. Then two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

**Theorem 3** *Let $(X, \preceq, p_b)$ be a $p_b$-complete ordered partial $b$-metric space with the coefficient $s \geq 1$, and let $f, g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that $(f, g)$ is a generalized $(\psi, \varphi)_s$-contraction pair for some altering distance functions $\psi$ and $\varphi$. If $f$ and $g$ are continuous, then $f$ and $g$ have a common fixed point.*

*Proof* Let us divide the proof into two parts as follows.

*First part*. We prove that $u \in X$ is a fixed point of $f$ if and only if it is a fixed point of $g$. Suppose that $u$ is a fixed point of $f$, that is, $fu = u$. As $u \preceq u$, by (4.1), we have

$$\begin{aligned}
\psi\left(s^2 p_b(u,gu)\right) &= \psi\left(s^2 p_b(fu,gu)\right) \\
&\leq \psi\left(\max\left\{p_b(u,u), p_b(u,fu), p_b(u,gu), \frac{1}{2s}\left(p_b(u,gu) + p_b(u,fu)\right)\right\}\right) \\
&\quad - \varphi\left(\max\left\{p_b(u,u), p_b(u,fu), p_b(u,gu), \frac{1}{2s}\left(p_b(u,gu) + p_b(u,fu)\right)\right\}\right) \\
&\leq \psi\left(p_b(u,gu)\right) - \varphi\left(\max\left\{p_b(u,u), p_b(u,fu), p_b(u,gu), \frac{1}{2s}\left(p_b(u,gu) + p_b(u,fu)\right)\right\}\right) \\
&\leq \psi\left(s^2 p_b(u,gu)\right) \\
&\quad - \varphi\left(\max\left\{p_b(u,u), p_b(u,fu), p_b(u,gu), \frac{1}{2s}\left(p_b(u,gu) + p_b(u,fu)\right)\right\}\right).
\end{aligned}$$

Therefore, $p_b(u, gu) = 0$ and hence $gu = u$. Similarly, we can show that if $u$ is a fixed point of $g$, then $u$ is a fixed point of $f$.

*Second part* (construction of a sequence by iterative technique).

Let $x_0 \in X$. We construct a sequence $\{x_n\}$ in $X$ such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all nonnegative integers $n$. As $f$ and $g$ are weakly increasing with respect to $\preceq$, we have

$$\begin{aligned} x_1 = fx_0 \preceq gfx_0 = x_2 = gx_1 \preceq fgx_1 = x_3 \preceq \cdots \\ \preceq x_{2n+1} = fx_{2n} \preceq gfx_{2n} = x_{2n+2} \preceq \cdots . \end{aligned}$$

If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}$, then $x_{2n} = fx_{2n}$. Thus $x_{2n}$ is a fixed point of $f$. By the first part, we conclude that $x_{2n}$ is also a fixed point of $g$.

If $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N}$, then $x_{2n+1} = gx_{2n+1}$. Thus, $x_{2n+1}$ is a fixed point of $g$. By the first part, we conclude that $x_{2n+1}$ is also a fixed point of $f$. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Now, we complete the proof in the following steps.

*Step* 1: We will prove that

$$\lim_{n\to\infty} p_b(x_n, x_{n+1}) = 0.$$

As $x_{2n+1}$ and $x_{2n+2}$ are comparable, by (4.1), we have

$$\begin{aligned} \psi\big(p_b(x_{2n+1}, x_{2n+2})\big) &\leq \psi\big(s^2 p_b(x_{2n+1}, x_{2n+2})\big) \\ &= \psi\big(s^2 p_b(fx_{2n}, gx_{2n+1})\big) \\ &\leq \psi\big(M_s^{f,g}(x_{2n}, x_{2n+1})\big) - \varphi\big(M_s^{f,g}(x_{2n}, x_{2n+1})\big), \end{aligned}$$

where

$$\begin{aligned} M_s^{f,g}(x_{2n}, x_{2n+1}) = {} & \max\bigg\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n}, fx_{2n}), p_b(x_{2n+1}, gx_{2n+1}), \\ & \frac{p_b(fx_{2n}, x_{2n+1}) + p_b(x_{2n}, gx_{2n+1})}{2s}\bigg\} \\ = {} & \max\bigg\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2}), \\ & \frac{p_b(x_{2n+1}, x_{2n+1}) + p_b(x_{2n}, x_{2n+2})}{2s}\bigg\} \\ \leq {} & \max\bigg\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2}), \\ & \frac{sp_b(x_{2n}, x_{2n+1}) + sp_b(x_{2n+1}, x_{2n+2})}{2s}\bigg\} \\ = {} & \max\big\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2})\big\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \psi\big(p_b(x_{2n+1}, x_{2n+2})\big) \leq {} & \psi\big(\max\big\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2})\big\}\big) \\ & - \varphi\big(\max\big\{p_b(x_{2n}, x_{2n+1}), p_b(x_{2n+1}, x_{2n+2})\big\}\big). \end{aligned} \tag{4.2}$$

If

$$\max\{p_b(x_{2n},x_{2n+1}),p_b(x_{2n+1},x_{2n+2})\} = p_b(x_{2n+1},x_{2n+2}),$$

then (4.2) becomes

$$\begin{aligned}\psi(p_b(x_{2n+1},x_{2n+2})) &\le \psi(p_b(x_{2n+1},x_{2n+2})) - \varphi(p_b(x_{2n+1},x_{2n+2})) \\ &< \psi(p_b(x_{2n+1},x_{2n+2})),\end{aligned}$$

which gives a contradiction. Hence,

$$\max\{p_b(x_{2n},x_{2n+1}),p_b(x_{2n+1},x_{2n+2})\} = p_b(x_{2n},x_{2n+1}),$$

and (4.2) becomes

$$\begin{aligned}\psi(p_b(x_{2n+1},x_{2n+2})) &\le \psi(p_b(x_{2n},x_{2n+1})) - \varphi(p_b(x_{2n},x_{2n+1})) \\ &\le \psi(p_b(x_{2n},x_{2n+1})).\end{aligned} \tag{4.3}$$

Similarly, we can show that

$$\psi(p_b(x_{2n+1},x_{2n})) \le \psi(p_b(x_{2n-1},x_{2n})) - \varphi(p_b(x_{2n-1},x_{2n})) \le \psi(p_b(x_{2n-1},x_{2n})). \tag{4.4}$$

By (4.3) and (4.4), we get that $\{p_b(x_n,x_{n+1}) : n \in \mathbb{N}\}$ is a nonincreasing sequence of positive numbers. Hence, there is $r \ge 0$ such that

$$\lim_{n\to\infty} p_b(x_n,x_{n+1}) = r.$$

Letting $n \to \infty$ in (4.3), we get

$$\psi(r) \le \psi(r) - \varphi(r) \le \psi(r),$$

which implies that $\varphi(r) = 0$ and hence $r = 0$. So, we have

$$\lim_{n\to\infty} p_b(x_n,x_n) \le \lim_{n\to\infty} p_b(x_n,x_{n+1}) = 0. \tag{4.5}$$

*Step* 2. We will prove that $\{x_n\}$ is a $p_b$-Cauchy sequence. Because of (4.5), it is sufficient to show that $\{x_{2n}\}$ is a $p_b$-Cauchy sequence. By Lemma 1, we should show that $\{x_{2n}\}$ is $b$-Cauchy in $(X,d_{p_b})$. Suppose the contrary, *i.e.*, that $\{x_{2n}\}$ is not a $b$-Cauchy sequence in $(X,d_{p_b})$. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{2m_i}\}$ and $\{x_{2n_i}\}$ of $\{x_{2n}\}$ such that $n_i$ is the smallest index for which

$$n_i > m_i > i, \quad d_{p_b}(x_{2m_i},x_{2n_i}) \ge \varepsilon. \tag{4.6}$$

This means that

$$d_{p_b}(x_{2m_i},x_{2n_i-2}) < \varepsilon. \tag{4.7}$$

From (4.6) and using the triangular inequality, we get

$$\varepsilon \leq d_{p_b}(x_{2m_i}, x_{2n_i}) \leq sd_{p_b}(x_{2m_i}, x_{2m_i+1}) + sd_{p_b}(x_{2m_i+1}, x_{2n_i}).$$

Using (4.5) and taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d_{p_b}(x_{2m_i+1}, x_{2n_i}).$$

On the other hand, we have

$$d_{p_b}(x_{2m_i}, x_{2n_i-1}) \leq sd_{p_b}(x_{2m_i}, x_{2n_i-2}) + sd_{p_b}(x_{2n_i-2}, x_{2n_i-1}).$$

Using (4.5), (4.7) and taking the upper limit as $i \to \infty$, we get

$$\limsup_{i\to\infty} d_{p_b}(x_{2m_i}, x_{2n_i-1}) \leq \varepsilon s. \tag{4.8}$$

Again, using the triangular inequality, we have

$$\begin{aligned} d_{p_b}(x_{2m_i}, x_{2n_i}) &\leq sd_{p_b}(x_{2m_i}, x_{2n_i-2}) + sd_{p_b}(x_{2n_i-2}, x_{2n_i}) \\ &\leq sd_{p_b}(x_{2m_i}, x_{2n_i-2}) + s^2 d_{p_b}(x_{2n_i-2}, x_{2n_i-1}) + s^2 d_{p_b}(x_{2n_i-1}, x_{2n_i}) \end{aligned}$$

and

$$d_{p_b}(x_{2m_i+1}, x_{2n_i-1}) \leq sd_{p_b}(x_{2m_i+1}, x_{2m_i}) + sd_{p_b}(x_{2m_i}, x_{2n_i-1}).$$

Taking the upper limit as $i \to \infty$ in the above inequalities and using (4.5), (4.7) and (4.8), we get

$$\limsup_{i\to\infty} d_{p_b}(x_{2m_i}, x_{2n_i}) \leq \varepsilon s$$

and

$$\limsup_{i\to\infty} d_{p_b}(x_{2m_i+1}, x_{2n_i-1}) \leq \varepsilon s^2.$$

From the definition of $d_{p_b}$ and (4.5), we have the following relations:

$$\frac{\varepsilon}{2s} \leq \limsup_{i\to\infty} p_b(x_{2m_i+1}, x_{2n_i}), \tag{4.9}$$

$$\frac{\varepsilon}{2s} \leq \liminf_{i\to\infty} p_b(x_{2m_i}, x_{2n_i-1}) \leq \limsup_{i\to\infty} p_b(x_{2m_i}, x_{2n_i-1}) \leq \frac{s\varepsilon}{2}, \tag{4.10}$$

$$\limsup_{i\to\infty} p_b(x_{2m_i}, x_{2n_i}) \leq \frac{s\varepsilon}{2}, \tag{4.11}$$

$$\limsup_{i\to\infty} p_b(x_{2m_i+1}, x_{2n_i-1}) \leq \frac{s^2\varepsilon}{2}. \tag{4.12}$$

Since $x_{2m_i}$ and $x_{2n_i-1}$ are comparable, using (4.1) we have

$$\begin{aligned}\psi\left(s^2 p_b(x_{2m_i+1},x_{2n_i})\right) &= \psi\left(s^2 p_b(fx_{2m_i},gx_{2n_i-1})\right)\\ &\le \psi\left(M_s^{f,g}(x_{2m_i},x_{2n_i-1})\right) - \varphi\left(M_s^{f,g}(x_{2m_i},x_{2n_i-1})\right),\end{aligned}\tag{4.13}$$

where

$$\begin{aligned}M_s^{f,g}(x_{2m_i},x_{2n_i-1}) = \max\bigg\{&p_b(x_{2m_i},x_{2n_i-1}),p_b(x_{2m_i},x_{2m_i+1}),p_b(x_{2n_i-1},x_{2n_i}),\\ &\frac{p_b(x_{2m_i},x_{2n_i})+p_b(x_{2m_i+1},x_{2n_i-1})}{2s}\bigg\}.\end{aligned}\tag{4.14}$$

Taking the upper limit in (4.14) and using (4.5) and (4.10)-(4.12), we get

$$\begin{aligned}\limsup_{i\to\infty} M_s^{f,g}(x_{2m_i},x_{2n_i-1}) &= \max\bigg\{\limsup_{i\to\infty} p_b(x_{2m_i},x_{2n_i-1}),0,0,\\ &\qquad\frac{\limsup_{i\to\infty} p_b(x_{2m_i},x_{2n_i})+\limsup_{i\to\infty} p_b(x_{2m_i+1},x_{2n_i-1})}{2s}\bigg\}\\ &\le \max\left\{\frac{s\varepsilon}{2},\frac{\frac{\varepsilon s+\varepsilon s^2}{2}}{2s}\right\} = \frac{s\varepsilon}{2}.\end{aligned}\tag{4.15}$$

Now, taking the upper limit as $i\to\infty$ in (4.13) and using (4.9) and (4.15), we have

$$\begin{aligned}\psi\left(\frac{s\varepsilon}{2}\right) = \psi\left(s^2\frac{\varepsilon}{2s}\right) &\le \psi\left(s^2\limsup_{i\to\infty} p_b(x_{2m_i+1},x_{2n_i})\right)\\ &\le \psi\left(\limsup_{i\to\infty} M_s^{f,g}(x_{2m_i},x_{2n_i-1})\right) - \varphi\left(\liminf_{i\to\infty} M_s^{f,g}(x_{2m_i},x_{2n_i-1})\right)\\ &\le \psi\left(\frac{s\varepsilon}{2}\right) - \varphi\left(\liminf_{i\to\infty} M_s^{f,g}(x_{2m_i},x_{2n_i-1})\right),\end{aligned}$$

which implies that $\varphi(\liminf_{i\to\infty} M_s^{f,g}(x_{2m_i},x_{2n_i-1}))=0$. By (4.14), it follows that

$$\liminf_{i\to\infty} p_b(x_{2m_i},x_{2n_i-1}) = 0,$$

which is in contradiction with (4.10). Thus, we have proved that $\{x_n\}$ is a $b$-Cauchy sequence in the metric space $(X,d_{p_b})$. Since $(X,p_b)$ is $p_b$-complete, then from Lemma 1, $(X,d_{p_b})$ is a $b$-complete $b$-metric space. Therefore, the sequence $\{x_n\}$ converges to some $z\in X$, that is, $\lim_{n\to\infty} d_{p_b}(x_n,z)=0$. Again, from Lemma 1,

$$\lim_{n\to\infty} p_b(z,x_n) = \lim_{n\to\infty} p_b(x_n,x_n) = p_b(z,z).$$

On the other hand, from (4.5) we get that

$$\lim_{n\to\infty} p_b(z,x_n) = \lim_{n\to\infty} p_b(x_n,x_n) = p_b(z,z) = 0.$$

*Step* 3 (Existence of a common fixed point). Using the triangular inequality, we get

$$p_b(z,fz) \le sp_b(z,fx_{2n}) + sp_b(fx_{2n},fz),$$
$$p_b(z,gz) \le sp_b(z,gx_{2n+1}) + sp_b(gx_{2n+1},gz).$$

Letting $n \to \infty$ and using the continuity of $f$ and $g$, we get

$$p_b(z,fz) \le s \lim_{n\to\infty} p_b(z,fx_{2n}) + s \lim_{n\to\infty} p_b(fx_{2n},fz) = sp_b(fz,fz),$$
$$p_b(z,gz) \le s \lim_{n\to\infty} p_b(z,gx_{2n+1}) + s \lim_{n\to\infty} p_b(gx_{2n+1},gz) = sp_b(gz,gz).$$

Therefore,

$$\max\{p_b(z,fz),p_b(z,gz)\} \le \max\{sp_b(fz,fz),sp_b(gz,gz)\} \le s^2 p_b(gz,fz). \tag{4.16}$$

From (4.1), we have

$$\psi\left(s^2 p_b(fz,gz)\right) \le \psi\left(M_s^{f,g}(z,z)\right) - \varphi\left(M_s^{f,g}(z,z)\right), \tag{4.17}$$

where

$$\begin{aligned} M_s^{f,g}(z,z) &= \max\left\{p_b(z,z),p_b(z,fz),p_b(z,gz),\frac{p_b(z,gz)+p_b(z,fz)}{2s}\right\} \\ &= \max\{p_b(z,fz),p_b(z,gz)\}. \end{aligned}$$

As $\psi$ is nondecreasing, we have $s^2 p_b(fz,gz) \le \max\{p_b(z,fz),p_b(z,gz)\}$. Hence, by (4.16) we obtain that $s^2 p_b(fz,gz) = \max\{p_b(z,fz),p_b(z,gz)\}$. But then, using (4.17), we get that $\varphi(M_s^{f,g}(z,z)) = 0$. Thus, we have $fz = gz = z$ and $z$ is a common fixed point of $f$ and $g$. □

The continuity of functions $f$ and $g$ in Theorem 3 can be replaced by another condition.

**Theorem 4** *Under the hypotheses of Theorem* 3, *without the continuity assumption on the functions $f$ and $g$, for any nondecreasing sequence $\{x_n\}$ in $X$ such that $x_n \to x \in X$, let us have $x_n \preceq x$ for all $n \in \mathbb{N}$. Then $f$ and $g$ have a common fixed point in $X$.*

*Proof* Reviewing the proof of Theorem 3, we construct an increasing sequence $\{x_n\}$ in $X$ such that $x_n \to z$ for some $z \in X$. Using the given assumption on $X$, we have $x_n \preceq z$ for all $n \in \mathbb{N}$. Now, we show that $fz = gz = z$. By (4.1), we have

$$\begin{aligned} \psi\left(s^2 p_b(x_{2n+1},gz)\right) &= \psi\left(s^2 p_b(fx_{2n},gz)\right) \\ &\le \psi\left(M_s^{f,g}(x_{2n},z)\right) - \varphi\left(M_s^{f,g}(x_{2n},z)\right), \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} M_s^{f,g}(x_{2n},z) &= \max\left\{p_b(x_{2n},z),p_b(x_{2n},fx_{2n}),p_b(z,gz),\frac{p_b(x_{2n},gz)+p_b(fx_{2n},z)}{2s}\right\} \\ &= \max\left\{p_b(x_{2n},z),p_b(x_{2n},x_{2n+1}),p_b(z,gz),\frac{p_b(x_{2n},gz)+p_b(x_{2n+1},z)}{2s}\right\}. \end{aligned} \tag{4.19}$$

Letting $n \to \infty$ in (4.19) and using Lemma 2, we get

$$\begin{aligned}\frac{p_b(z,gz)}{s^2} &\leq \max\left\{p_b(z,gz), \frac{\frac{p_b(z,gz)}{s}}{2s}\right\} \leq \liminf_{n\to\infty} M_s^{f,g}(x_{2n},z)\\ &\leq \limsup_{n\to\infty} M_s^{f,g}(x_{2n},z) \leq \max\left\{p_b(z,gz), \frac{sp_b(z,gz)}{2s}\right\} = p_b(z,gz).\end{aligned} \tag{4.20}$$

Again, taking the upper limit as $n \to \infty$ in (4.18) and using Lemma 2 and (4.20), we get

$$\begin{aligned}\psi\left(p_b(z,gz)\right) &= \psi\left(s^2\frac{1}{s^2}p_b(z,gz)\right) \leq \psi\left(s^2 \limsup_{n\to\infty} d(x_{2n+1},gz)\right)\\ &\leq \psi\left(\limsup_{n\to\infty} M_s^{f,g}(x_{2n},z)\right) - \varphi\left(\liminf_{n\to\infty} M_s^{f,g}(x_{2n},z)\right)\\ &\leq \psi\left(p_b(z,gz)\right) - \varphi\left(\liminf_{n\to\infty} M_s^{f,g}(x_{2n},z)\right).\end{aligned}$$

Therefore, $\varphi(\liminf_{n\to\infty} M_s^{f,g}(x_{2n},z)) \leq 0$, equivalently, $\liminf_{n\to\infty} M_s^{f,g}(x_{2n},z) = 0$. Thus, from (4.20) we get $z = gz$ and hence $z$ is a fixed point of $g$. On the other hand, similar to the first part of the proof of Theorem 3, we can show that $fz = z$. Hence, $z$ is a common fixed point of $f$ and $g$. □

Also, we have the following results.

**Corollary 3** *Let $(X,\preceq,p_b)$ be a $p_b$-complete ordered partial b-metric space with the coefficient $s \geq 1$, and let $f,g : X \to X$ be two weakly increasing mappings with respect to $\preceq$. Suppose that there exists $k \in [0,1)$ such that*

$$p_b(fx,gy) \leq \frac{k}{s^2}\max\left\{p_b(x,y), p_b(x,fx), p_b(y,gy), \frac{p_b(x,gy)+p_b(fx,y)}{2s}\right\}$$

*for all comparable elements $x,y \in X$. If $f$ and $g$ are continuous, then $f$ and $g$ have a common fixed point.*

**Corollary 4** *Under the hypotheses of Corollary 3, without the continuity assumption on the functions $f$ and $g$, assume that whenever $\{x_n\}$ is a nondecreasing sequence in $X$ such that $x_n \to x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Then $f$ and $g$ have a common fixed point in $X$.*

**Remark 1** Recall that a subset $W$ of a partially ordered set $X$ is said to be well ordered if every two elements of $W$ are comparable. Note that in Theorems 1 and 2, it can be proved in a standard way that $f$ has a unique fixed point provided that the fixed points of $f$ are comparable. Similarly, in Theorems 3 and 4, the set of common fixed points of $f$ and $g$ is well ordered if and only if $f$ and $g$ have one and only one common fixed point.

The usability of these results is demonstrated by the following example.

**Example 5** Let $X = \{0,1,2,3,4\}$ be equipped with the following partial order $\preceq$:

$$\preceq := \left\{(0,0),(1,1),(1,2),(2,2),(3,3),(4,2),(4,4)\right\}.$$

Define a partial $b$-metric $p_b : X \times X \to \mathbb{R}^+$ by

$$p_b(x,y) = \begin{cases} 0 & \text{if } x = y, \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

It is easy to see that $(X, p_b)$ is a $p_b$-complete partial $b$-metric space, with $s = 49/25$.

Define self-maps $f$ and $g$ by

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 1 & 2 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 1 & 1 \end{pmatrix}.$$

We see that $f$ and $g$ are weakly increasing mappings with respect to $\preceq$ and that $f$ and $g$ are continuous.

Define $\psi, \varphi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \sqrt{t}$ and $\varphi(t) = \frac{t}{300}$. In order to check that $(f,g)$ is a generalized $(\psi,\varphi)_s$-contractive pair, only the case $x = 2$, $y = 4$ is nontrivial (when $x$ and $y$ are comparable and the left-hand side of condition (4.1) is positive). Then

$$\psi\big(s^2 p_b(f2, g4)\big) = \sqrt{s^2 \cdot 3^2} = \frac{147}{25} = \sqrt{36} - \frac{36}{300} = \psi\big(M_s^{f,g}(2,4)\big) - \varphi\big(M_s^{f,g}(2,4)\big).$$

Thus, all the conditions of Theorem 3 are satisfied and hence $f$ and $g$ have common fixed points. Indeed, 0 and 2 are two common fixed points of $f$ and $g$. Note that the ordered set $(\{0,2\}, \preceq)$ is not well ordered.

Note that if the same example is considered in the space without order, then the contractive condition is not satisfied. For example,

$$\begin{aligned} \psi\big(s^2 p_b(f1, g4)\big) &= \sqrt{s^2 \cdot 3^2} = \frac{147}{25} \\ &> \frac{59}{12} = \sqrt{25} - \frac{25}{300} = \psi\big(M_s^{f,g}(1,4)\big) - \varphi\big(M_s^{f,g}(1,4)\big). \end{aligned}$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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